BULL. AUSTRAL. MATH. SOC. 90C30
VOL. 46 (1992) [385–389]

ON SUFFICIENCY OF THE KUHN-TUCKER CONDITIONS IN NONDIFFERENTIABLE PROGRAMMING

FUAN ZHAO

Some generalised invex conditions are given for a nondifferentiable constrained optimisation problem, generalising those of Hanson and Mond for differentiable problems. Some duality results are obtained.

1. INTRODUCTION

Let $f, g_1, g_2, \ldots, g_m$ be local Lipschitz functions defined on an open subset C of R^n. Consider the problem

$$\text{(P)} \qquad \text{MIN}\, f(x) \text{ subject to } g_i(x) \leqslant 0 \qquad (i = 1, 2, \ldots, m),\, x \in C.$$

Several necessary conditions have been established for a local optimal solution of (P) to satisfy Kuhn-Tucker conditions, for example [6, 7, 2]. Denote by $f^0(x; d)$ the Clarke generalised directional derivative of f at x in the direction d (see [2]), and by $\partial f(x)$ the Clarke generalised subgradient of f at x. Then assuming a constraint qualification, the Kuhn-Tucker necessary conditions for a minimum at $\overline{x}$ are:

$$\text{(KT)} \qquad (\exists \lambda_i \geqslant 0, i = 1, 2, \ldots, m)\ 0 \in \partial f(\overline{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\overline{x}),\ (\forall i) \lambda_i g_i(\overline{x}) = 0.$$

If all the functions are continuously differentiable, then $\partial f(\overline{x}) = \{\nabla f(\overline{x})\}$ and $\partial g_i(\overline{x}) = \{\nabla g_i(\overline{x})\}$. These necessary conditions at a feasible point $\overline{x}$ become also sufficient if all the functions are convex, or under weaker conditions given by Hanson [4], or by Hanson and Mond [5].

Received 13 November 1991
The author would like to thank Dr. B.D. Craven of the University of Melbourne for his invaluable help in the preparation of this paper.

2. A simple proof of Hanson and Mond's theorem

Hanson and Mond [5] say that the objective f and the constraint functions $g_i (i = 1, 2, \ldots, m)$ are *Type I invex* with respect to η at $\overline{x}$ if there exists a function $\eta: R^n \to R^n$ such that, for all x feasible for (P),

$$(1) \qquad f(x) - f(\overline{x}) \geqslant [\nabla f(\overline{x})]^T \eta(x) \text{ and } -g_i(\overline{x}) \geqslant [\nabla g_i(\overline{x})]^T \eta(x).$$

Ben-Israel and Mond [1] have characterised such functions η.

Lemma 1. *If $v_0, v_1, v_2, \ldots, v_k$ are vectors in R^n, with $k < n$, and t_i $(i = 0, 1, \ldots, k)$ are nonnegative real numbers, then the linear inequality system $v_i^T x \leqslant t_i$ $(i = 0, 1, \ldots, k)$ has a nonzero solution x.*

Proof: It is sufficient to prove the case when all $t_i = 0$. If $k = n - 1$ and the v_i are linearly independent, then the matrix M whose columns are the v_i is invertible. Let $e = (1, 1, \ldots, 1)$; then $x = -M^{-1}e \neq 0$ satisfies $(\forall i)$ $v_i^T x = -1 < 0$. Suppose that the v_i are linearly dependent, say $\overline{v}_s = \sum_{j \neq s} \lambda_j v_j$ for some numbers λ_j. There is a nonzero x satisfying $v_i^T x = 0$ $(i = 0, 1, 2, \ldots, k, i \neq s)$ since $k < n$; then $v_s^T x = 0$ also. A nonzero x is found in either case. □

Theorem 2. [5, Theorem 2.2]. *If (P) has $k < n$ active constraints at an optimal point $\overline{x}$ for (P), then f and g_i $(i = 1, 2, \ldots, m)$ are Type I invex with respect to a common vector function $\eta \not\equiv 0$.*

Proof: At a feasible point x, $t_0 = f(x) - f(\overline{x}) \geqslant 0$ and each $t_i = -g_i(\overline{x}) \geqslant 0$. Let $v_0 = \nabla f(\overline{x})$ and $v_i = \nabla g_i(\overline{x})$. By Lemma 1, (1) has a nonzero solution $\eta \equiv \eta(x)$. □

3. Sufficiency of the Kuhn-Tucker conditions

If (KT) holds, then

$$\text{(KT2)} \qquad (\exists \zeta_0 \in \partial f(\overline{x}), \zeta_i \in \partial g_i(\overline{x}))\ \zeta_0 + \sum_{i>0} \lambda_i \zeta_i;\ (\forall i > 0)\ \lambda_i \geqslant 0,\ \lambda_i g_i(\overline{x}) = 0;$$

thus $\lambda_i = 0$ for inactive constraints. The function f and g_i will now be called *Type I invex* with respect to a vector function η at $\overline{x}$ if, for each feasible x,

$$(2) \qquad f(x) - f(\overline{x}) \geqslant \zeta_0^T \eta(x) \text{ and } -g_i(\overline{x}) \geqslant \zeta_i^T \eta(x)\, (i = 1, 2, \ldots, m).$$

Inactive constraints may be omitted from (2).

Theorem 3. *Let (KT2) hold at a feasible point $\overline{x}$ of (P), where the number of active constraints is $k < n$. Then $\overline{x}$ is optimal if and only if f and g_i are Type I invex with respect to a common vector function η.*

PROOF: If (2) holds, and x is feasible, then

$$f(x) - f(\overline{x}) = f(x) - f(\overline{x}) + \sum_{i>0} \lambda_i(-g_i(\overline{x})) \geqslant \left(\zeta_0 + \sum_{i>0} \zeta_i\right)^T \eta(x) = 0$$

using (KT2) and (3). Conversely, if $\overline{x}$ is a minimum, then (2) is proved in the same manner as Theorem 2. ∎

REMARKS. The first part of this proof does not use the hypothesis $k < n$.

The vectors ζ_0 and ζ_i cannot be replaced here by *arbitrary* elements of $\partial f(\overline{x})$ and $\partial g_i(\overline{x})$ respectively, because then $\zeta_0 + \sum_{i>0} \zeta_i$ is no longer zero.

In [**3**], a *generalised invex* property was defined in terms of Clarke generalised directional derivatives, namely

(3) $$f(x) - f(\overline{x}) \geqslant f^0(\overline{x}; \eta(x)),\ g_i(x) - g_i(\overline{x}) \geqslant g_i^0(\overline{x}; \eta(x))\ (i = 1, 2, \ldots, m).$$

It was shown in [**3**] that (KT2) at a feasible point $\overline{x}$, together with (3), imply a minimum at $\overline{x}$. Consider now a weakened version of (3), that for all feasible x,

(4) $$f(x) - f(\overline{x}) \geqslant f^0(\overline{x}; \eta(x)),\ g_i(\overline{x}) \geqslant \quad g_i^0(\overline{x}; \eta(x)) \quad (i = 1, 2, \ldots, m).$$

However, (4) is *not* a consequence of (2), whether or not $k < n$.

THEOREM 4. *If (KT2) and (4) hold at a feasible point $\overline{x}$, then $\overline{x}$ is a minimum of (P).*

PROOF: If x is feasible for (P), then

$$\begin{aligned} f(x) - f(\overline{x}) &= f(x) - f(\overline{x}) + \sum_{i>0} \lambda_i(-g_i(\overline{x})) && \text{by (KT2)} \\ &\geqslant f^0(\overline{x}; \eta(x)) + \sum_{i>0} \lambda_i g_i^\circ(\overline{x}; \eta(x)) && \text{by (4)} \\ &\geqslant \theta_0^T \eta(x) + \sum_{i>0} \lambda_i \theta_i^T \eta(x) \\ &\qquad \text{for all } \theta_0 \in \partial f(\overline{x}) \text{ and all } \theta_i \in \partial g_i(\overline{x}) \\ &\geqslant 0 \quad \text{by (KT2), substituting } \theta_i = \zeta_i\ (i \geqslant 0). \end{aligned}$$

∎

https://doi.org/10.1017/S0004972700012041 Published online by Cambridge University Press

4. Subgradient duality

Schechter [8, 9] proposed a dual problem for convex nondifferentiable problems, and proved a *subgradient duality* result. This dual to (P) has the form

$$\text{(D)} \qquad \text{MAX} f(z) + \sum_i \theta_i g_i(z) \text{ subject to } (\forall i) \quad \theta_i \geqslant 0,\, 0 \in \partial f(z) + \sum_i \theta_i \partial g_i(z).$$

If (z, θ) is feasible for (D), then there exist $\omega_0 \in \partial f(z)$ and $\omega_i \in \partial g_i(z)$ $(i > 0)$ such that $0 = \omega_0 + \sum_{i>0} \theta_i \omega_i$. Consider the modified invex property, generalising the *Type II invex* of Hanson and Mond [5], that for some function $\eta(.,.)$,

$$(5) \qquad f(y) - f(z) \geqslant \omega_0^T \eta(y, z), \quad (\forall i > 0) - g_i(z) \geqslant \omega_i^T \eta(y, z).$$

Remark. The definition in [5] assumes the more restrictive $\eta(z)$.

Theorem 5. *Assume that (2) holds whenever x is feasible for (P), and (5) holds whenever is feasible for (P) and (z, θ) is feasible for (D). Then weak duality holds:*

$$f(x) \geqslant f(z) + \sum \theta_i g_i(z).$$

If also (KT2) holds for (P) at $\overline{x}$, with multipliers λ_i, then zero duality gap holds: $(\overline{x}, \lambda)$ is feasible for (D), and $f(\overline{x}) = f(\overline{x}) + \sum \lambda_i g_i(\overline{x})$.

Proof: *Weak duality*

$$\begin{aligned} f(x) - [f(z) + \sum \theta_i g_i(z)] &\geqslant \omega_0^T \eta(y, z) - \sum \theta_i g_i(z) && \text{by (5)} \\ &= -\sum \theta_i \omega_i - \sum \theta_i g_i(z) && \text{by a constraint of (D)} \\ &\geqslant +\sum \theta_i g_i(z) - \sum \theta_i g_i(z) && \text{by (5)} \\ &= 0. \end{aligned}$$

Zero duality gap The statements follow from (KT2). ∎

Theorem 6. *Let $(\overline{x}, \lambda)$ be feasible for (D), let (5) hold at $y = \overline{x}$, and let $\lambda^T g(\overline{x}) = 0$. Then $(\overline{x}, \lambda)$ is optimal for (D).*

Proof: There are $\omega_0 \in \partial f(z)$ and $\omega_i \in \partial g_i(z)$ such that $0 = \omega_0 + \sum_{i>0} \lambda_i \omega_i$. A similar calculation to the *weak duality* proof of Theorem 5 shows that, if (z, θ) is feasible for (D), then

$$[f(\overline{x}) + \lambda^T g(\overline{x})] - [f(z) + \theta^T g(z)] \geqslant \lambda^T g(\overline{x}) - 0 \text{ by hypotheses.}$$

∎

References

[1] A. Ben-Israel and B. Mond, 'What is invexity?', *J. Austral. Math. Soc.*, series B **28** (1968), 1–9.

[2] F.H. Clarke, *Nonsmooth analysis and optimization* (Wiley, New York, 1983).

[3] B.D. Craven, 'Nondifferentiable approximation by smooth approximations', *Optimization* **17** (1986), 3–17.

[4] M.A. Hanson, 'On sufficiency of the Kuhn-Tucker conditions', *J. Math. Anal. Appl.* **80** (1981), 545–550.

[5] M.A. Hanson and B. Mond, 'Necessary and sufficient conditions in constrained optimization', *Math. Programming* **37** (1987), 51–58.

[6] J.-P. Hiriart-Urruty, 'On necessary optimality conditions in nondifferentiable programming', *Math. Programming* 14 (1978), 73–86.

[7] J.-P. Hiriart-Urruty, 'Refinements of necessary optimality conditions in nondifferentiable programming', *Appl. Math. Optim.* **5** (1979), 63–83.

[8] M. Schechter, 'A subgradient duality theorem', *J. Math. Anal. Appl.* **61** (1977), 850–855.

[9] M. Schechter, 'More on subgradient duality', *J. Math. Anal. Appl.* **71** (1979), 252–262.

Institute of Applied Mathematics
Academia Sinica
Beijing 100080
Peoples Republic of China