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CARLESON MEASURES ON SPACES OF HARDY-SOBOLEV TYPE

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ABSTRACT. We study positive measures on $\mathbb{B}^n$ satisfying that $\int_{\mathbb{B}^n} |f(z)|^p \, d\mu(z) \leq C\|f\|^p_{H^p_\alpha}$, for any $f \in H^p_\alpha$, where H^p_α is the Hardy-Sobolev space in the unit ball. We obtain several computable sufficient conditions as well as some necessary conditions and establish their sharpness. We study the same problem for Besov-Sobolev spaces and give some applications to multipliers.

1. **Introduction.** Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$, and $\mathbb{S}^n$ its boundary. We will denote by dV the normalized Lebesgue volume measure on $\mathbb{B}^n$, and by $d\sigma$ the normalized Lebesgue measure on $\mathbb{S}^n$. For $\alpha \in \mathbb{R}$ and $0 < p < +\infty$, the Hardy-Sobolev space H^p_α consists of holomorphic functions f in $\mathbb{B}^n$ so that $R^\alpha f \in H^p(\mathbb{B}^n)$, where if $f = \sum_k f_k$ is its homogeneous expansion, $R^\alpha f = \sum_k (k+1)^\alpha f_k$.

For $n = 1$, $\alpha = 0$, and $p \geq 1$, Carleson [C] proved that the finite positive Borel measures on $\mathbb{B}^1 = \mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \leq C\|f\|^p_{H^p(\mathbb{D})},$$

are characterized by what is now known as Carleson condition: there exists a constant $K > 0$ with $\mu\big(T(I)\big) \leq K|I|$ for any interval I in the unit circle, where $T(I)$ is the corresponding "Carleson box" over I.

If G is a region, μ a finite positive Borel measure and B a Banach space of continuous functions in G, we say that μ is a *Carleson measure* for B if there exists a constant $C > 0$ such that for any f in B,

$$\int_G |f(z)|^p \, d\mu(z) \leq C\|f\|^p_B. \tag{1.1}$$

The purpose of this work is to study Carleson measures for Hardy-Sobolev spaces and other related spaces.

In some particular cases, these measures have been treated by different authors, ([C], [St], [Lu2], [N-R-S], [A-Bo], [A-J], [F-S], [Ke-S]), as well as their connection with problems about multipliers, interpolation, solution of the $\bar{\partial}$-problem and duality theory for H^1. We will mention briefly the results closely related to our work.

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For $n = 1$, and $p > 1$, Stegenga [St] characterized the Carleson measures for $H^p_\alpha(\mathbb{D})$, $\alpha p \leq 1$ (when $\alpha p > 1$ these spaces consist of continuous functions on $\bar{\mathbb{D}}$, and any finite Borel measure is then Carleson for $H^p_\alpha(\mathbb{D})$). He showed that those measures are the ones satisfying that $\mu\big(T(A)\big) \leq KC_{\alpha p}(A)$ for any open set A in the unit circle, where $C_{\alpha p}$ is an appropriate Bessel capacity depending on α and p. If $p \leq 1$, the characterization in [A-J] is simpler: any finite positive Borel measure is Carleson for $H^p_\alpha(\mathbb{D})$ if and only if $\mu\big(T(I)\big) \leq K|I|^{1-\alpha p}$, for every open interval I in the unit circle. Since $C_{\alpha p}(I) \simeq |I|^{1-\alpha p}$, the condition in [A-J] is Stegenga's condition only for intervals. The proof in [St] goes roughly in the following way: in dimension one, Carleson measures for $H^p_\alpha(\mathbb{D})$ coincide with Carleson measures for the space of Poisson transforms of Bessel potentials of L^p functions. The key point in the argument is then a strong type capacity inequality.

In fact, Hansson's theorem extending this strong capacitary inequality to dimension $n > 1$, shows ([N-R-S]) that Stegenga's theorem extends to measures μ on $\mathbb{R}^{n+1}_+$, for the spaces $B = P[J_\alpha * L^p]$, where P is the Poisson kernel on $\mathbb{R}^{n+1}_+$ and J_α is the Bessel kernel in $\mathbb{R}^n$.

In [A-C] it is introduced a non-isotropic Bessel kernel given by

$$K_\alpha(z,\zeta) = \frac{1}{|1 - z\bar{\zeta}|^{n-\alpha}}, \quad z \in \bar{\mathbb{B}}^n,\ \zeta \in \mathbb{S}^n,$$

where $0 < \alpha < n$. For $1 \leq p < +\infty$, and $f \in L^p(d\sigma)$, the non-isotropic convolution is denoted by

$$K_\alpha * f(z) = \int_{\mathbb{S}^n} K_\alpha(z,\zeta) f(\zeta)\, d\sigma(\zeta), \quad z \in \bar{\mathbb{B}}^n.$$

It is also introduced a non-isotropic Bessel capacity, $C_{\alpha p}$, associated to these kernels, and a non-isotropic version of Hansson's theorem is proved. If $P[K_\alpha * L^p]$ is the space of Poisson-Szegő transforms of convolutions of L^p functions with K_α, following the methods of Stegenga, it is easy to show that a positive finite Borel measure μ in $\mathbb{B}^n$ is a Carleson measure for $P[K_\alpha * L^p]$, if and only if there exists a constant $M > 0$ so that $\mu\big(T(A)\big) \leq MC_{\alpha p}(A)$ for any open set $A \subset \mathbb{S}^n$. Here $T(A)$ is an admissible tent over A.

The paper is organized as follows. In Section 2 we study Carleson measures for $P[K_\alpha * L^p]$. As we have already said, Stegenga's condition still characterizes, but being difficult to check, the purpose is to find computable sufficient conditions. We have found several such conditions in terms of duality, moduli of continuity and geometric estimates. Those last kind of conditions improve the one obtained in [N-R-S] for $P[J_\alpha * L^p]$ in $\mathbb{R}^{n+1}_+$. We also give some examples which show, in a certain sense, the "sharpness" of the sufficient conditions, as well as the relations among them.

In Section 3 we deal with the holomorphic case, where, in general, no necessary and sufficient size conditions are known. We begin proving the non equivalence, for $n > 1$, of the problems for $P[K_\alpha * L^p]$ and H^p_α, and observing that the sufficient conditions for the first space are also sufficient for the second one. We give a necessary and sufficient condition for a measure to be Carleson for H^p_α, in terms of atomic representation of those spaces. Some particular cases are presented where Stegenga's condition is necessary and sufficient.

We also study the problem for A^p_{qs}, the holomorphic Besov-Sobolev space, and give similar sufficient conditions. Finally, we apply the previous theorems giving examples of multipliers for A^p_{qs}, which could not be obtained as a consequence of the sufficient condition in [N-R-S].

As final remarks on notation, we will adopt the usual convention of writing by the same letter various absolute constants whose values may differ in each occurrence. Also $A \lesssim B$ will mean that there exists C so that $A \leq CB$.

2. **Carleson measures for $P[K_\alpha * L^p]$.** As we have already said in the introduction, Stegenga's condition characterizes Carleson measures for $P[K_\alpha * L^p]$, $p > 1$, $\alpha p \leq n$: a finite positive Borel measure μ in $\mathbb{B}^n$ is a Carleson measure for $P[K_\alpha * L^p]$, if and only if there exists $K > 0$ such that

$$\mu\big(T(A)\big) \leq KC_{\alpha p}(A), \tag{2.1}$$

for any open set $A \subset \mathbb{S}^n$, where $T(A)$ is the admissible tent over A given by

$$T(A) = T_\beta(A) = \mathbb{B}^n \setminus \bigcup_{\zeta \notin A} D_\beta(\zeta), \tag{2.2}$$

$D_\beta(\zeta) = \{z \,;\, |1 - z\bar{\zeta}| < \beta(1 - |z|)\}$, and $C_{\alpha p}$ is the non-isotropic Bessel capacity defined by

$$C_{\alpha p}(A) = \inf\{\|f\|_p^p \,; f \in L^p_+(d\sigma), K_\alpha * f \geq 1 \text{ on } A\}. \tag{2.3}$$

If $A = B(\zeta, r)$ is a non-isotropic ball, $C_{\alpha p}\big(B(\zeta, r)\big) \simeq r^{n-\alpha p}$, and then the condition

$$\mu(T(B(\zeta, r)) \leq Kr^{n-\alpha p},$$

is necessary but not sufficient (see [St]).

It is easy to see that if we define "weak" Carleson measures for $P[K_\alpha * L^p]$ as the finite positive Borel measures μ on $\mathbb{B}^n$ such that

$$\sup_{\lambda > 0} \lambda \mu\big(\{z \in \mathbb{B}^n \,;\, |F(z)| > \lambda\}\big)^{1/p} \leq C\|F\|_{p,\alpha}, \tag{2.4}$$

then μ is a Carleson measure for $P[K_\alpha * L^p]$ if and only if it is a weak Carleson measure. Indeed, let A be any open set in $\mathbb{S}^n$ and let f be any test function for $C_{\alpha p}(A)$. Then, (see [N-R-S, Lemma 3.4]) there exists $b > 0$, depending only on n but not on A, with

$$P[f](z) \geq b, \quad \text{for } z \in T(A).$$

Thus $T(A) \subseteq \{z \,;\, P[K_\alpha * f](z) \geq b\}$, and if μ satisfies (2.4), $\mu\big(T(A)\big) \leq C\|f\|_p^p$. Taking infimum on f, we get the desired conclusion.

In this section we will study the Carleson measures for $P[K_\alpha * L^p]$, $p > 1$ and $\alpha p \leq n$.

Our first result gives a first sufficient condition, which is deduced using duality, and does not involve capacity.

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THEOREM 2.1. *Let $1 < p < +\infty$, $0 < \alpha$, $\alpha p \leq n$. If a positive Borel measure μ in $\mathbb{B}^n$ satisfies that*

$$\sup_{\omega\in\mathbb{B}^n}\int_{\mathbb{S}^n}\frac{1}{|1-\omega\bar{\zeta}|^{n-\alpha}}\left(\int_{\mathbb{B}^n}\frac{d\mu(z)}{|1-z\bar{\zeta}|^{n-\alpha}}\right)^{\frac{1}{p-1}}d\sigma(\zeta)<+\infty, \tag{2.5}$$

then μ is a Carleson measure for $P[K_\alpha * L^p]$.

PROOF OF THEOREM 2.1. We begin with the following simple lemma:

LEMMA 2.1. *Suppose $0 < \alpha < n$. Then*

$$P[K_\alpha * f](z) \simeq K_\alpha * f(z) \quad \textit{for each } z \in \mathbb{B}^n, \textit{ and } f \in L^1_+(d\sigma).$$

PROOF OF LEMMA 2.1. Lemma 1.7 in [A-C] shows that $K_\alpha * f(z) \lesssim P[K_\alpha * f](z)$. The other estimate will hold if we check that

$$\int_{\mathbb{S}^n}\frac{(1-|z|^2)^n}{|1-z\bar{\omega}|^{2n}}\frac{d\sigma(\omega)}{|1-\omega\bar{\zeta}|^{n-\alpha}}\lesssim\frac{1}{|1-z\bar{\zeta}|^{n-\alpha}}.$$

This will be obtained by breaking the integral in two pieces corresponding to $|1-\omega\bar{\zeta}| \geq \varepsilon|1-z\bar{\zeta}|$ and $|1-\omega\bar{\zeta}| \leq \varepsilon|1-z\bar{\zeta}|$, with $\varepsilon > 0$ to be chosen. Then

$$\int_{|1-\omega\bar{\zeta}|\geq\varepsilon|1-z\bar{\zeta}|}\frac{(1-|z|^2)^n}{|1-z\bar{\omega}|^{2n}}\frac{d\sigma(\omega)}{|1-\omega\bar{\zeta}|^{n-\alpha}}\lesssim\frac{1}{|1-z\bar{\zeta}|^{n-\alpha}},$$

and if $\varepsilon > 0$ is small enough, we obtain that if $|1-\omega\bar{\zeta}| \leq \varepsilon|1-z\bar{\zeta}|$, then $|1-z\bar{\omega}| \simeq |1-z\bar{\zeta}|$. Thus

$$\begin{aligned}\int_{|1-\omega\bar{\zeta}|<\varepsilon|1-z\bar{\zeta}|}&\frac{(1-|z|^2)^n}{|1-z\bar{\omega}|^{2n}}\frac{d\sigma(\omega)}{|1-\omega\bar{\zeta}|^{n-\alpha}}d\sigma(\omega)\\&\lesssim\frac{1}{|1-z\bar{\zeta}|^n}\int_{|1-\omega\bar{\zeta}|<\varepsilon|1-z\bar{\zeta}|}\frac{d\sigma(\omega)}{|1-\omega\bar{\zeta}|^{n-\alpha}}\simeq\frac{1}{|1-z\bar{\zeta}|^{n-\alpha}},\end{aligned}$$

and we obtain the lemma. ∎

Returning to the proof of the theorem, we just need to show that the linear operator given by

$$Tf(z)=\int_{\mathbb{S}^n}\frac{f(\zeta)}{|1-z\bar{\zeta}|^{n-\alpha}}d\sigma(\zeta),$$

is bounded from $L^p(d\sigma)$ to $L^p(d\mu)$. This is equivalent to show that the adjoint operator T^* defined by

$$T^*f(\zeta)=\int_{\mathbb{B}^n}\frac{f(z)}{|1-z\bar{\zeta}|^{n-\alpha}}d\mu(z),$$

is bounded from $L^{p'}(d\mu)$ to $L^{p'}(d\sigma)$, $\frac{1}{p}+\frac{1}{p'}=1$. Applying Hölder's inequality and Fubini's theorem,

$$\begin{aligned}\|T^*f\|^{p'}_{L^{p'}(d\sigma)} &= \int_{\mathbb{S}^n}\left|\int_{\mathbb{B}^n}\frac{f(z)\,d\mu(z)}{|1-z\bar{\zeta}|^{n-\alpha}}\right|^{p'} d\sigma(\zeta)\\ &\le \int_{\mathbb{S}^n}\int_{\mathbb{B}^n}\frac{|f(\omega)|^{p'}}{|1-\omega\bar{\zeta}|^{n-\alpha}}\,d\mu(\omega)\Big(\int_{\mathbb{B}^n}\frac{d\mu(z)}{|1-z\bar{\zeta}|^{n-\alpha}}\Big)^{\frac{p'}{p}} d\sigma(\zeta)\\ &= \int_{\mathbb{B}^n}|f(\omega)|^{p'}\left(\int_{\mathbb{S}^n}\frac{1}{|1-\omega\bar{\zeta}|^{n-\alpha}}\Big(\int_{\mathbb{B}^n}\frac{d\mu(z)}{|1-z\bar{\zeta}|^{n-\alpha}}\Big)^{\frac{1}{p-1}}\right) d\sigma(\zeta)\,d\mu(\omega).\end{aligned}$$

Now (2.5) finishes the proof. ■

REMARK 2.1. When $p=2$, it is easy to see that condition (2.5) is equivalent to

$$\sup_{\omega\in\mathbb{B}^n}\int_{\mathbb{B}^n}\frac{d\mu(z)}{|1-z\bar{\omega}|^{n-2\alpha}}<+\infty, \tag{2.6}$$

just because

$$\int_{\mathbb{S}^n}\frac{d\sigma(\zeta)}{|1-\omega\bar{\zeta}|^{n-\alpha}|1-z\bar{\zeta}|^{n-\alpha}}\simeq\frac{1}{|1-z\bar{\omega}|^{n-2\alpha}}.$$

REMARK 2.2. It is also easy to show that condition (2.6) can be rewritten in terms of the function $\varphi_\zeta(r)=\mu\Big(T\big(B(\zeta,r)\big)\Big)$, as

$$\sup_{\zeta\in\mathbb{S}^n}\int_0^{+\infty}\frac{\varphi_\zeta(r)}{r^{n-2\alpha}}\frac{dr}{r}<+\infty.$$

But if $p\neq 2$ there is no a similar expression of (2.5) in terms of φ_ζ. Instead, we will see in the following theorem a sufficient condition involving a modulus of continuity of μ.

THEOREM 2.2. *Let μ be a finite positive Borel measure in $\mathbb{B}^n$. For $0<\delta\le 2$, let*

$$\varphi(\delta)=\sup_{\zeta\in\mathbb{S}^n}\mu\Big(T\big(B(\zeta,\delta)\big)\Big).$$

Assume $1<p<+\infty$, $m=n-\alpha p\ge 0$, $\alpha>0$ and that

$$\int_0^{+\infty}\big(\varphi(\delta)\delta^{-m}\big)^{\frac{1}{p-1}}\frac{d\delta}{\delta}<+\infty. \tag{2.7}$$

Then μ is a Carleson measure for $P[K_\alpha * L^p]$.

PROOF OF THEOREM 2.2. In [M-K, Theorem 6.1] it is proved that

$$\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n}\frac{d\mu(z)}{|y-z|^{n-\alpha}}\Big)^{\frac{1}{p-1}}\frac{dy}{|x-y|^{n-\alpha}}\le C\int_0^{+\infty}\Big(\frac{\omega(\mu,\delta)}{\delta^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{d\delta}{\delta},$$

where $\omega(\mu,\delta)=\sup_{x\in\mathbb{R}^n}\mu(\{y\,;\,|x-y|\le\delta\})$. Their methods can be used to show that

$$\sup_{\omega\in\mathbb{B}^n}\int_{\mathbb{S}^n}\Big(\int_{\mathbb{B}^n}\frac{d\mu(z)}{|1-z\bar{\zeta}|^{n-\alpha}}\Big)^{\frac{1}{p-1}}\frac{d\sigma(\zeta)}{|1-\omega\bar{\zeta}|^{n-\alpha}}\lesssim\int_0^{+\infty}\Big(\frac{\varphi(\delta)}{\delta^{n-\alpha p}}\Big)^{\frac{1}{p-1}}\frac{d\delta}{\delta}.$$ ■

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REMARK 2.3. There are many examples of functions φ satisfying (2.7):
(1) $\varphi(\delta)=\delta^m(\log\frac{1}{\delta})^{1-q}$, $q>p$.
(2) $\varphi(\delta)=\delta^m(\log\frac{1}{\delta})^{1-p}(\log\log\frac{1}{\delta})^{1-q}$, $q>p$.
The following proposition will show that, in terms of modulus of continuity, (2.7) is sharp.

PROPOSITION 2.1. *Let $\varphi\colon[0,+\infty)\longrightarrow\mathbb{R}$ be a non-decreasing function such that $\varphi(0)=0$, constant for $x\geq x_0$, and*

$$\int_0^{x_0}\left(\varphi(\delta)\delta^{-m}\right)^{\frac{1}{p-1}}\frac{d\delta}{\delta}=+\infty.$$

Assume that $\ell_j\in[0,+\infty)$ defined by $\varphi(\ell_j)=\frac{1}{2^{jn}}$, $j=1,2,\ldots$, satisfies $2\ell_{j+1}<\ell_j$.

Then there exists a finite positive Borel measure ν in $\mathbb{B}^n$, so that

$$\nu\Big(T\big(B(\zeta,\delta)\big)\Big)\leq\varphi(\delta),\quad \zeta\in\mathbb{S}^n,\ \delta>0,$$

but ν is not a Carleson measure for $P[K_\alpha*L^p]$.

REMARK 3.4. It is easy to show that if $\sqrt[n]{\varphi}$ is strictly concave on $[0,+\infty)$, the condition on the sequence $(\ell_j)_j$ holds.

PROOF OF PROPOSITION 2.1. In [M-K, Lemma 7.2] it is shown that the n-dimensional Cantor set $\tilde{E}\subset\mathbb{R}^n$ associated to the sequence $(\ell_j)_j$ (see [K-S] for a construction of such sets), has zero Bessel capacity in $\mathbb{R}^n$, and that there exists a measure $\tilde{\mu}$, supported in $\tilde{E}$ so that for any $r>0$, $x_0\in\mathbb{R}^n$, $\tilde{\mu}(\{x\in\mathbb{R}^n\ ;\ |x-x_0|<r\})\leq\varphi(r)$, whereas if $x_0\in\tilde{E}$, $\tilde{\mu}(\{x\in\mathbb{R}^n\ ;\ |x-x_0|<r\})\simeq\varphi(r)$.

Identifying $\tilde{E}$ with a compact subset of an n-dimensional transverse variety included in $\mathbb{S}^n$, we obtain a set $E\subset\mathbb{S}^n$ of zero non-isotropic Bessel capacity. The measure μ defined in $\mathbb{S}^n$ by transporting $\tilde{\mu}$, has its support in E and satisfies similar estimates than $\tilde{\mu}$, with non isotropic balls.

Now, for $\delta>0$ let $E_\delta=\{\zeta\in\mathbb{S}^n, d(\zeta,E)<\delta\}$ where $d(\zeta,E)$ is the non-isotropic Koranyi distance from ζ to E. Since $C_{\alpha p}(E)=0$, and $C_{\alpha p}$ is an outer capacity ([Me]), we obtain that the increasing function $g(\delta)=C_{\alpha p}(E_\delta)$ tends to zero as $\delta\longrightarrow 0$. It is easy to construct an integrable positive function h on $[0,1)$ satisfying that for any $m\in\mathbb{N}$,

$$\int_{1-\frac{c_1}{m}}^1 h(r)\,dr\geq\sqrt{g\Big(\frac{1}{m}\Big)}.$$

We define now a positive finite Borel measure in $\mathbb{B}^n$ by

$$\nu(f)=\int_0^1\int_{\mathbb{S}^n}f(r\zeta)h(r)\,d\mu(\zeta)\,dr,\qquad\text{for any } f\in\mathcal{C}(\mathbb{B}^n),$$

which will be the desired measure. If $\zeta\in\mathbb{B}^n$ and $\delta>0$, any $z\in T\big(B(\zeta,\delta)\big)$ satisfies $\frac{z}{|z|}\in B(\zeta,\delta)$. Then

$$\nu\Big(T\big(B(\zeta,\delta)\big)\Big)\leq\int_0^1 h(r)\mu\big(B(\zeta,\delta)\big)\,dr\lesssim\varphi(\delta),\qquad\text{whereas}$$

$$\nu\big(T(E_{\frac{1}{m}})\big) \geq \int_{1-\frac{c_1}{m}}^{1} h(r) \int_{\mathbb{S}^n} \chi_{E_{\frac{c}{m}}}(\zeta)\, d\mu(\zeta)\, dr = \int_{1-\frac{c_1}{m}}^{1} h(r)\mu(E)\, dr$$
$$\geq C\sqrt{g\Big(\frac{1}{m}\Big)} = C\sqrt{C_{\alpha p}\Big(E_{\frac{1}{m}}\Big)}.$$

Thus ν does not satisfy Stegenga's condition. ∎

The third kind of sufficient conditions will be of geometric type. Before stating them we need some more definitions. Let $1 < p < +\infty$, $m = n - \alpha p > 0$, and $1 \leq \tau \leq \frac{n}{m}$. If $\zeta \in \mathbb{S}^n$ and $\beta > 0$ let

$$\Omega_\tau(\zeta) = \Omega_{\tau\beta}(\zeta) = \{z \in \mathbb{B}^n \,;\, |1 - z\bar{\zeta}|^\tau < \beta(1 - |z|)\}$$

and for any $E \subset \mathbb{S}^n$, the "tangential" tent over E is defined by

$$T_\tau(E) = T_\tau^\beta(E) = \mathbb{B}^n \setminus \bigcup_{\zeta \notin E} \Omega_\tau(\zeta).$$

(when $\tau = 1, \Omega_1(\zeta) = D(\zeta)$, $T_1(E) = T(E)$).

In [N-R-S] it is shown that the geometric condition, when $\tau = \frac{n}{m}$, $\mu\Big(T_{\frac{n}{m}}^C\big(B(\zeta, r)\big)\Big) \lesssim r^n = r^{m\frac{n}{m}}$ is sufficient for a measure μ to be Carleson for $P[K_\alpha * L^p]$. The same condition, when $\tau = 1$, $\mu\Big(T_1^C\big(B(\zeta, r)\big)\Big) \lesssim r^m$ is the necessary condition we have already mentioned. It is then natural to ask whether the "intermediate" conditions $\mu\Big(T_\tau^C\big(B(\zeta, r)\big)\Big) \lesssim r^{m\tau}$, are necessary or sufficient.

The next theorem establishes that these conditions are also sufficient for a measure to be Carleson. Since Hausdorff content is not additive, the proof of the theorem needs a completely different approach of the one used in [N-R-S] for $\tau = \frac{n}{m}$. In that case, the corresponding Hausdorff content is just Lebesgue measure.

Before stating it, if $f: \mathbb{B}^n \longrightarrow \mathbb{C}$, we will denote by Nf the admissible maximal function given by $Nf(\zeta) = \sup_{z \in D(\zeta)} |f(z)|$.

THEOREM 2.3. *Let $1 < p < +\infty$, $m = n - \alpha p > 0$, and let μ be a finite positive Borel measure μ in $\mathbb{B}^n$ so that there exists $1 < \tau \leq \frac{n}{m}$ and $C > 0$ with*

$$\mu\Big(T_\tau^C\big(B(\zeta, r)\big)\Big) \lesssim r^{m\tau}. \tag{2.8}$$

Then μ is a Carleson measure for $P[K_\alpha * L^p]$.

PROOF OF THEOREM 2.3. We need the following:

PROPOSITION 2.2. *Let $0 < p < +\infty$, $0 < \alpha$ and $m = n - \alpha p > 0$. Let μ be a finite positive Borel measure in $\mathbb{B}^n$ satisfying that there exists $C > 0$, $K > 0$, $\beta > 0$, and $1 < \tau \leq \frac{n}{m}$ with*

(i) $\operatorname{supp}\mu \subset \{z\,;\,(1 - |z|) < \varepsilon C^{\frac{1}{\tau-1}}\}$, $(\varepsilon = \frac{1}{4}\frac{\tau}{\tau-1})$,

(ii) $\mu\Big(T_\tau^C\big(B(\zeta, r)\big) \cap T^\beta\big(B(\zeta, r)\big)\Big) \leq K r^{m\tau}$.

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Then there exists $M>0$ so that for any $f\colon \mathbb{B}^n \longrightarrow \mathbb{C}$

$$\int_{\mathbb{B}^n}(1-|z|)^{\alpha p}|f(z)|^p\,d\mu(z)\le MC^m\|Nf\|^p_{L^p(d\sigma)}.$$

REMARK 2.4. Observe that the above proposition gives that if μ satisfies (i) and (ii), then $(1-|z|)^{\alpha p}\,d\mu(z)$ is a Carleson measure for $H^p(\mathbb{B}^n)$, with a control on the constant expressing the continuity of the mapping from $H^p(\mathbb{B}^n)$ to $L^p\big((1-|z|)^{\alpha p}\,d\mu(z)\big)$.

It suffices to prove the proposition for $p=1$, since $\|Nf\|^p_{L^p(d\sigma)}=\|Nf^p\|_{L^1(d\sigma)}$. We claim that

$$(2.9)\quad \int_{\mathbb{B}^n}(1-|z|)^\alpha|f(z)|\,d\mu(z)\lesssim\int_0^{+\infty}\int_0^{\varepsilon C^{\frac{\alpha}{\tau-1}}}\mu\big(\{z\,;\,|f(z)|>t,(1-|z|)^\alpha>\ell\}\big)\,d\ell\,dt.$$

In fact, similar inequalities were treated in [Ci-Do-Su], and we briefly sketch the proof of (2.9). If $t>0$ and $\ell>0$, let $E(t,\ell)=\mu\big(\{z\,;\,|f(z)|>t,(1-|z|)^\alpha>\ell\}\big)$. Let $z\in\operatorname{supp}\mu$ satisfying that $|f(z)|(1-|z|)^\alpha>t$. There exists $j\in\mathbb{Z}^-$, $k\in\mathbb{Z}$ so that $2^j\varepsilon C^{\frac{\alpha}{\tau-1}}<(1-|z|)^\alpha\le 2^{j+1}\varepsilon C^{\frac{\alpha}{\tau-1}}$, $2^k<|f(z)|\le 2^{k+1}$, and in particular $z\in E(2^k,2^j\varepsilon C^{\frac{\alpha}{\tau-1}})$, and $t\le 4 2^{k+j}\varepsilon C^{\frac{\alpha}{\tau-1}}$. Thus

$$\begin{aligned}\int_0^{+\infty}\mu\big(\{z\,;\,(1-|z|)^\alpha|f(z)|>t\}\big)\,dt&\le\sum_{k=-\infty}^{+\infty}\sum_{j=-\infty}^{-1}\int_0^{2^{j+k+2}\varepsilon C^{\frac{\alpha}{\tau-1}}}\mu\big(E(2^k,2^j\varepsilon C^{\frac{\alpha}{\tau-1}})\big)\,dt\\&\lesssim\sum_{k=-\infty}^{+\infty}2^k\sum_{j=-\infty}^{-1}2^jC^{\frac{\alpha}{\tau-1}}\mu\big(E(2^k,2^j\varepsilon C^{\frac{\alpha}{\tau-1}})\big)\\&\simeq\int_0^{+\infty}\int_0^{\varepsilon C^{\frac{\alpha}{\tau-1}}}\mu\big(E(t,\ell)\big)\,d\ell\,dt,\end{aligned}$$

which gives (2.9).

Next, since (2.9) holds, in order to prove the proposition, we just need to check that

$$(2.10)\qquad \int_0^{\varepsilon C^{\frac{\alpha}{\tau-1}}}\mu\big(\{z\,;\,|f(z)|>t,(1-|z|)^\alpha>\ell\}\big)\,d\ell\lesssim C^m\sigma\big(\{\zeta\,;\,Nf(\zeta)>t\}\big).$$

Let $\mathcal{B}=(B_k)_k$, with $B_k=B(\zeta_k,r_k)$ be a Whitney decomposition of the set $\{\zeta\,;\,Nf(\zeta)>t\}$. Thus: (a) $\{Nf>t\}=\bigcup_k B_k$; (b) there exists $h\in\mathbb{N}$, only depending on n, so that no point in $\mathbb{S}^n$ lies in more than h distinct balls B_k; (c) $B(\zeta_k,hr_k)\not\subset\{Nf>t\}$ for each k.

Let $z\in\mathbb{B}^n$ with $(1-|z|)<\varepsilon C^{\frac{1}{\tau-1}}$ and $|f(z)|>t$. Then there exists $M>0$ with $B\big(z_0,M(1-|z|)\big)\subset\{Nf>t\}$, where $z_0=\frac{z}{|z|}$. In particular there exists $k\in\mathbb{N}$ and $z_0\in B_k$, and by property (c) of the Whitney decomposition $(1-|z|)\lesssim r_k$. We want to show that there exists $m>0$ and

$$(2.11)\qquad z\in T^C_\tau\Big(B\big(\zeta_k,m(C^{\frac{1}{\tau}}r_k^{\frac{1}{\tau}}+r_k)\big)\Big)\cap T^\beta\Big(B\big(\zeta_k,m(C^{\frac{1}{k}}r_k^{\frac{1}{\tau}}+r_k)\big)\Big).$$

Indeed let $\zeta\in\mathbb{S}^n$ with $z\in\Omega_\tau(\zeta)$ (the condition $1-|z|\le\varepsilon C^{\frac{1}{\tau-1}}$ with $\varepsilon=\frac{1}{4^{\frac{\tau}{\tau-1}}}$ gives that is always possible to find such ζ). Then,

$$|1-\zeta\bar\zeta_k|\lesssim|1-\zeta\bar z|+(1-|z|^2)+|1-z_0\bar\zeta_k|\lesssim C^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}+r_k\lesssim C^{\frac{1}{\tau}}r_k^{\frac{1}{\tau}}+r_k.$$

Thus if ζ is any point such that $z \in \Omega_\tau(\zeta)$, $\zeta \in B\big(\zeta_k, m(C^{\frac{1}{\tau}} r_k^{\frac{1}{\tau}} + r_k)\big)$, and we deduce from the definition that $z \in T_\tau^C\Big(B\big(\zeta_k, m(C^{\frac{1}{\tau}} r_k^{\frac{1}{\tau}} + r_k)\big)\Big)$.

The other inclusion in (2.11) is proved in a similar way.

Now we decompose $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$, where any ball B_k in $\mathcal{B}^1$ satisfies $r_k \le (Cr_k)^{\frac{1}{\tau}}$, and if $B_k \in \mathcal{B}^2$, $(Cr_k)^{\frac{1}{\tau}} \le r_k$. Applying (2.10) we get

$$\int_0^{\varepsilon C^{\frac{\alpha}{\tau-1}}} \mu\big(\{z\,;\,|f(z)| > t, (1-|z|)^\alpha > \ell\}\big)\,d\ell$$
$$\le \sum_{B_k \in \mathcal{B}^1} \int_0^{Mr_k^\alpha} \mu\Big(T_\tau^C\big(B(\zeta_k, 2mC^{\frac{1}{\tau}} r_k^{\frac{1}{\tau}})\big) \cap T^\beta\big(B(\zeta_k, 2mC^{\frac{1}{\tau}} r_k^{\frac{1}{\tau}})\big)\Big)\,d\ell$$
$$+ \sum_{B_k \in \mathcal{B}^2} \int_0^{\varepsilon C^{\frac{\alpha}{\tau-1}}} \mu\Big(T_\tau^C\big(B(\zeta_k, 2mr_k)\big) \cap T^\beta\big(B(\zeta_k, 2mr_k)\big)\Big)\,d\ell,$$

where in the first estimate we have also used that if $\ell < (1-|z|)^\alpha$ and $(1-|z|) \lesssim r_k$, then $\ell \lesssim r_k^\alpha$. Finally, we use the hypothesis on μ to obtain that the last sums are bounded by

$$C^m \sum_{B_k \in \mathcal{B}^1} r_k^{m+\alpha} + C^m \sum_{B_k \in \mathcal{B}^2} C^{\frac{\alpha}{\tau-1} - m} r_k^{m\tau} \lesssim C^m \sigma(\{Nf > t\}).$$

Here we have used that if $B_k \in \mathcal{B}^2$, $C \le r_k^{\tau-1}$, and then $C^{\frac{\alpha}{\tau-1} - m} \le r_k^{\alpha - (\tau-1)m}$. Thus we have seen (2.10), and that finishes the proof of the proposition. ∎

Going back to the proof of the theorem, Lemma 2.1 gives that we just need to show that there exists $M > 0$ such that if

$$F(z) = \int_{\mathbb{S}^n} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-\alpha}}\,d\sigma(\zeta),$$

with $f \in L_+^p(d\sigma)$, then

$$\int_{\mathbb{B}^n} |F|^p\,d\mu \le M \int_{\mathbb{S}^n} f^p\,d\sigma. \tag{2.12}$$

Since

$$\frac{1}{|1 - z\bar{\zeta}|^{n-\alpha}} \lesssim \int_0^1 \frac{(1-t)^{n+\alpha-1}}{|1 - tz\bar{\zeta}|^{2n}}\,dt,$$

we then have:

$$F(z) = \int_{\mathbb{S}^n} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-\alpha}}\,d\sigma(\zeta) \lesssim \int_{\mathbb{S}^n} f(\zeta) \int_0^1 \frac{(1-t^2)^{n+\alpha-1}}{|1 - tz\bar{\zeta}|^{2n}}\,dt\,d\sigma(\zeta)$$

$$\lesssim \int_0^1 (1-t)^{\alpha-1} \int_{\mathbb{S}^n} \frac{(1 - t^2|z|^2)^n}{|1 - tz\bar{\zeta}|^{2n}} f(\zeta)\,d\sigma(\zeta)\,dt = \int_0^1 (1-t)^{\alpha-1} g(tz)\,dt,$$

where $g = P[f]$. Breaking up the integral in the right hand side in two pieces, from 0 to A and from A to 1 ($0 < A < 1$ to be chosen), we denote the corresponding integrals by g_1 and g_2. We will show that both functions satisfy an estimate like (2.12).

First, since the supremum of g over any compact in $\mathbb{B}^n$ is bounded by the L^p-norm of g, we may assume that $\operatorname{supp}\mu \subset \{z\,;\,1-|z|\le \min(\varepsilon C^{\frac{1}{\tau-1}},\frac{1}{\beta})\}$, where $\beta>2$ will be chosen later. By the same argument the estimate for g_1 is immediate. So we need only to deal with the contribution of g_2 to (2.12). Let $1=t_0>t_1>\cdots>t_\ell$, where $\ell\in\mathbb{N}$ is the first integer so that $2^\ell(1-|z|)>\frac{2}{\beta}$, and where for each $k\le\ell$, $1-t_k|z|=2^k(1-|z|)$. If we choose $A=(1-\frac{2}{\beta})(1-\frac{1}{\beta})^{-1}$, then $t_\ell\le A$, and

$$\begin{aligned} g_2(z) &\le \int_{t_\ell}^1 (1-t)^{\alpha-1}g(tz)\,dt \le \sum_{k=0}^{\ell-1}\int_{t_{k+1}}^{t_k}(1-t)^{\alpha-1}g(tz)\,dt\\ &\lesssim \int_{t_1}^1 (1-t|z|)^\alpha g(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha}\,dt\\ &\quad+\sum_{k=1}^{\ell-1}\int_{t_{k+1}}^{t_k}(1-t|z|)^\alpha g(tz)\frac{dt}{(1-t)}. \end{aligned}$$

We denote by $g_{2,k}(z)$, $k=0,\dots,\ell-1$, the integrals that appear on the right hand side. Let $\mu_0,\dots,\mu_\ell$ be the measures in $\mathbb{B}^n$ defined by:

$$\mu_0(h)=\mu\left(\int_{t_1}^1 h(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha}\,dt\right)$$
$$\mu_k(h)=\mu\left(\int_{t_{k+1}}^{t_k} h(tz)\frac{dt}{(1-t)}\right),\quad k=1\cdots\ell,\ h\in C(\mathbb{B}^n).$$

Each μ_k is a finite measure. Indeed, for $k=0$, and since $1-t_1|z|=2(1-|z|)$, we have that if $t_1\le t\le 1$, $1-t|z|\simeq 1-|z|$ and

$$\int_{t_1}^1\frac{(1-t)^{\alpha-1}}{(1-t|z|)^\alpha}\,dt\simeq\frac{1}{(1-|z|)^\alpha}\int_{t_1}^1(1-t)^{\alpha-1}\,dt\simeq 1, \tag{2.13}$$

and if $1\le k\le\ell$,

$$\begin{aligned}\int_{t_{k+1}}^{t_k}\frac{dt}{(1-t)}&=\log\frac{1-t_{k+1}}{1-t_k}=\log\frac{1-t_{k+1}|z|-t_{k+1}(1-|z|)}{1-t_k|z|-t_k(1-|z|)}\\ &\le\log\frac{2^{k+1}(1-|z|)}{2^k(1-|z|)-2^{k-1}(1-|z|)}=\log 4,\end{aligned}\tag{2.14}$$

where in the last inequality we have used that since $k\ge1$, $t_k<2^{k-1}$.

We check that the measures μ_k, $0\le k\le\ell$, are in the hypothesis of Proposition 2, for appropriate constants. We first check the size estimate. We will show that there exists $M>0$ so that

$$\mu_k\left(T_\tau^{2^{-k}M}\big(B(\zeta_0,r)\big)\cap T^\beta\big(B(\zeta_0,r)\big)\right)\lesssim r^{m\tau}\tag{2.15}$$

for any $\zeta_0\in\mathbb{S}^n, r>0$.

Indeed, let $z\in\operatorname{supp}\mu$ and assume $tz\in T_\tau^{2^{-k}M}\big(B(\zeta_0,r)\big)\cap T^\beta\big(B(\zeta_0,r)\big)$, $t\le t_k$. Let $\zeta\in\mathbb{S}^n$ so that $z\in\Omega_\tau^C(\zeta)$ (again, since $\operatorname{supp}\mu\subset\{z\,;\,1-|z|\le\varepsilon C^{\frac{1}{\tau-1}}\}$ there exists such

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point). If $tz \in D(\zeta)$, we have since $tz \in T^{\beta}\big(B(\zeta_0, r)\big)$ that $|1-\zeta_0\bar{\zeta}| \le r$ and consequently that $z \in T_\tau^C\big(B(\zeta_0, r)\big)$. Widening, if necessary, the aperture of the admissible region $D(\zeta)$, it is immediate to show that if $k \le l-1$, $t \le t_k$ and if $tz \notin D(\zeta)$, then $tz \in \Omega_\tau^{2^{-k}M}(\zeta)$. Hence, since $tz \in T_\tau^{2^{-k}M}\big(B(\zeta_0, r)\big)$, we have that $|1-\zeta_0\bar{\zeta}| \le r$, and $z \in T_\tau^C\big(B(\zeta_0, r)\big)$. Now, this inclusion, (2.13), (2.14) and the hypothesis on μ gives that in any case:

$$\mu_0\Big(T_\tau^M\big(B(\zeta_0,r)\big) \cap T^{\beta}\big(B(\zeta_0,r)\big)\Big) \le \mu\bigg(\chi_{T_\tau^C(B(\zeta_0,r))}\bigg[\sup_{z\in T_\tau^C(B(\zeta_0,r))} \int_1^{t_1} \frac{(1-t)^{\alpha-1}}{(1-t|z|)^{\alpha}}\,dt\bigg]\bigg) \lesssim r^{m\tau},$$

and if $1 \le k \le \ell$

$$\mu_k\Big(T_\tau^{2^{-k}M}\big(B(\zeta_0,r)\big) \cap T^{\beta}\big(B(\zeta_0,r)\big)\Big) \le \mu\bigg(\chi_{T_\tau^C(B(\zeta_0,r))}\bigg[\sup_{z\in T_\tau^C(B(\zeta_0,r))} \int_{t_{k+1}}^{t_k} \frac{dt}{(1-t)}\bigg]\bigg) \lesssim r^{m\tau}.$$

We deal now with the condition on the support of the measures μ_k, $0 \le k \le \ell$. Let h be any continuous function supported in $\{z\,;\,(1-|z|) > \varepsilon\frac{C^{\frac{1}{\tau-1}}}{2^k}\}$, $0 \le k \le \ell$. For any $t \le t_k$ we have $1-t|z| \ge 1-t_k|z| = 2^k(1-|z|) > \varepsilon C^{\frac{1}{\tau-1}}$, and consequently $tz \notin \operatorname{supp}\mu$ and $\mu_k(h) = 0$. Thus if $k \le \ell$,

$$\operatorname{supp}\mu_k \subset \Big\{z\,;\,(1-|z|) < \varepsilon\frac{C^{\frac{1}{\tau-1}}}{2^k}\Big\}.$$

Assuming first that $\frac{1}{\tau-1} \le 1$, we finish the proof of the theorem. We can choose $M > 0$ big enough so that $\frac{C^{\frac{1}{\tau-1}}}{2^k} \le (\frac{M}{2^k})^{\frac{1}{\tau-1}}$, $k \le \ell$, and we then have that the measures μ_k are in the hypothesis of Proposition 2.2 with constant $2^{-k}M$. Hölder's inequality together with (2.12) and (2.13) gives

$$\begin{aligned}
\|g_2\|_{L^p(d\mu)} &\le \sum_{k\ge0}\|g_{2,k}\|_{L^p(d\mu)} = \bigg(\int_{\mathbb{B}^n}\bigg|\int_{t_1}^{1}(1-t|z|)^{\alpha}g(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^{\alpha}}\,dt\bigg|^p d\mu(z)\bigg)^{\frac1p}\\
&\quad+\sum_{k\ge1}\bigg(\int_{\mathbb{B}^n}\bigg|\int_{t_{k+1}}^{t_k}(1-t|z|)^{\alpha}g(tz)\frac{dt}{1-t}\bigg|^p d\mu(z)\bigg)^{\frac1p}\\
&\le \bigg(\int_{\mathbb{B}^n}\int_{t_1}^{1}(1-t|z|)^{\alpha p}g^p(tz)\frac{(1-t)^{\alpha-1}}{(1-t|z|)^{\alpha}}\,dt\,d\mu(z)\bigg)^{\frac1p}\\
&\quad+\sum_{k\ge1}\bigg(\int_{\mathbb{B}^n}\int_{t_{k+1}}^{t_k}(1-t|z|)^{\alpha p}g^p(tz)\frac{dt}{1-t}\,d\mu(z)\bigg)^{\frac1p}\\
&= \sum_{k\ge0}\bigg(\int_{\mathbb{B}^n}(1-|z|)^{\alpha p}g^p(z)\,d\mu_k(z)\bigg)^{\frac1p}.
\end{aligned}$$

Applying now Proposition 2.2 to each μ_k we deduce that the above sum is bounded by $\sum_{k=0}^{\ell}(2^{-k}M)^{\frac{m}{p}}\|Ng\|_p \lesssim \|f\|_{L^p(d\sigma)}$, and we have ended the proof when $\frac{1}{\tau-1} \le 1$.

If $\frac{1}{\tau-1} \ge 1$, we may choose k_0 so that for $k \ge k_0$, $2^{-k}M \le (\frac{1}{2^k})^{\tau-1}C$. Applying Proposition 2.2 to μ_k, $k \ge k_0$, now with constant $(\frac{1}{2^k})^{\tau-1}C$, a similar argument would give that

$$\sum_{k\ge k_0}\|g_{2,k}\|_{L^p(d\mu)} \lesssim \sum_{k\ge k_0}\Big(\frac{1}{2^k}\Big)^{\frac{m(\tau-1)}{p}}\|f\|_{L^p(d\sigma)}.$$

Finally for $0 \leq k < k_0$, $(\frac{1}{2^k})^{\tau-1} C < 2^{-k} M$ and we proceed as in the case $\frac{1}{\tau-1} \leq 1$. ∎

Before going further we give a definition of tangential "Carleson box".

DEFINITION. For $k > 0$ and $1 \leq \tau \leq \frac{n}{m}$, $\zeta_0 \in \mathbb{S}^n$ and $r > 0$,

$$V_\tau^k\big(B(\zeta_0, r)\big) = \left\{z \,;\, \left|1 - \frac{z}{|z|}\bar{\zeta}_0\right| < r 1 - |z| < kr^\tau\right\}.$$

We then have

LEMMA 2.2. *(i) Let $k > 0$ and $1 \leq \tau \leq \frac{n}{m}$. Then there exist $k_1 > 1$, $k_2 < 1$ so that for any $r > 0$, $\zeta_0 \in \mathbb{S}^n$,*

$$T_\tau^{k_1}\big(B(\zeta_0, k_2 r)\big) \subset V_\tau^k\big(B(\zeta_0, r)\big).$$

(ii) Analogously, let $k > 0$ and $1 \leq \tau \leq \frac{n}{m}$. Then there exist $k_1 > 0$, $k_2 > 0$ so that for any $r < 1$, $\zeta_0 \in \mathbb{S}^n$,

$$V_\tau^{k_1}\big(B(\zeta_0, k_2 r)\big) \subset T_\tau^k\big(B(\zeta_0, r)\big).$$

PROOF OF LEMMA 2.2. Let $z \in T_\tau^{k_1}\big(B(\zeta_0, k_2 r)\big)$, where $k_1 > 1$, $k_2 > 0$ are to be chosen. Since $z \in \Omega_\tau^{k_1}(z_0)$, $z_0 = \frac{z}{|z|}$, we must then have $|1 - \zeta_0 \bar{z}_0| < k_2 r < r$ if $k_2 < 1$.

Now choose $\zeta \in \mathbb{S}^n$ so that $|1 - \zeta\bar{\zeta}_0| = 2k_2 r$ (if k_2 is small enough it always exists). Then since $\zeta \notin B(\zeta_0, k_2 r)$, we have that $z \notin \Omega_\tau^{k_1}(\zeta)$. Thus

$$1 - |z| \leq \frac{1}{k_1}|1 - z\bar{\zeta}|^\tau \leq \frac{C}{k_1}\Big((1 - |z|) + |1 - z_0\bar{\zeta}_0| + |1 - \zeta_0\bar{\zeta}|\Big)^\tau \leq \frac{C}{k_1}\big((1 - |z|) + r^\tau\big),$$

and if k_1 is big enough, $1 - |z| \leq kr^\tau$, and we have proved (i).

The proof of (ii) goes similarly. ∎

COROLLARY 2.1. *Let μ be a finite positive Borel measure in $\mathbb{B}^n$. Then the following are equivalent:*

(i) There exists $k, C_1 > 0$ such that $\mu\Big(T_\tau^k\big(B(\zeta_0, r)\big)\Big) \leq C_1 r^{m\tau}$ for any $\zeta_0 \in \mathbb{S}^n$, $r > 0$.

(ii) There exists k, $C_2 > 0$ such that

$$\mu\Big(V_\tau^k\big(B(\zeta_0, r)\big)\Big) \leq C_2 r^{m\tau} \quad \textit{for any } \zeta_0 \in \mathbb{S}^n,\ r > 0.$$

We can now show that any measure satisfying the sufficient condition of [N-R-S], also satisfies the sufficient condition in Theorem 2.3, for any $1 < \tau \leq \frac{n}{m}$. Nevertheless, there are measures satisfying such sufficient condition, but do not satisfy the condition of [N-R-S].

PROPOSITION 2.3. *Assume* $1 < p < +\infty$ *and* $m = n - \alpha p > 0$. *Then:*
(i) Let μ *be a finite positive Borel measure in* $\mathbb{B}^n$, *so that, there exists* $C > 0$ *with*

$$\mu\Big(T_{\frac{n}{m}}\big(B(\zeta_0, r)\big)\Big) \leq Cr^n, \quad \text{for any } \zeta_0 \in \mathbb{S}^n,\ r > 0.$$

Then for any $1 < \tau \leq \frac{n}{m}$, *there exists* $C_1 > 0$ *so that for any* $\zeta_0 \in \mathbb{S}^n$, $r > 0$

$$\mu\Big(T_\tau\big(B(\zeta_0, r)\big)\Big) \leq C_1 r^{m\tau}$$

(ii) For any $\tau < \frac{n}{m}$ *there exists a positive Borel measure* μ *in* $\mathbb{B}^n$ *satisfying*

$$\mu\Big(T_\tau\big(B(\zeta, r)\big)\Big) \lesssim r^{m\tau}, \quad \text{but} \quad \mu\Big(T_{\frac{n}{m}}\big(B(\zeta, r)\big)\Big) \not\lesssim r^n.$$

PROOF OF PROPOSITION 2.3. By Corollary 2.1 it is enough to show the proposition with tents changed for the equivalent regions V_τ. Since $V_\tau\big(B(\zeta_0, r)\big) \subset V_{\frac{n}{m}}\big(B(\zeta_0, r^{\frac{\tau m}{n}})\big)$, we have (i). For the proof of (ii), let ν be a m-measure on $\mathbb{S}^n$, *i.e.*, there exists $E \subset \mathbb{S}^n$ so that for any $\zeta_0 \in E$, $r > 0$ $\nu\big(B(\zeta_0, r)\big) \simeq r^m$, whereas for any $\zeta_0 \in \mathbb{S}^n$ $\nu\big(B(\zeta_0, r)\big) \lesssim r^m$ (for the existence of such measures, see [K-S] and the argument in Proposition 2.1).

Let μ be the measure in $\mathbb{B}^n$ defined by

$$\mu(f) = \int_0^1 \int_{\mathbb{S}^n} f(r\zeta)(1 - r)^t \, d\mu(\zeta) \, dr,$$

for $f \in \mathcal{C}(\mathbb{B}^n)$ and where $t > -1$ is to be chosen later.

Next, let $\zeta_0 \in \mathbb{S}^n$, $r > 0$. Then

$$\mu\Big(V_\tau^k\big(B(\zeta_0, r)\big)\Big) = \int_{1-kr^\tau}^1 (1 - x)^t \int_{B(\zeta_0, r)} d\mu(\zeta) \, dx \leq r^{\tau(t+1)+m}.$$

If $\zeta_0 \in E$, $r > 0$ we have

$$\mu\Big(V_{\frac{n}{m}}^k\big(B(\zeta_0, r)\big)\Big) \simeq \delta^{\frac{n}{m}(t+1)+m}.$$

Since $\tau < \frac{n}{m}$ we can choose $t > -1$ so that $\tau m < \tau(t + 1) + m$ but $\frac{n}{m}(t + 1) + m < n$, and that finishes the proof. ∎

The following two examples will show the "sharpness" of the sufficient condition of Theorem 2.3 as well as the relation between the different sufficient conditions. As a consequence we will see that none of them are necessary.

EXAMPLE 2.1. Let $1 < p < +\infty$, $m = n - \alpha p > 0$, and let $\Psi\colon [0, 2) \longrightarrow \mathbb{R}^+$ be a differentiable function such that $\Psi(0) = 0$ and $\frac{\Psi(r)}{r^m} \longrightarrow +\infty$, as $r \longrightarrow 0$. Then there exists a finite positive Borel measure μ in $\mathbb{B}^n$ satisfying:

$$\mu\Big(T_\tau\big(B(\zeta_0, r)\big)\Big) \leq \Psi(Cr^\tau),$$

but μ is not a Carleson measure for $P[K_\alpha * L^p]$.

Let $\zeta_0 \in \mathbb{S}^n$ be a fixed point, and define $d\mu = \Psi'(1-r)\,dr\delta_{\zeta_0}$, where δ_{ζ_0} is the Dirac measure at ζ_0. Then $\mu\Big(T_\tau\big(B(\zeta,r)\big)\Big) \lesssim \Psi(Cr^\tau)$, but for any $r > 0$, $\mu\Big(T\big(B(\zeta_0,r)\big)\Big) \simeq \Psi(Cr)$. This last estimate, the condition of growth of Ψ at zero, and the fact that $C_{\alpha p}\big(B(\zeta_0,r)\big) \simeq r^m$ shows that μ does not satisfy Stegenga's condition. ∎

EXAMPLE 2.2. Let φ be the modulus of continuity of μ defined in Theorem 2.2.

(i) For $1 < \tau \leq \frac{n}{m}$, there exists a finite Borel measure μ in $\mathbb{B}^n$ such that

$$\mu\Big(T_\tau\big(B(\zeta,r)\big)\Big) \lesssim r^{m\tau},$$

but

$$\int_0^{+\infty} \big(\varphi(\delta)\delta^{-m}\big)^{\frac{1}{p-1}} \frac{d\delta}{\delta} = +\infty.$$

(ii) There exists a positive finite Borel measure μ in $\mathbb{B}^n$ so that

$$\int_0^{+\infty} \big(\varphi(\delta)\delta^{-m}\big)^{\frac{1}{p-1}} \frac{d\delta}{\delta} < +\infty$$

and there exists $\zeta \in \mathbb{S}^n$, so that for any $1 < \tau \leq \frac{n}{m}$,

$$\mu\Big(T_\tau\big(B(\zeta,r)\big)\Big) \not\lesssim r^{m\tau}.$$

For (i) it is enough to take $d\mu = (1-r)^{m-1}\delta_{\zeta_0}$, $\zeta_0 \in \mathbb{S}^n$. For the second example, let $E \subset \mathbb{S}^n$, be an m-set and ν be a measure on $\mathbb{S}^n$, supported in E satisfying: $\nu\big(B(\zeta,r)\big) \simeq r^m$ for each $\zeta \in E$, $r < 1$, and $\nu\big(B(\zeta,r)\big) \lesssim r^m$, $\forall \zeta \in \mathbb{S}^n, r > 0$. Let $q > p$ be fixed, and let $f(r) = (\log\frac{1}{r})^{1-q}$, and define μ by

$$\mu(g) = \int_0^1 \int_{\mathbb{S}^n} f'(1-r) g(r)\, d\nu(\zeta)\, dr, \quad \text{for any } f \in C(\mathbb{B}^n).$$

It is then immediate to show that μ is the desired measure.

REMARK 2.5. In [A-Bo] the so-called β-Carleson measures are studied. They are defined as the finite positive Borel measures μ in $\mathbb{B}^n$ so that there exists $C > 0$ with

$$\mu\big(T(A)\big) \leq C\sigma(A)^\beta, \quad \text{for any open set } A \subset \mathbb{S}^n.$$

Following [Me, Theorem 20] it is easy to show that there exists $M > 0$ so that $\sigma(A)^{\frac{n-\alpha p}{n}} \leq MC_{\alpha p}(A)$. Thus any $\frac{n-\alpha p}{n}$-Carleson measure μ satisfies Stegenga's condition and is a Carleson measure for $P[K_\alpha * L^p], p > 1$. The following example will show that there are Carleson measures for $P[K_\alpha * L^p]$ which are not $\frac{n-\alpha p}{n}$-Carleson. Consider a transverse curve in $\mathbb{S}^n$, $\gamma\colon I = [a,b] \longrightarrow \mathbb{S}^n$, and denote by dx the unidimensional Lebesgue measure on γ. Let μ be the positive measure in $\mathbb{B}^n$, supported in $\{r\gamma(x)\ ;\ x \in I, r < 1\}$ given by $d\mu(r,x) = (\log\frac{1}{r})^{1-q}\,dr\,dx$, with $q > p$. Then for any $\zeta \in \mathbb{S}^n$, $r > 0$, $\mu\Big(T\big(B(\zeta,r)\big)\Big) \lesssim r(\log\frac{1}{r})^{1-q}$, whereas $\mu\Big(T\big(B(\gamma(t),r)\big)\Big) \simeq r(\log\frac{1}{r})^{1-q}$. Theorem 2.2 shows that μ is a Carleson measure for $P[K_\alpha * L^p]$, if $1 = n - \alpha p$.

Assume that μ is also an $\frac{1}{n}$-Carleson measure, let $\varepsilon > 0$ and consider $A_\varepsilon = \bigcup_{i=1}^{N_\varepsilon} B\big(\gamma(x_i),\varepsilon\big)$ the union of a maximal number of disjoint non-isotropic balls centered at points in γ. Since A_ε is open and we have

$$\sum_{i=1}^{N_\varepsilon} \mu\Big(T\big(B(\gamma(x_i),\varepsilon)\big)\Big) = \mu\big(T(A_\varepsilon)\big) \le C(N_\varepsilon \varepsilon^n)^{\frac{1}{n}},$$

this estimate together with the fact that $\mu\Big(T\big(B(\gamma(x_i),\varepsilon)\big)\Big) \simeq \varepsilon(\log\frac{1}{\varepsilon})^{1-q}$ and $N_\varepsilon \simeq \frac{1}{\varepsilon}$ gives a contradiction.

3. **Carleson measures for holomorphic spaces.** In this section we will study conditions on a measure μ in $\mathbb{B}^n$ to be Carleson for Hardy-Sobolev and Besov-Sobolev spaces. The first example, which is based on one included in [A-C, p. 448] shows that, unlike what happens in dimension one, Stegenga's condition is, in general, not necessary for a measure μ to be Carleson for H^p_α, $n > 1$.

PROPOSITION 3.1. *If $1 < p \le 2$, $n - \alpha p \ge 1$, there exists a finite positive Borel measure μ in $\mathbb{B}^n$, such that μ is a Carleson measure for H^p_α, but μ is not Carleson for $P[K_\alpha * L^p]$.*

PROOF OF PROPOSITION 3.1. By Theorem 3.1 and Corollary 3.1 in [A-C] there exists a compact set $E \subset \mathbb{S}^n$ with $C_{\alpha p}(E) = 0$, and an invariant positive measure ν in $\mathbb{S}^n$, supported in E so that

$$\int_{\mathbb{S}^n} |Nf|^p \, d\nu \le C\|R^\alpha f\|_p^p \quad \text{for any } f \in H(\mathbb{B}^n). \tag{3.1}$$

For any $\delta > 0$, let $E_\delta = \{\zeta \in \mathbb{S}^n \ ; \ d(\zeta, E) < \delta\}$, and $g(\delta) = C_{\alpha p}(E_\delta)$. This function tends to zero as δ tends to zero since $C_{\alpha p}(E) = 0$. Let h be the function constructed in Proposition 2.1, and define a measure μ in $\mathbb{B}^n$ given by $\int_{\mathbb{B}^n} f\, d\mu = \int_0^1 \int_{\mathbb{S}^n} f(r\zeta)h(r)\, d\nu(\zeta)\, dr$.

Then (3.1) gives that μ is a Carleson measure for H^p_α. Indeed, if $f \in H^p_\alpha$

$$\begin{aligned}\int_0^1 \int_{\mathbb{S}^n} |f(r\zeta)|^p h(r)\, d\nu(\zeta)\, dr &\le \int_0^1 \int_{\mathbb{S}^n} |Nf(\zeta)|^p h(r)\, d\nu(\zeta)\, dr \\ &\lesssim \|R^\alpha f\|_p^p = \|f\|_{p,\alpha}^p.\end{aligned}$$

Now the same argument given in Proposition 2.1 shows that μ does not satisfy Stegenga's condition.

REMARK 3.1. Since any $f \in H^p_\alpha$ satisfies that there exists $g \in L^p_+(d\sigma)$ with $|f(z)| \le P[K_\alpha * g](z)$ and $\|f\|_{p,\alpha}^p \simeq \|g\|_{L^p(d\sigma)}^p$ ([A-C, Lemma 1.7]), all the sufficient conditions in Section 2 also hold for the Hardy-Sobolev spaces.

From the following representation theorem for H^p_α, ([Lu1, Theorem 5.5]), follows an equivalent formulation of the definition of a Carleson measure.

THEOREM [LU1]. *There exists $\delta > 0$, $\varepsilon > 0$, so that for any $(r_n)_n \subset (0,1)$, with $r_n \longrightarrow 1$, and for any $n \in \mathbb{N}$, a finite sequence $(a_{nk})_{k=1}^{k(n)} \subset \{z \in \mathbb{B}^n \; ; \; |z| = r_n\}$, $n \geq 1$, satisfying*

(1) $\bigcup_k \{z \; ; \; \rho(z, a_{nk}) < \delta\} \supset \{z \in \mathbb{B}^n \; ; \; |z| = r_n\}$ (here $\rho(z,\omega)$ is the pseudohyperbolic metric in $\mathbb{B}^n$, that is, $\rho(z,\omega) = |h_\omega(z)|$), where h_ω is the automorphism of $\mathbb{B}^n$ taking 0 to ω.

(2) If $k, j \in \{1 \cdots k(n)\}$ and $k \neq j$ then $\rho(a_{nj}, a_{nk}) > \varepsilon$.

Then any $f \in H^p$ can be written as

$$f(z) = \sum_{n,k} c_{nk} \frac{(1 - |a_{nk}|)^{n(1-\frac{1}{p})}}{(1 - z\bar{a}_{nk})^n},$$

where $\sum_{n=1}^{\infty} (\sum_{k=1}^{k(n)} |c_{nk}|^p)^{\frac{1}{p}} < +\infty$. Furthermore the mapping Ψ defined by $\Psi(C) = \Psi\big((c_{nk})_{nk}\big) = \sum_{n,k} c_{nk} \frac{(1-|a_{nk}|)^{n(1-\frac{1}{p})}}{(1-z\bar{a}_{nk})^n}$ is continuous.

With the same notations we then have

PROPOSITION 3.2. *Let $1 < p < +\infty$, $\alpha p \leq n$. Then μ is a Carleson measure for H^p_α if and only if there exists $M > 0$ so that for any finite sequence $(c_{nk})_{nk} \subset \mathbb{C}$,*

$$\left(\int_{\mathbb{B}^n} \left| \sum_{n,k} c_{nk} \frac{(1 - |a_{nk}|)^{n(1-\frac{1}{p})}}{(1 - z\bar{a}_{nk})^{n-\alpha}} \right|^p d\mu(z) \right)^{\frac{1}{p}} \leq M \sum_n \Big(\sum_{k=1}^{k(n)} |c_{nk}|^p \Big)^{\frac{1}{p}}. \tag{3.2}$$

PROOF OF PROPOSITION 3.2. It is proved in [Ca-O, Theorem 2.1] that the linear functional $\phi \colon H^p \longrightarrow H^p_\alpha$ given by

$$\phi(f)(z) = \int_{\mathbb{S}^n} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}} \, d\sigma(\zeta)$$

is onto. Thus the composition with Ψ is also onto. But for any $a_{nk} \in \mathbb{B}^n$, Cauchy's formula gives

$$\phi\left(\frac{1}{(1 - \bar{a}_{nk} \cdot)^n} \right)(z) = \int_{\mathbb{S}^n} \frac{1}{(1 - \zeta\bar{a}_{nk})^n} \frac{d\sigma(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}} = \frac{1}{(1 - z\bar{a}_{nk})^{n-\alpha}},$$

and we obtain the desired conclusion. ■

REMARK 3.2. If we just take one term in (3.2) we deduce a necessary condition, namely $\mu\Big(T\big(B(\zeta, r)\big)\Big) \leq r^m$, which, as is shown in [St] is not sufficient.

We have seen that, in general, Stegenga's condition is no longer necessary for a measure to be Carleson for H^p_α. Nevertheless there are some particular cases where it still characterizes. The first case is given by the following proposition.

PROPOSITION 3.3. *Assume $p > 1$, $\alpha p \leq n$ and $n - \alpha < 1$. Then a finite positive Borel measure μ in $\mathbb{B}^n$ is Carleson for H^p_α if and only if*

$$\mu\big(T(A)\big) \leq k C_{\alpha p}(A),$$

for any open set $A \subset \mathbb{S}^n$.

PROOF OF PROPOSITION 3.3. Applying again [Ca-O, Theorem 2.1], any $F \in H^p_\alpha$ can be written as

$$F(z) = \int_{\mathbb{S}^n} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}}\, d\sigma(\zeta), \quad \text{with } \|F\|_{p,\alpha} \simeq \|f\|_{L^p(d\sigma)}.$$

Since μ is Carleson for H^p_α, there exists $C > 0$ such that

$$\int_{\mathbb{B}^n} \left| \int_{\mathbb{S}^n} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}}\, d\sigma(\zeta) \right|^p d\mu(z) \leq C\|f\|_p^p, \quad f \in L^p_+(d\sigma).$$

But since $n - \alpha < 1$, this is equivalent to

$$\int_{\mathbb{B}^n} \left(\int_{\mathbb{S}^n} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-\alpha}}\, d\sigma(\zeta) \right)^p d\mu(z) \leq C\|f\|_p^p,$$

and since the integral on the left is equivalent to $\int_{\mathbb{B}^n} (P[K_\alpha * |f|])^p(z)\, d\mu(z)$, we are done. ∎

Before giving our next result we need some definitions. If $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$, and $\gamma \colon I \longrightarrow \mathbb{S}^n$ is a smooth non-singular map, $\Gamma = \gamma(I)$ is complex-tangential if and only if $\gamma'(x)(h) \cdot \overline{\gamma(x)} = 0$ for any $x \in I$ and $h \in \mathbb{R}^k$. Let $\pi(\Gamma) = \{r\zeta \,;\, \zeta \in \Gamma 0 < r \leq 1\}$, and denote by $d\tilde{\nu}$ the k-dimensional Lebesgue measure on Γ.

PROPOSITION 3.4. *If μ is a finite positive Borel measure in $\mathbb{B}^n$, supported in $\pi(\Gamma)$, then μ is a Carleson measure for H^p_α if and only if there exists $\delta > 0$, $M > 0$ so that for any open set $A \subset \Gamma_\delta = \{\zeta \,;\, d(\zeta, \Gamma) < \delta\}$,*

$$\mu\big(T(A)\big) \leq MC_{\alpha p}(A).$$

PROOF OF PROPOSITION 3.4. The proof is based in Lemma 2.7 and Theorem 2.8 in [A-C], where it is proved that if E is a subset of a complex tangential manifold with $C_{\alpha p}(E) = 0$, E is an exceptional set for H^p_α. For the sake of completeness, we give a brief sketch of the proof.

Let μ be a measure supported in $\pi(\Gamma)$, which is Carleson for H^p_α, and let $A \subseteq \Gamma_\delta, \delta > 0$ to be chosen. Take $g \in L^p_+(d\sigma)$ any test function. Then $K_\alpha * g \geq 1$ on A, and there exists $C > 0$ so that $P[K_\alpha * g] \geq C$, on $T(A)$. Since $K_\alpha * g \simeq P[K_\alpha * g]$, we deduce that

$$\mu\big(T(A)\big) \leq C \int_{T(A)} (K_\alpha * g)^p(z)\, d\mu(z) = C \int_{T(A) \cap \pi(\Gamma)} (K_\alpha * g)^p(r\zeta)\, d\mu(r\zeta). \tag{3.3}$$

Now the argument in Lemma 2.7 in [A-C] can be adapted to show that $K_\alpha * g(r\zeta) \lesssim \int_{\pi(\Gamma)} P[g](\eta) K_\alpha(r\zeta, \eta)\, d\nu(\eta)$, where ν is the measure in $\pi(\Gamma)$ defined by $d\nu(r\zeta) = (1 - r)^{n - \frac{k}{2} - 1}\, d\tilde{\nu}(\zeta)$. Thus (3.3) is bounded by

$$\int_{\pi(\Gamma)} \left(\int_{\pi(\Gamma)} P[g](\eta) K_\beta(r\zeta, \eta)\, d\nu(\eta) \right)^p d\mu(r\zeta).$$

Given $r\zeta \in \pi(\Gamma)$, let $V_{r\zeta} = \{\eta \in \pi(\Gamma) ; |1 - r\zeta\bar{\eta}| < \delta\}$. In [A-C, (2.6)] is shown that for any $\eta \in V_{r\zeta}$, $\operatorname{Re} \frac{1}{(1-r\zeta\bar{\eta})^{n-\alpha}} \gtrsim \frac{1}{|1-r\zeta\bar{\eta}|^{n-\alpha}}$, and breaking the inner integral in two pieces we obtain

$$\begin{aligned}\mu\big(T(A)\big) &\lesssim \int_{\pi(\Gamma)} \left(\operatorname{Re} \int_{V_{r\zeta}} P[g](\eta) \frac{d\nu(\eta)}{(1 - r\zeta\bar{\eta})^{n-\alpha}}\right)^p d\mu(r\zeta) \\ &\quad + \int_{\pi(\Gamma)} \left(\int_{\pi(\Gamma)\setminus V_{r\zeta}} P[g](\eta) \frac{d\nu(\eta)}{|1 - r\zeta\bar{\eta}|^{n-\alpha}}\right)^p d\mu(r\zeta) \\ &\lesssim \int_{\pi(\Gamma)} \left|\int_{\pi(\Gamma)} P[g](\eta) \frac{d\nu(\eta)}{(1 - r\zeta\bar{\eta})^{n-\alpha}}\right|^p d\mu(r\zeta) \\ &\quad + \int_{\pi(\Gamma)} \left(\int_{\pi(\Gamma)\setminus V_{r\zeta}} P[g](\eta) \frac{d\nu(\eta)}{|1 - r\zeta\bar{\eta}|^{n-\alpha}}\right|^p d\mu(r\zeta).\end{aligned}$$

Next, Corollary 2.6 in [A-C] shows that the function

$$G(z) = \int_{\pi(\Gamma)} P[g](\eta) \frac{d\nu(\eta)}{(1 - z\bar{\eta})^{n-\alpha}}$$

is in H^p_α, and $\|R^\alpha G\|_p^p \lesssim \int_{\mathbb{B}^n} P[g]^p \, d\nu(\eta)$. Then the above is bounded by

$$\int_{\mathbb{B}^n} P[g]^p \, d\nu,$$

which in turn is bounded by $\|g\|_p^p$, since $d\nu$ satisfies by Proposition 2.3 in [A-C] that $\int_{\mathbb{B}^n} |h| \, d\nu \lesssim \int_{\mathbb{S}^n} Nh \, d\sigma$. ■

The last case is also based in [A-C] with slight modifications, and we refer there for the proof.

PROPOSITION 3.5. *Let $p = 2, n - 2\alpha < 1$. Then a finite positive Borel measure in $\mathbb{B}^n$ is a Carleson measure for H^p_α if and only if*

$$\mu\big(T(A)\big) \le KC_{\alpha p}(A),$$

for any open set A in $\mathbb{S}^n$.

In the following, we will deal with Carleson measures for Besov-Sobolev spaces. If $1 < p < +\infty$, $q > 0$ and $s \in \mathbb{N}$, the Besov-Sobolev space A^p_{qs} is given by

$$A^p_{qs} = \left\{f \in H(\mathbb{B}^n) ; \|f\|^p_{A^p_{qs}} = |f(0)| + \int_{\mathbb{B}^n} (1 - |z|)^{q-1} |R^s f(z)|^p \, dV(z) < +\infty\right\}.$$

It is well known ([Be, Theorem 1.2]) that if $s_j \in \mathbb{N}$, $q_j > 0, j = 1, 2$ and $0 < p < +\infty$, and if $q_2 - q_1 = p(s_2 - s_1)$, then $A^p_{q_1 s_1} = A^p_{q_2 s_2}$, with equivalent norm. In particular $A^p_{qs} = A^p_{q+ps+1}$ and we may always assume that $q \ge 1$. It is also known ([Be]) that if $p \le 2$, $A^p_{qs} \subset H^p_{s-\frac{q}{p}}$, whereas if $2 \le p < +\infty$, $H^p_{s-\frac{q}{p}} \subset A^p_{qs}$. In what follows, we will denote $\alpha = s - \frac{q}{p}$, $m = n - \alpha p$.

From the following representation theorem for A^p_{qs}, ([O-F, Theorem 4.1]) we deduce an equivalent formulation for a measure μ to be Carleson for A^p_{qs}.

THEOREM [O-F]. *If $0 < \eta_1 < \eta_0$ are small enough, and $\varepsilon < 1$, there exists a lattice $(a_{kj}) \subset \mathbb{B}^n$ so that*

(i) $|a_{kj}| = 1 - \varepsilon^k$, $j = 1, \ldots, j_k$.

(ii) $\bigcup_{k,j}\{\omega\ ;\ \rho(a_{kj}, \omega) < \eta_0\} = \mathbb{B}^n$.

(iii) $\{\omega\ ;\ \rho(a_{kj}, \omega) < \eta_1\} \cap \{\omega\ ;\ \rho(a_{k'j'}, \omega) < \eta_1\} \neq \emptyset$ *iff* $k = k'$, $j = j'$; *so that the mapping*

$$T : \Big\{ C = (c_{kj})_{kj}\ ;\ \sum_{k,j} |c_{kj}|^p \varepsilon^{km} < +\infty \Big\} \longrightarrow A^p_{qs} \quad \text{given by}$$

$$T(C) = \sum_{k=0}^{+\infty} \sum_{j=0}^{j_k} c_{kj} \frac{(1 - |a_{kj}|^2)^{n+1+\gamma}}{(1 - \bar{a}_{kj} z)^{n+1+\gamma}} \text{ is onto, provided } \gamma > \frac{q}{p}.$$

With the above notations we then have:

PROPOSITION 3.6. *Let $1 < p < +\infty$, $q > 0$ and $s \in \mathbb{N}$ so that $0 < \alpha = s - \frac{q}{p}$, $m = n - \alpha p \geq 0$. Then a finite positive Borel measure μ in $\mathbb{B}^n$ is a Carleson measure for A^p_{qs}, if and only if there exists $M > 0$ such that*

$$\int_{\mathbb{B}^n} \Big| \sum_{k,j} c_{kj} \frac{(1 - |a_{kj}|^2)^{n+1+\gamma}}{(1 - \bar{a}_{kj} z)^{n+1+\gamma}} \Big|^p d\mu(z) \leq M \sum_{k,j} |c_{kj}|^p \varepsilon^{km},$$

for any finite sequence $(c_{kj})_{k,j} \subset \mathbb{C}$.

Our next type of results will be obtained, as in the previous cases, by duality.

THEOREM 3.1. *Assume $1 < p < +\infty$, $q > 0$ and $s \in \mathbb{N}$ so that $0 < \alpha = s - \frac{q}{p}$, $m = n - \alpha p \geq 0$. Let μ be a finite Borel positive measure in $\mathbb{B}^n$ satisfying*

$$\sup_{z \in \mathbb{B}^n} \int_{\mathbb{B}^n} \frac{(1 - |\zeta|^2)^{sp'+q-1}}{|1 - z\bar{\zeta}|^{n+q}} \Big(\int_{\mathbb{B}^n} \frac{d\mu(\omega)}{|1 - \omega\bar{\zeta}|^{n+q}} \Big)^{\frac{1}{p-1}} dV(\zeta) < +\infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Then μ is a Carleson measure for A^p_{qs}.

PROOF OF THEOREM 3.1. In [Be, Corollary 2.3] it is shown that any $f \in A^1_{q0}$ can be written as

$$f(z) = \sum_{j=0}^{s} \int_{\mathbb{B}^n} R^j f(\zeta) K_j(z, \zeta)(1 - |\zeta|^2)^{q+s-1}\, dV(\zeta),$$

where the kernels K_j are holomorphic in z, and satisfy for any $\beta \in \mathbb{N}^n$,

$$|\partial_z^\beta K_j(z, \zeta)| \leq \frac{C_\beta}{|1 - z\bar{\zeta}|^{n+q+|\beta|}}.$$

The above estimates on the kernel, and Theorem 3.1 in [Be-Bu] gives that the operator defined by

$$K_j f(z) = \int_{\mathbb{B}^n} (1 - |\zeta|^2)^{q+s-1} K_j(z, \zeta) f(\zeta)\, dV(\zeta)$$

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is bounded from $L^p_q = L^p\big((1-|z|)^{q-1}\,dV(z)\big)$ to itself.

On the other hand, there exists a linear bounded operator, $\phi\colon L^p_q \longrightarrow \overbrace{L^p_q \times \cdots \times L^p_q}^{s}$, $\phi(h) = (h_0,\dots,h_s)$ with $h_s = h$, and $Rh_{i-1} = h_i$. All together, the operator $T\colon L^p_q \longrightarrow A^p_{qs}$ defined by

$$Th(z) = \int_{\mathbb{B}^n} \sum_{j=0}^{s} h_j(\zeta)K_j(z,\zeta)(1-|\zeta|^2)^{q+s-1}\,dV(\zeta)$$

is bounded and onto. Thus μ is a Carleson measure for A^p_{qs} if and only if T is bounded from L^p_q to $L^p(d\mu)$. Equivalently, the adjoint operator $T^*\colon L^{p'}(d\mu) \longrightarrow L^{p'}_q$, $\frac{1}{p}+\frac{1}{p'} = 1$, given by

$$T^*f(\zeta) = \int_{\mathbb{B}^n} \sum_{j=0}^{s} f(z)K_j(z,\zeta)(1-|\zeta|^2)^s\,d\mu(z),$$

must be bounded. Now, Hölder's inequality and the hypothesis on μ give

$$\begin{aligned}
&\int_{\mathbb{B}^n}\left|\int_{\mathbb{B}^n}\frac{(1-|\zeta|^2)^s}{|1-z\bar\zeta|^{n+q}}f(z)\,d\mu(z)\right|^{p'}(1-|z|^2)^{q-1}\,dV(\zeta)\\
&\quad\le \int_{\mathbb{B}^n}|f(z)|^{p'}\left(\int_{\mathbb{B}^n}\frac{(1-|\zeta|^2)^{sp'+q-1}}{|1-z\bar\zeta|^{n+q}}\left(\int_{\mathbb{B}^n}\frac{d\mu(\omega)}{|1-\omega\bar\zeta|^{n+q}}\right)^{\frac{1}{p-1}}dV(\zeta)\right)d\mu(z)\\
&\quad\lesssim \int_{\mathbb{B}^n}|f(z)|^{p'}\,d\mu(z),
\end{aligned}$$

which together with the estimates on $K_j(z,\zeta)$, show that T^* is bounded. ∎

REMARK 3.2. As we have already said $H^2_\alpha = A^2_{qs}$ with $q = 2(s-\alpha)$. It is easy to check that in that case the sufficient condition (2.5) given in Remark 2.1 and the one in that theorem coincides. In fact, a change to polar coordinates shows that

$$\int_{\mathbb{B}^n}\frac{(1-|\zeta|^2)^{2s+q+1}}{|1-z\bar\zeta|^{n+q}}\frac{dV(\zeta)}{|1-\omega\bar\zeta|^{n+q}} \simeq \frac{1}{|1-z\bar\omega|^{n-2\alpha}},$$

which gives the desired equivalence.

We also can state, as in the case of potentials, a sufficient condition in terms of a modulus of continuity of μ.

THEOREM 3.2. *Assume* $1 < p < +\infty$, $q > 0$ *and* $s \in \mathbb{N}$ *so that* $0 < \alpha = s - \frac{q}{p}$, $m = n - \alpha p \ge 0$. *Let* μ *be a finite Borel measure in* $\mathbb{B}^n$ *satisfying.*

$$\int_0^{+\infty}\left(\varphi(\delta)\delta^{-m}\right)^{\frac{1}{p-1}}\frac{d\delta}{\delta} < +\infty,$$

with $\varphi(\delta) = \sup_{\zeta\in\mathbb{B}^n}\mu\Big(T\big(B(\zeta,\delta)\big)\Big)$. *Then* μ *is a Carleson measure for* A^p_{qs}.

PROOF OF THEOREM 3.2. An integration in polar coordinates gives

$$\begin{aligned}
&\int_{\mathbb{B}^n}\frac{(1-|\zeta|^2)^{p's+q-1}}{|1-z\bar\zeta|^{n+q}}\left(\int_{\mathbb{B}^n}\frac{d\mu(\omega)}{|1-\omega\bar\zeta|^{n+q}}\right)^{\frac{1}{p-1}}dV(\zeta)\\
&\qquad\lesssim \int_{\mathbb{S}^n}\frac{1}{|1-z\bar\eta|^{n-\alpha}}\left(\int_{\mathbb{B}^n}\frac{d\mu(\omega)}{|1-\eta\bar\omega|^{n-\alpha}}\right)^{\frac{1}{p-1}}d\sigma(\eta).
\end{aligned}$$

The main estimate in Theorem 2.2 shows that the integral on the right hand side is bounded. Hence, Theorem 3.1 finishes the proof. ∎

THEOREM 3.3. *Let* $1 < p < +\infty$, $q > 0$ *and* $s \in \mathbb{N}$ *so that* $\alpha = s - \frac{q}{p}$, $m = n - \alpha p > 0$. *Let* μ *be a finite positive Borel measure in* $\mathbb{B}^n$ *satisfying that there exist* $C > 0$ *and* $1 < \tau \leq \frac{n}{m}$ *with*

$$\mu\Big(T^C_\tau\big(B(\zeta, r)\big)\Big) \lesssim r^{m\tau},$$

for any $\zeta \in \mathbb{S}^n$, $r > 0$. *Then* μ *is a Carleson measure for* A^p_{qs}.

PROOF OF THEOREM 3.3. The proof is similar to Theorem 2.3 replacing Proposition 2.2 by

PROPOSITION 3.7. *Let* μ *be a positive finite Borel measure in* $\mathbb{B}^n$, *and assume there exists* $\beta > 2$, $C > 0$ *and* $1 < \tau \leq \frac{n}{m}$ *so that*

(i) $\operatorname{supp}\mu \subset \{z\,;\,(1-|z|) < \varepsilon C^{\frac{1}{\tau-1}}\}$, *where* $\varepsilon = \frac{1}{4^{\frac{\tau}{\tau-1}}}$,

(ii) $\mu\Big(T^C_\tau\big(B(\zeta, r)\big) \cap T^\beta\big(B(\zeta, r)\big)\Big) \lesssim r^{m\tau}$.

Then

$$\int_{\mathbb{B}^n} (1-|z|^2)^{sp} |f(z)|^p \, d\mu(z) \lesssim C^m \int_{\mathbb{B}^n} (1-|z|^2)^{q-1} |f(z)|^p \, dV(z),$$

for any $f \in H(\mathbb{B}^n)$.

PROOF OF PROPOSITION 3.7. Let $B_\rho(z,r) = \{\omega \in \mathbb{B}^n\;;\;\rho(z,\omega) < r\}$, where $z \in \mathbb{B}^n, r > 0$ and $\rho(z,\omega)$ is the pseudodistance in $\mathbb{B}^n$. It is then easy to show that there exist constants $k, K > 0$ so that for any $z \in \mathbb{B}^n$, and if $z = z_0|z|$, $1-|z| < \varepsilon C^{\frac{1}{\tau-1}}$,

$$\mathbb{B}_\rho\big(z, k(1-|z|)\big) \subset T^C_\tau\Big(B\big(z_0, KC^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}\big)\Big) \cap T^\beta\Big(B\big(z_0, KC^{\frac{1}{\tau}}(1-|z|)^{\frac{1}{\tau}}\big)\Big), \tag{3.4}$$

and that if $\omega \in B_\rho\big(z, k(1-|z|)\big)$ then $1-|\omega| \simeq 1-|z|$.

Now the mean-value inequality gives that for any $z \in \mathbb{B}^n$

$$|f(z)|^p \lesssim \frac{1}{(1-|z|)^{n+1}} \int_{B_\rho(z,k(1-|z|))} |f(z)|^p \, dV(\omega).$$

Applying Fubini's theorem, the fact that $1-|\omega| \simeq 1-|z|$ and (3.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{B}^n} (1-|z|)^{sp} |f(z)|^p \, d\mu(z) \\
&\quad\lesssim \int_{\{z;(1-|z|)<\varepsilon C^{\frac{1}{\tau}}\}} (1-|z|)^{sp} \frac{1}{(1-|z|)^{n+1}} \int_{B_\rho(z,k(1-|z|))} |f(z)|^p \, dV(\omega)\, d\mu(z) \\
&\quad\lesssim \int_{\mathbb{B}^n} (1-|\omega|)^{sp-(n+1)} |f(\omega)|^p \int_{z \in B_\rho(\omega, k'(1-|z|))} d\mu(z)\, dV(\omega) \\
&\quad\lesssim \int_{\mathbb{B}^n} (1-|z|)^{sp-(n+1)} |f(\omega)|^p \\
&\qquad\qquad \mu\Big(T^C_\tau B\big(\omega_0, kC^{\frac{1}{\tau}}(1-|\omega|)^{\frac{1}{\tau}}\big) \cap T^\beta\Big(B\big(\omega_0, kC^{\frac{1}{\tau}}(1-|\omega|)^{\frac{1}{\tau}}\big)\Big)\Big).
\end{aligned}$$

The hypotheses on μ finally give that the above is bounded by

$$C^m \int_{\mathbb{B}^n} (1-|\omega|)^{m+sp-(n+1)} |f(\omega)|^p \, dV(\omega) = C^m \int_{\mathbb{B}^n} (1-|\omega|)^{q-1} |f(\omega)|^p dV(\omega). \qquad \blacksquare$$

Our final result needs some more definitions. A holomorphic function f in the unit disc $\mathbb{D}$ is a multiplier for $A^p_{qs}(\mathbb{D})$ if for any $g \in A^p_{qs}(\mathbb{D})$, $f \cdot g \in A^p_{qs}(\mathbb{D})$. In [Ve] it is proved the following characterization of inner multipliers:

THEOREM [VE]. *If $p > 1$, $s \in \mathbb{N}$, $q > 0$, $0 < \alpha = s - \frac{q}{p}$, and $m = 1 - \alpha p = 1 + q - sp > 0$, and if f is an inner function in $\mathbb{D}$, then f is a multiplier for $A^p_{qs}(\mathbb{D})$ if and only if f is a Blaschke product, whose sequence $(a_k)_k$ of zeroes satisfies that the measure $\mu = \sum_k (1-|a_k|)^{1+q-sp} \delta_{a_k}$ is a Carleson measure for $A^p_{qs}(\mathbb{D})$.*

In particular, all the sufficient conditions for a measure to be Carleson for $A^p_{qs}(\mathbb{D})$ that we have obtained, can be used to give examples of multipliers. The following example will produce a multiplier which we are not able to obtain from the sufficient conditions for a measure to be Carleson, known up to now.

EXAMPLE 3.1. There exists a sequence $(a_k)_k \subset \mathbb{D}$ so that if $0 < m = 1+q-sp < 1$, $p > 1$, and if $\mu = \sum (1-|a_k|)^m \delta_{a_k}$, then μ is a Carleson measure for A^p_{qs}. In addition, there exists $\zeta_0 \in \mathbb{S}$ so that

$$\mu\Big(T_{\frac{1}{m}}\big(B(\zeta_0, r)\big)\Big) \not\lesssim r^n. \tag{3.5}$$

In particular, the Blaschke product whose zeros are the sequence $(a_k)_k$ is a multiplier for A^p_{qs}.

We will construct the required measure in the upper half-plane, being easier to deal with the computations. We need to show that there exists a sequence $(a_k)_k \subset \mathbb{R}^2_+$ with $a_k = (x_k, y_k)$ so that the measure $\mu = \sum y_k^m \delta_{a_k}$ satisfies the hypothesis on Theorem 3.3 but does not satisfy (3.5). This will follow once we prove that there exists some $\tau < \frac{1}{m}$ so that

(i) $\sum_{a_k \in \mathbb{Q}_\tau(B(x_0,r))} y_k^m \lesssim r^m$ but

(ii) $\sum_{a_k \in \mathbb{Q}_{\frac{1}{m}}(B(x_0,r))} y_k^m \not\lesssim r^m$,

Where $\mathbb{Q}_\tau\big(B(x_0,r)\big) = \{(x,y)\ ;\ |x-x_0| < r^{\frac{1}{\tau}}, y < r\}$, and $\mathbb{Q}_{\frac{1}{m}}$ is defined similarly, and where $x_0 \in \mathbb{R}$, $r > 0$.

Now, if $\tau < \frac{1}{m}$, let $t \in (\tau, \frac{1}{m})$ be fixed, and let $\varphi\colon [0,+\infty) \longrightarrow \mathbb{R}$ be the function defined by $\varphi(x) = e^{\frac{-\sqrt{x}}{tm}}$. For $k \in \mathbb{N}$, let $a_k = \big(\varphi(k), \varphi(k)^t\big)$. We will show that the sequence $(a_k)_k$ satisfies (i) and (ii). First assume that $x_0 = 0$, and let $0 < r < 1$.

Since $\tau < t < \frac{1}{m}$, we have that conditions (i) and (ii) can be rewritten as:

(i′) $\sum_{\varphi(k) < r^{\frac{1}{\tau}}} \varphi(k)^{tm} \lesssim r^m$

(ii′) $\sum_{\varphi(k)^t < r} \varphi(k)^{tm} \not\lesssim r^m$,

conditions which are immediate to verify for the selected φ. In particular, (ii′) says that (i) holds, which gives that (3.5) is not satisfied.

Thus in order to finish we need to show that (i) is also satisfied for any $x_0 \in \mathbb{R}$. Since (i′) holds, it is enough to prove (i) for the regions $\{(x,y)\ ;\ s \le x \le s + r^{\frac{1}{\tau}}, 0 < y < r\}$

with $r^{\frac{1}{\tau}} \leq s$. We must then show that

$$\sum_{s<\varphi(k)<s+r^{\frac{1}{\tau}}} \varphi(k)^{tm} \lesssim r^m.$$

Expressing the above sum as an integral, using the definition of φ and the mean-value theorem, we have

$$\begin{aligned}\sum_{s<\varphi(k)<s+r^{\frac{1}{\tau}}} \varphi(k)^{tm} &\lesssim \left(1+\log\frac{1}{(s+r^{\frac{1}{\tau}})^{tm}}\right)(s+r^{\frac{1}{\tau}})^{tm} - \left(1+\log\frac{1}{s^{tm}}\right)s^{tm} \\ &= tmr^{\frac{1}{\tau}}\left[\log\frac{1}{h^{tm}}\cdot h^{tm-1}\right],\end{aligned}$$

where $h \in [s, s+r^{\frac{1}{\tau}}]$.

Now the function in brackets is decreasing ($tm \leq 1$), and since $r^{\frac{1}{\tau}} \leq s \leq h$, the above is bounded by $r^{\frac{tm}{\tau}} \log\frac{1}{r}$, which is, due to the fact that $t > \tau$, bounded by r^m. ∎

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