A UNIFIED APPROACH TO GENERATE RISK MEASURES

BY

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ABSTRACT

The paper derives many existing risk measures and premium principles by minimizing a Markov bound for the tail probability. Our approach involves two exogenous functions $v(S)$ and $\phi(S,\pi)$ and another exogenous parameter $\alpha \leq 1$. Minimizing a general Markov bound leads to the following unifying equation:

$$E[\phi(S,\pi)] = \alpha E[v(S)].$$

For any random variable, the risk measure π is the solution to the unifying equation. By varying the functions ϕ and v, the paper derives the mean value principle, the zero-utility premium principle, the Swiss premium principle, Tail VaR, Yaari's dual theory of risk, mixture of Esscher principles and more. The paper also discusses combining two risks with super-additive properties and sub-additive properties. In addition, we recall some of the important characterization theorems of these risk measures.

KEYWORDS

Insurance premium principle, Risk measure, Markov inequality

1. INTRODUCTION

In the economic and actuarial financial literature the concept of insurance premium principles (risk measures) has been studied from different angles. An insurance premium principle is a mapping from the set of risks to the reals, cf. e.g. Gerber (1979). The reason to study insurance premium principles is the well-known fact in the actuarial field that if the premium income equals the expectation of the claim size or less, ruin is certain. In order to keep the ruin probability restricted one considers a risk characteristic or a risk measure for calculating premiums that includes a safety loading. This concept is essential for the economics of actuarial evaluations. Several types of insurance premium principles have been studied and characterized by means of axioms as in

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Goovaerts *et al.* (1984). On the other hand, desirable properties for premiums relevant from an economic point of view have been considered. An insurance premium principle is often considered as the "price" of a risk (or of a tail risk in reinsurance), as the value of a stochastic reserve, or as an indication of the maximal probable loss. This gives the relation to ordering of risks that is recently developed in the actuarial literature. In Artzner (1999), see also Artzner *et al.* (1999), a risk measure is also defined as a mapping from the set of r.v.'s to the reals. It could be argued that a risk measure is a broader concept than an insurance premium calculation principle. Indeed, for a risk X, the probability $\rho(X) = \Pr[X > 1.10\mathrm{E}[X]]$ is a risk measure, but this is not a premium calculation principle because tacitly it is assumed that premiums are expressed in monetary units. However, assuming homogeneity for a risk measure, hence $\rho(aX) = a\rho(X)$ for all real $a > 0$ and all risks X, implies that $\rho(X)$ allows changing the monetary units. On the other hand, because the parameters appearing in the insurance premium principles may depend on monetary units, the class of insurance premiums contains the risk measures that are homogeneous as a special case. In addition, let X be a risk variable with finite expectation and let u be an initial surplus. Defining a transformed random variable describing risk as

$$Y = X + \frac{\alpha}{u}(X - \mathrm{E}[X])^2$$

also allows risk measures to depend on other monetary quantities. Consequently it is difficult to give a distinction between insurance premium principles and homogeneous risk measures. Different sets of axioms lead to different risk measures. The choice of the relevant axioms of course depends on the economics of the situations for which it is used. Desirable properties might be different for actual calculation of premiums, for reinsurance premiums, or for allocation, and so on.

In this paper we present a unified approach to some important classes of premium principles as well as risk measures, based on the Markov inequality for tail probabilities. We prove that most well-known insurance premium principles can be derived in this way. In addition, we will refer to some of the important characterization theorems of these risk measures.

Basic material on utility theory and insurance goes back to Borch (1968, 1974), using the utility concept of von Neumann and Morgenstern (1944). The foundation of premium principles was laid by Bühlmann (1970) who introduced the zero-utility premium, Gerber (1979) and comprehensively by Goovaerts *et al.* (1984). The utility concept, the mean-value premium principle as well as the expected value principle can be deduced from certain axioms. An early source is Hardy *et al.* (1952). The Swiss premium calculation principle was introduced by Gerber (1974) and De Vijlder and Goovaerts (1979). A multiplicative equivalent of the utility framework has led to the Orlicz principle as introduced by Haezendonck and Goovaerts (1982). A characterization for additive premiums has been introduced by Gerber and Goovaerts (1981), and led to the so-called mixture of Esscher premium principles. More recently, Wang (1996) introduced in the actuarial literature the distortion functions into the framework of risk measures, using Yaari's (1987) dual theory of choice under risk. This approach can also be introduced in an axiomatic way. Artzner (1999) restricted the class of Orlicz premium principles by adding the requirement of translation invariance to its axioms, weakened by Jarrow (2002). This has mathematical consequences that are sometimes contrary to practical insurance applications. In the 1980's the practical significance of the basic axioms has been discussed; see Goovaerts *et al.* (1984). On the same grounds Artzner (1999) provided an argumentation for selecting a set of desirable axioms. In Goovaerts *et al.* (2003) it is argued that there are no sets of axioms generally valid for all types of risky situations. There is a difference in desirable properties when one considers a risk measure for allocation of capital, a risk measure for regulating purposes or a risk measure for premiums. There is a parallel with mathematical statistics, where characteristics of distributions may have quite different meanings and uses, like e.g. the mean to measure central tendency, the variance to measure spread, the skewness to reflect asymmetry and the peakedness to measure the thickness of the tails. In an actuarial context, risk measures might have different properties than in other economic contexts. For instance, if we cannot assume that there are two different reinsurers willing to cover both halves of a risk separately, the risk measure (premium) for the entire risk should be larger than twice the original risk measure.

This paper aims to introduce many different risk measures (premium principles) now available, each with their desirable properties, within a unified framework based on the Markov inequality. To give an idea how this is achieved, we give a simple illustration.

Example 1.1. *The exponential premium is derived as the solution to the utility equilibrium equation*

$$\mathrm{E}\left[-e^{-\beta(w-X)}\right] = \mathrm{E}\left[-e^{-\beta(w-\pi)}\right], \tag{1.1}$$

where w is the initial capital and $u(x) = -e^{-\beta x}$ *is the utility attached to wealth level x. This is equivalent to*

$$\mathrm{E}\left[e^{-\beta(\pi-X)}\right] = 1, \tag{1.2}$$

hence we get the explicit solution

$$\pi = \frac{1}{\beta}\log \mathrm{E}\left[e^{\beta X}\right]. \tag{1.3}$$

Taking $Y = e^{\beta X}$ *and* $y = e^{\beta\pi}$ *and applying* $\Pr[Y > y] \le \frac{1}{y}\mathrm{E}[Y]$ *(Markov inequality), we get the following inequality for the survival probabilities with X:*

$$\Pr[X > \pi] \le \frac{1}{e^{\beta\pi}}\mathrm{E}\left[e^{\beta X}\right]. \tag{1.4}$$

For this Markov bound to be non-trivial, the r.h.s. of (1.4) must be at most 1. *It equals* 1 *when π is equal to the exponential(β) premium with X. This procedure leads to an equation which gives the premium for X from a Markov bound.*

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Two more things must be noted. First, for fixed π*, we write the bound in (1.4) as* $f(\beta) = \mathrm{E}[e^{\beta(X-\pi)}]$*. Since* $f''(\beta) = \mathrm{E}[(X-\pi)^2 e^{\beta(X-\pi)}] > 0$ *the function* $f(\beta)$ *is convex in some relevant* β*-region. The risk aversion* β_0 *for which this bound* $f(\beta)$ *is minimal has* $f'(\beta_0) = 0$*, hence* $\pi = \mathrm{E}[Xe^{\beta_0 X}] / \mathrm{E}[e^{\beta_0 X}]$*, which is the Esscher premium for X with parameter* β_0*. This way, also the Esscher premium has been linked to a Markov bound. The r.h.s. of (1.4) equals* 1 *for* $\beta = 0$ *as well, and is less than or equal to* 1 *for* β *in the interval* $[0,\beta_1]$*, where* β_1 *is the risk aversion for which the exponential premium equals* π*. Second, if* $\pi = \frac{1}{\beta}\log \mathrm{E}[e^{\beta X}]$ *holds, we have the following exponential upper bound for tail probabilities: for any* $k > 0$*,*

$$\Pr[X > \pi + k] \leq \frac{1}{e^{\beta(\pi + k)}} \mathrm{E}\left[e^{\beta X}\right] = e^{-\beta k}. \tag{1.5}$$

Using variations of the Markov bound above, the various equations that generate various premium principles (or risk measures) can be derived. Section 2 presents a method to do this, Section 3 applies this method to many such principles, and discusses their axiomatic foundations as well as some other properties; Section 4 concludes.

2. Generating Markovian risk measures

Throughout this paper, we denote the cumulative distribution function (cdf) of a random variable S by F_S. For any non-negative and non-decreasing function $v(s)$ satisfying

$$\mathrm{E}[v(S)] < +\infty, \tag{2.1}$$

we define an associated r.v. S^* having a cdf with differential

$$\mathrm{d}F_{S^*}(s) = \frac{v(s)\mathrm{d}F_S(s)}{\mathrm{E}[v(S)]}, \quad -\infty < s < +\infty. \tag{2.2}$$

It is easy to prove that

$$\Pr[S > \pi] \leq \Pr\left[S^* > \pi\right], \quad -\infty < \pi < +\infty. \tag{2.3}$$

For any Lebesgue measurable bivariate function $\phi(\cdot,\cdot)$ satisfying

$$\phi(s,\pi) \geq \mathrm{I}_{(s>\pi)}, \tag{2.4}$$

we have the following inequalities:

$$\Pr\left[S^* > \pi\right] = \mathrm{E}\left[\mathrm{I}_{(S^*>\pi)}\right] \leq \mathrm{E}\left[\phi\left(S^*,\pi\right)\right]. \tag{2.5}$$

Then it follows from (2.3) that

$$\Pr[S > \pi] \leq \frac{\mathrm{E}[\phi(S,\pi)v(S)]}{\mathrm{E}[v(S)]}. \tag{GMI}$$

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This is a generalized version of the Markov inequality, which has $S \geq 0$ with probability 1 and $\pi \geq 0$, $\phi(s,\pi) = s/\pi$ and $v(s) \equiv 1$. Therefore, we denote it by the acronym [GMI]. Similar discussions can be found in Runnenburg and Goovaerts (1985), where the functions $v(\cdot)$ and $\phi(\cdot,\cdot)$ are specified as $v(\cdot) \equiv 1$ and $\phi(s,\pi) = f(s)/f(\pi)$, respectively, for some non-negative and non-decreasing function $f(\cdot)$.

For the inequality [GMI] to make sense, the bivariate function $\phi(s,\pi)$ given in (2.4) and the r.v. S should satisfy

$$\mathrm{E}[\phi(S,\pi)\, v(S)] < +\infty \tag{2.6}$$

for all relevant π. Note that if (2.6) holds for some $-\infty < \pi < +\infty$ then (2.1) does as well. By assuming (2.6), it is clear that the family of r.v.'s S considered in the inequality [GMI] is restricted, in the sense that the right tail of S can not be arbitrarily heavy. For the given functions $\phi(\cdot,\cdot)$ and $v(\cdot)$ as above, we introduce below a family of all admissible r.v.'s that satisfy (2.6):

$$\mathbb{S}_{\phi,v} = \{S : \mathrm{E}[\phi(S,\pi)v(S)] < +\infty \quad \text{for all large } \pi\}. \tag{2.7}$$

Sometimes we are interested in the case that there exists a minimal value $\pi_M^{(\alpha)}$ such that [GMI] gives a bound

$$\Pr[S > \pi_M] \leq \frac{\mathrm{E}[\phi(S,\pi_M)\, v(S)]}{\mathrm{E}[v(S)]} \leq \alpha \leq 1. \tag{2.8}$$

Note that (2.8) produces an upper bound for the α-quantile $q_\alpha(S)$ of S. For each $0 \leq \alpha \leq 1$, the restriction (2.4) on $\phi(\cdot,\cdot)$ allows us further to introduce a subfamily of $\mathbb{S}_{\phi,v}$ as follows:

$$\mathbb{S}_{\phi,v,\alpha} = \left\{S : \frac{\mathrm{E}[\phi(S,\pi)\, v(S)]}{\mathrm{E}[v(S)]} \leq \alpha \quad \text{for all large } \pi\right\}. \tag{2.9}$$

If in (2.4) the function $\phi(s,\pi)$ is strictly smaller than 1 for at least one point (s,π), then it is not difficult to prove that there are some values of $0 \leq \alpha < 1$ such that the subfamilies $\mathbb{S}_{\phi,v,\alpha}$ are not empty. We also note that $\mathbb{S}_{\phi,v,\alpha}$ increases in $\alpha \geq 0$.

Hereafter, for a real function $f(\cdot)$ defined on an interval D and a constant b in the range of the function $f(\cdot)$, we write an equation $f(\pi) = b$ with the understanding that its root is the minimal value of π satisfying the inequalities $f(\pi) \leq b$ and $\max\{f(x) \mid x \in (\pi - \varepsilon, \pi + \varepsilon) \cap D\} \geq b$ for any $\varepsilon > 0$. With this convention, the minimal value $\pi_M^{(\alpha)}$ such that the second inequality in (2.8) holds is simply the solution of the equation

$$\frac{\mathrm{E}[\phi(S,\pi_M)\, v(S)]}{\mathrm{E}[v(S)]} = \alpha. \tag{UE_α}$$

When $\alpha = 1$, we call

$$\frac{\mathrm{E}\left[\phi(S,\pi_M)\,v(S)\right]}{\mathrm{E}\left[v(S)\right]} = 1 \qquad \text{[UE]}$$

the unifying equation, or [UE] in acronym. This equation will act as the unifying form to generate many well-known risk measures. The equation [UE] gives the minimal percentile for which the upper bound for the tail probability of S still makes sense. It will turn out that these minimal percentiles correspond to several well-known premium principles (risk measures). It is clear that the solution of the equation [UE] is not smaller than the minimal value of the r.v. S.

Definition 2.1. *Let S be an admissible r.v. from the family* $\mathbb{S}_{\phi,v,\alpha}$ *for some* $0 \le \alpha \le 1$, *where* $\phi = \phi(\cdot,\cdot)$ *and* $v = v(\cdot)$ *are two given measurable functions with* ϕ *satisfying (2.4) and v non-negative and non-decreasing. The solution* $\pi_M^{(\alpha)}$ *of the equation [*UE_α*] is called a* Markovian risk measure *of the r.v. S at level* α.

Remark 2.1. *About the actuarial meaning of the ingredients* $\phi(\cdot,\cdot)$, $v(\cdot)$ *and* α *in Definition 2.1 we remark that* α *represents a confidence bound, which in practical situations is determined by the regulator or the management of an insurance company. In principle the actuarial risk measures considered are intended to be approximations (on the safe side) for the VaR of order* α, *and to have some desirable actuarial properties such as additivity, subadditivity, or superadditivity, according to actuarial applications for calculating solvency margins, for RBC calculations, as well as for the top-down approach of premiums calculations. The functions* $\phi(\cdot,\cdot)$ *and* $v(\cdot)$ *are introduced to derive bounds for the VaR, so that these bounds have some desirable properties for applications. In addition, because a risk measure provides an upper bound for the VaR, it might be interesting to determine the minimal value of the risk measure attached by the different choices of* $\phi(\cdot,\cdot)$ *and* $v(\cdot)$.

Remark 2.2. *Clearly, given the ingredients* $\phi(\cdot,\cdot)$, $v(\cdot)$ *and* α, *the Markovian risk measure* $\pi_M^{(\alpha)}(S)$ *involves only the distribution of the admissible r.v. S. A Markovian risk measure provides an upper bound for the VaR at the same level. By selecting appropriate functions* ϕ *the Markovian risk measures can reflect desirable properties when adding r.v.'s in addition to their dependence structure.*

Remark 2.3. *Let* X_1 *and* X_2 *be two admissible r.v.'s, with Markovian risk measures* $\pi_M^{(\alpha)}(X_1)$ *and* $\pi_M^{(\alpha)}(X_2)$. *Then we have*

$$\Pr\left[X_1 > \pi_M^{(\alpha)}(X_1)\right] \le \alpha, \qquad \Pr\left[X_2 > \pi_M^{(\alpha)}(X_2)\right] \le \alpha. \qquad (2.10)$$

We can obtain from the equation [UE_α*] that*

$$\Pr\left[X_1 + X_2 > \pi_M^{(\alpha)}(X_1) + \pi_M^{(\alpha)}(X_2)\right] \le \alpha \qquad (2.11)$$

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- *in case X_1 and X_2 are independent when the risk measure $\pi_M^{(\alpha)}$ involved is subadditive for sums of independent risks;*
- *in case X_1 and X_2 are comonotonic when the risk measure $\pi_M^{(\alpha)}$ involved is subadditive for sums of comonotonic risks;*
- *for any X_1 and X_2, regardless of their dependence structure, in case a subadditive risk measure $\pi_M^{(\alpha)}$ is applied.*

3. Some Markovian risk measures

In what follows we will provide a list of important insurance premium principles (or risk measures) and show how they can be derived from the equation [UE]. We will also list a set of basic underlying axioms. In practice, for different situations different sets of axioms are needed.

3.1. The mean value principle

The mean value principle has been characterized by Hardy *et al.* (1952); see also Goovaerts *et al.* (1984), Chapter 2.8, in the framework of insurance premiums.

Definition 3.1. *Let S be a risk variable. For a given non-decreasing and non-negative function $f(\cdot)$ such that* $\mathrm{E}[f(S)]$ *converges, the* mean value risk measure $\pi = \pi_f$ *is the root of the equation* $f(\pi) = \mathrm{E}[f(S)]$.

Clearly, we can obtain the mean value risk measure by choosing in the equation [UE] the functions $\phi(s,\pi) = f(s)/f(\pi)$ and $v(\cdot) \equiv 1$. As verified in Goovaerts *et al.* (1984), p. 57-61, this principle can be characterized by the following axioms (necessary and sufficient conditions):

A1.1. $\pi(c) = c$ for any degenerate risk c;

A1.2. $\Pr[X \le Y] = 1 \Longrightarrow \pi(X) \le \pi(Y)$;

A1.3. If $\pi(X) = \pi(X')$, Y is a r.v. and I is a Bernoulli variable independent of the vector $\{X, X', Y\}$, then $\pi(IX + (1-I)Y) = \pi(IX' + (1-I)Y)$.

Remark 3.1. *This last axiom can be expressed in terms of distribution functions by assuming that mixing F_Y with F_X or with $F_{X'}$ leads to the same risk measure, as long as the mixing weights are the same.*

Remark 3.2. *Under the condition that* $\mathrm{E}[f(S)]$ *converges one obtains as an upper bound for the survival probability*

$$\Pr[S > \pi + u] \le \frac{\mathrm{E}[f(S)]}{f(\pi + u)} = \frac{f(\pi)}{f(\pi + u)}. \tag{3.1}$$

Specifically, when $\mathrm{E}[e^{\lambda S}] < \infty$ *for some* $\lambda > 0$ *one obtains* $e^{-\lambda u}$ *as an upper bound for the probability* $\Pr[S > \pi + u]$*, see the example in Section 1. In case π_α is the root of* $\mathrm{E}[f(S)] = \alpha f(\pi)$*, by the inequality [GMI] one gets* $\Pr[S \ge \pi_\alpha] \le \alpha$.

3.2. The zero-utility premium principle

The zero-utility premium principle was introduced by Bühlmann (1970).

Definition 3.2. *Let* $u(\cdot)$ *be a non-decreasing utility function. The* zero-utility premium $\pi(S)$ *is the solution of* $u(0) = \mathrm{E}[u(\pi - S)]$.

We assume that either the risk or the utility function is bounded from above. Because $u(\cdot)$ and $u(\cdot) + c$ define the same ordering in expected utility, the utility is determined such that $u(x) \to 0$ as $x \to +\infty$. To obtain the zero-utility premium principle, one chooses in the equation [UE] the functions $\phi(s,\pi) = u(\pi - s) / u(0)$ and $v(\cdot) \equiv 1$.

In order to relate the utility to the VaR one should proceed as follows. By the inequality [GMI], we get

$$\Pr\left[S > \pi_\alpha\right] \le \mathrm{E}\left[\frac{-u\left(\pi_\alpha - S\right)}{-u(0)}\right] = \alpha, \tag{3.2}$$

where π_α is the solution of the equation $\mathrm{E}[u(\pi - S)] = \alpha u(0)$. The result obtained here requires that the utility function $u(\cdot)$ is bounded from below. However, this restriction can be weakened by considering limits for translated utility functions.

Let the symbol $\preceq_{eu}$ represent the weak order with respect to the zero-utility premium principle, that is, $X \preceq_{eu} Y$ means that X is preferable to Y. We write $X \sim_{eu} Y$ if both $X \preceq_{eu} Y$ and $Y \preceq_{eu} X$. It is well-known that the preferences of a decision maker between risks can be described by means of comparing expected utility as a measure of the risk if they fulfill the following five axioms which are due to von Neumann and Morgenstern (1944) (combining Denuit *et al.* (1999) and Wang and Young (1998)):

A2.1. If $F_X = F_Y$ then $X \sim_{eu} Y$;

A2.2. The order $\preceq_{eu}$ is reflexive, transitive and complete;

A2.3. If $X_n \preceq_{eu} Y$ and $F_{X_n} \to F_X$ then $X \preceq_{eu} Y$;

A2.4. If $F_X \ge F_Y$ then $X \preceq_{eu} Y$;

A2.5. If $X \preceq_{eu} Y$ and if the distribution functions of X'_p and Y'_p are given by $F_{X'_p}(x) = pF_X(x) + (1-p)F_Z(x)$ and $F_{Y'_p}(x) = pF_Y(x) + (1-p)F_Z(x)$ where F_Z is an arbitrary distribution function, then $X'_p \preceq_{eu} Y'_p$ for any $p \in [0,1]$.

From these axioms, the existence of a utility function $u(\cdot)$ can be proven, with the property that $X \preceq_{eu} Y$ if and only if $\mathrm{E}[u(-X)] \ge \mathrm{E}[u(-Y)]$.

3.3. The Swiss premium calculation principle

The Swiss premium principle was introduced by Gerber (1974) to put the mean value principle and the zero-utility principle in a unified framework.

Definition 3.3. *Let $w(\cdot)$ be a non-negative and non-decreasing function on $\mathbb{R}$ and $0 \le z \le 1$ be a parameter. Then the* Swiss premium principle $\pi = \pi(S)$ *is the root of the equation*

$$\mathrm{E}\left[w\left(S - z\pi\right)\right] = w\left((1 - z)\,\pi\right). \tag{3.3}$$

This equation is the special case of [UE] with $\phi(s, \pi) = w(s - z\pi) / w((1 - z)\pi)$ and $v(\cdot) \equiv 1$. It is clear that $z = 0$ provides us with the mean value premium, while $z = 1$ gives the zero-utility premium. Recall that by the inequality [GMI], the root π_α of the equation $\mathrm{E}[w(S - z\pi)] = \alpha w((1 - z)\pi)$ determines an upper bound for the VaR_α.

Remark 3.3. *Because one still may choose $w(\cdot)$, it can be arranged to have supplementary properties for the risk measures. Indeed if we assume that $w(\cdot)$ is convex, we have*

$$X \le_{cx} Y \Longrightarrow \pi(X) \le \pi(Y). \tag{3.4}$$

See for instance Dhaene et al. (2002a, b) for the definition of $\le_{cx}$ (convex order). For two random pairs (S_1, S_2) and $(\widetilde{S}_1, \widetilde{S}_2)$ with the same marginal distributions, we call (S_1, S_2) more related than $(\widetilde{S}_1, \widetilde{S}_2)$ if the probability $\Pr[S_1 \le x, S_2 \le y]$ *that S_1 and S_2 are both small is larger than that for $\widetilde{S}_1$ and $\widetilde{S}_2$, for all x and y; see e.g. Kaas et al. (2001), Chapter 10.6. In this case one gets from* (3.4)

$$\pi(\widetilde{S}_1 + \widetilde{S}_2) \le \pi(S_1 + S_2). \tag{3.5}$$

The risk measure of the sum of a pair of r.v.'s with the same marginal distributions depends on the dependence structure, and in this case increases with the degree of dependence between the terms of the sum.

Remark 3.4. *Gerber (1974) proves the following characterization: Let $w(\cdot)$ be strictly increasing and continuous, then the Swiss premium calculation principle generated by $w(\cdot)$ is additive for independent risks if and only if $w(\cdot)$ is exponential or linear.*

3.4. The Orlicz premium principle

The Orlicz principle was introduced by Haezendonck and Goovaerts (1982) as a multiplicative equivalent of the zero-utility principle. To introduce this premium principle, they used the concept of a Young function ψ, which is a mapping from $\mathbb{R}_0^+$ into $\mathbb{R}_0^+$ that can be written as an integral of the form

$$\psi(x) = \int_0^x f(t)\,\mathrm{d}t, \qquad x \ge 0, \tag{3.6}$$

where f is a left-continuous, non-decreasing on $\mathbb{R}_0^+$ satisfying $f(0) = 0$ and $\lim_{x \to +\infty} f(x) = +\infty$. It is seen that a Young function ψ is absolutely continuous, convex and strictly increasing, and has $\psi'(0) = 0$. We say that ψ is normalized if $\psi(1) = 1$.

Definition 3.4. *Let ψ be a normalized Young function. The root of the equation*

$$\mathrm{E}[\psi(S/\pi)] = 1 \tag{3.7}$$

is called the Orlicz premium principle *of the risk S.*

The unified approach follows from the equation [UE] with $\phi(s,\pi)$ replaced by $\psi(s/\pi)$ and $v(s) \equiv 1$. The Orlicz premium satisfies the following properties:

A4.1. $\Pr[X \le Y] = 1 \Longrightarrow \pi(X) \le \pi(Y)$;

A4.2. $\pi(X) = 1$ when $X \equiv 1$;

A4.3. $\pi(aX) = a\pi(X)$ for any $a > 0$ and any risk X;

A4.4. $\pi(X+Y) \le \pi(X) + \pi(Y)$.

Remark 3.5. *A4.3 above says that the Orlicz premium principle is positively homogenous. In the literature, positive homogeneity is often confused with currency independence. As an example, we look at the standard deviation principle* $\pi_1(X) = \mathrm{E}[X] + \alpha \cdot \sigma[X]$ *and the variance principle* $\pi_2(X) = \mathrm{E}[X] + \beta \cdot \mathrm{Var}[X]$, *where α and β are two positive constants, α is dimension-free but the dimension of* $1/\beta$ *is money. Clearly $\pi_1(X)$ is positive homogenous but $\pi_2(X)$ is not. But it stands to reason that when applying a premium principle, if the currency is changed, so should all constants having dimension money. So going from BFr to Euro, where* 1 *Euro* $\approx$ 40 *BFr, the value of β in $\pi_2(X)$ should be adjusted by the same factor. In this way both $\pi_1(X)$ and $\pi_2(X)$ are independent of the monetary unit.*

Remark 3.6. *These properties remain exactly the same for risks that may also be negative, such as those used in the definition of coherent risk measures by Artzner (1999). Indeed if $\pi(-1) = -1$ and one extends these properties to r.v.'s supported on the whole line $\mathbb{R}$, then*

$$\pi(X + a - a) \le \pi(X + a) - a. \tag{3.8}$$

Hence $\pi(X+a) \ge \pi(X) + a$ and consequently $\pi(X+a) = \pi(X) + a$.

The interested reader is referred to Haezendonck and Goovaerts (1982). If in addition translation invariance is imposed for non-negative risks, it turns out that the only coherent risk measure for non-negative risks within the class of Orlicz principles is an expectation $\pi(X) = \mathrm{E}[X]$.

Remark 3.7. *The Orlicz principle can also be generalized to cope with VaR_α. Actually, from the inequality [GMI], the solution π_α of the equation* $\mathrm{E}[\psi(S/\pi)]$ $= \alpha$ *gives* $\Pr[S > \pi_\alpha] \le \alpha$.

3.5. More general risk measures derived from Markov bounds

For this section, we confine to risks with the same mean. We consider more general risk measures derived from Markov bounds, applied to sums of pairs

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of r.v.'s, which may or may not be independent. The generalization consists in the fact that we consider the dependence structure to some extent in the risk premium, letting the premium for the sum $X + Y$ depend both on the distribution of the sum $X + Y$ and on the distribution of the sum $X^c + Y^c$ of the comonotonic (maximally dependent) copies of the r.v.'s X and Y. Because of this, we denote the premium for the sum $X + Y$ by $\pi(X, Y)$ rather than by $\pi(X + Y)$. When the r.v.'s X and Y are comonotonic, however, there is no difference in understanding between the two symbols $\pi(X, Y)$ and $\pi(X + Y)$.

Taking $\pi(X)$ simply equal to $\pi(X, 0)$, we consider the following properties:

A5.1. $\pi(aX) = a\pi(X)$ for any $a > 0$;

A5.2. $\pi(X + b) = \pi(X) + b$ for any $b \in \mathbb{R}$;

A5.3$_a$. $\pi(X, Y) \leq \pi(X) + \pi(Y)$;

A5.3$_b$. $\pi(X, Y) \geq \pi(X) + \pi(Y)$;

A5.3$_c$. $\pi(X, Y) = \pi(X) + \pi(Y)$.

Remark 3.8. *A5.3$_a$ describes the subadditivity property, which is realistic only in case diversification of risks is possible. However, this is rarely the case in insurance. Subadditivity gives rise to easy mathematics because distance functions can be used. The superadditivity property for a risk measure (that is not Artzner coherent) is redundant in the following practical situation of capital allocation or solvency assessment. Suppose that two companies with risks X_1 and X_2 merge and form a company with risk $X_1 + X_2$. Let d_1, d_2 and d denote the allocated capitals or solvency margins. Then, with probability* 1*,*

$$(X_1 + X_2 - d_1 - d_2)_+ \leq (X_1 - d_1)_+ + (X_2 - d_2)_+. \tag{3.9}$$

This inequality expresses the fact that, with probability 1*, the residual risk of the merged company is smaller than the risk of the split company. In case $d \geq d_1 + d_2$, one gets, also with probability* 1*,*

$$(X_1 + X_2 - d)_+ \leq (X_1 - d_1)_+ + (X_2 - d_2)_+. \tag{3.10}$$

Hence in case one calculates the capitals d_1, d_2 and d by means of a risk measure it should be superadditive (or additive) to describe the economics in the right way. Subadditivity is only based on the idea that it is easier to convince the shareholders of a conglomerate in failure to provide additional capital than the shareholders of some of the subsidiaries. Recent cases indicate that for companies in a financial distress situation splitting is the only way out.

Remark 3.9. *It should also be noted that subadditivity cannot be used as an argument for a merger of companies to be efficient. The preservation* (3.9) *of the inequality of risks with probability one expresses this fact; indeed*

$$\pi\big((X_1 + X_2 - d_1 - d_2)_+\big) \leq \pi\big((X_1 - d_1)_+ + (X_2 - d_2)_+\big) \tag{3.11}$$

expresses the efficiency of a merger. It has nothing to do with the subadditivity. A capital $d < d_1 + d_2$, for instance derived by a subadditive risk measure, can only *be considered if the dependence structure allows it. For instance if d is determined as the minimal root of the equation*

$$\pi\big((X_1 + X_2 - d)_+\big) = \pi\big((X_1 - d_1)_+ + (X_2 - d_2)_+\big), \tag{3.12}$$

d obviously depends on the dependence between X_1 *and* X_2*. Note that in* $(X_1 - d_1)_+ + (X_2 - d_2)_+$*, the two terms are dependent. Taking this dependence into account, the risk measure providing the capitals d,* d_1 *and* d_2 *will not always be subadditive, nor always superadditive, but may instead exhibit behavior similar to the VaR, see Embrechts et al. (2002).*

Let $\psi(\cdot)$ be a non-decreasing, non-negative, and convex function on $\mathbb{R}$ satisfying $\lim_{x\to+\infty} \psi(x) = +\infty$. For fixed $0 < p < 1$, we get, by choosing $v(\cdot) = 1$, the equality

$$\phi(X, Y, \pi) = \frac{1}{\psi(1)} \cdot \psi\left(\frac{\left(X + Y - F^{-1}_{X^c+Y^c}(p)\right)_+}{\pi - F^{-1}_{X^c+Y^c}(p)}\right) \tag{3.13}$$

and by solving [UE] for π, the following risk measure for the sum of two r.v.'s:

$$\mathrm{E}\left[\psi\left(\frac{\left(X + Y - F^{-1}_{X^c+Y^c}(p)\right)_+}{\pi(X,Y) - F^{-1}_{X^c+Y^c}(p)}\right)\right] = \psi(1) \tag{3.14}$$

for some parameter $0 < p < 1$. Hereafter, the pth quantile of a r.v. X with d.f. F_X is, as usual, defined by

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \ge p\}, \quad p \in [0,1]. \tag{3.15}$$

It is easily seen that there exists a unique constant $a(p) > 0$ such that

$$\mathrm{E}\left[\psi\left(\frac{\left(X + Y - F^{-1}_{X^c+Y^c}(p)\right)_+}{a(p)}\right)\right] = \psi(1). \tag{3.16}$$

Thus $\pi(X,Y) = F^{-1}_{X^c+Y^c}(p) + a(p)$. Especially, letting Y be degenerate at 0 we get $\pi(X) > F_X^{-1}(p)$.

Now we check that A5.1, A5.2 and A5.3$_a$ are satisfied by π (subadditive case). In fact, the proofs for the first two axioms are trivial. As for A5.3$_a$, we derive

$$\mathrm{E}\left[\psi\left(\frac{\left(X + Y - F^{-1}_{X^c+Y^c}(p)\right)_+}{\pi(X) + \pi(Y) - F^{-1}_{X^c+Y^c}(p)}\right)\right]$$

$$
\begin{aligned}
&\leq \mathrm{E}\left[\psi\left(\frac{\left(X-F_X^{-1}(p)\right)_+ + \left(Y-F_Y^{-1}(p)\right)_+}{\pi(X)+\pi(Y)-F_X^{-1}(p)-F_Y^{-1}(p)}\right)\right] \\
&= \mathrm{E}\left[\psi\left(\frac{\pi(X)-F_X^{-1}(p)}{\pi(X)+\pi(Y)-F_X^{-1}(p)-F_Y^{-1}(p)}\cdot\frac{X-F_X^{-1}(p)}{\pi(X)-F_X^{-1}(p)}\right.\right. \\
&\qquad \left.\left. + \frac{\pi(Y)-F_Y^{-1}(p)}{\pi(X)+\pi(Y)-F_X^{-1}(p)-F_Y^{-1}(p)}\cdot\frac{Y-F_Y^{-1}(p)}{\pi(Y)-F_Y^{-1}(p)}\right)\right] \\
&\leq \mathrm{E}\left[\frac{\pi(X)-F_X^{-1}(p)}{\pi(X)+\pi(Y)-F_X^{-1}(p)-F_Y^{-1}(p)}\cdot\psi\left(\frac{X-F_X^{-1}(p)}{\pi(X)-F_X^{-1}(p)}\right)\right] \\
&\qquad + \mathrm{E}\left[\frac{\pi(Y)-F_Y^{-1}(p)}{\pi(X)+\pi(Y)-F_X^{-1}(p)-F_Y^{-1}(p)}\cdot\psi\left(\frac{Y-F_Y^{-1}(p)}{\pi(Y)-F_Y^{-1}(p)}\right)\right] \\
&= \psi(1) \\
&= \mathrm{E}\left[\psi\left(\frac{\left(X+Y-F_{X^c+Y^c}^{-1}(p)\right)}{\pi(X,Y)-F_{X^c+Y^c}^{-1}(p)}\right)\right].
\end{aligned} \tag{3.17}
$$

This proves A5.3_a.

Remark 3.10. *If the function $\psi(\cdot)$ above is restricted to satisfy $\psi(1) = \psi'(1)$, then it can be proven that the risk measure*

$$\pi_l(X) = F_X^{-1}(p) + \mathrm{E}\left[\left(X - F_X^{-1}(p)\right)_+\right] \tag{3.18}$$

gives the lowest generalized Orlicz measure. In fact, since ψ is convex on $\mathbb{R}$ and satisfies $\psi(1) = \psi'(1)$, we have

$$\psi\left((x)_+\right) \geq \psi(1)\cdot(x)_+ \quad \text{for any } x \in \mathbb{R}. \tag{3.19}$$

Let $\pi(X)$ be a generalized Orlicz risk measure of the risk X, that is, $\pi(X)$ is the solution of the equation

$$\mathrm{E}\left[\psi\left(\frac{\left(X-F_X^{-1}(p)\right)_+}{\pi(X)-F_X^{-1}(p)}\right)\right] = \psi(1). \tag{3.20}$$

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By (3.19) and recalling that $\pi(X) > F_X^{-1}(p)$, *we have*

$$\psi(1) = \mathrm{E}\left[\psi\left(\frac{\left(X - F_X^{-1}(p)\right)_+}{\pi(X) - F_X^{-1}(p)}\right)\right] \geq \psi(1) \cdot \mathrm{E}\left[\frac{\left(X - F_X^{-1}(p)\right)_+}{\pi(X) - F_X^{-1}(p)}\right], \qquad (3.21)$$

which implies that

$$\pi(X) \geq F_X^{-1}(p) + \mathrm{E}\left[\left(X - F_X^{-1}(p)\right)_+\right] = \pi_l(X). \qquad (3.22)$$

Remark 3.11. *Now we consider the risk measure*

$$\mathrm{E}\left[\psi\left(\frac{\left(X - F_X^{-1}(p)\right)_+}{\pi(X) - F_X^{-1}(p)}\right)\right] = 1 - p \qquad (3.23)$$

for some parameter $0 < p < 1$. *Similarly as in Remark 3.12, if the function* $\psi(\cdot)$ *is restricted to satisfy* $\psi(1) = \psi'(1)$, *we obtain the lowest risk measure as*

$$\pi(X) = F_X^{-1}(p) + \frac{1}{1-p}\mathrm{E}\left[\left(X - F_X^{-1}(p)\right)_+\right] = \mathrm{E}\left[X \,\middle|\, X > F_X^{-1}(p)\right]. \qquad (3.24)$$

Remark 3.12. *Another choice is to consider the root of the equation*

$$\mathrm{E}\left[\frac{1}{\psi(1)} \cdot \psi\left(\frac{|X - \mathrm{E}[X]|}{\pi(X) - \mathrm{E}[X]}\right)\right] = \alpha, \qquad (3.25)$$

defining, in general terms, a risk measure for the deviation from the expectation. As a special case when $\psi(t) \equiv t^2 \mathrm{I}_{(t \geq 0)}$ *one gets*

$$\pi(X) = \mathrm{E}[X] + \frac{\sigma[X]}{\sqrt{\alpha}}. \qquad (3.26)$$

Note that both (3.24) *and* (3.26) *produce an upper bound for the* α*-quantile* $q_\alpha(X)$ *of X.*

Remark 3.13. *For a risk variable X, one could consider a risk measure* $\pi_c(X)$ *which is additive, and define another risk measure* $\rho(X)$, *where the deviation* $a(X) = \rho(X) - \pi_c(X)$ *is determined by*

$$\mathrm{E}\left[\psi\left(\frac{|X - \pi_c(X)|}{a(X)}\right)\right] = \psi(1). \qquad (3.27)$$

Here the role of $\pi_c(X)$ *is to measure central tendency while* $a(X)$ *measures the deviation of the risk variable X from* $\pi_c(X)$. *If* $\pi_c(X)$ *is positively homogenous,* *translation invariant and additive, then $\rho(X)$ is positively homogenous and translation invariant. The measure $\rho(X)$ may be subadditive or superadditive, depending on the convexity or concavity of the function $\psi(\cdot)$.*

3.6. Yaari's dual theory of choice under risk

Yaari (1987) introduced the dual theory of choice under risk. It was used by Wang (1996), who introduced distortion functions in the actuarial literature. A distortion function is defined as a non-decreasing function $g : [0,1] \longrightarrow [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

Definition 3.5. *Let S be a non-negative r.v. with d.f. F_S, and $g(\cdot)$ be a distortion function defined as above. The* distortion risk measure *associated with the distortion function g is defined by*

$$\pi = \int_0^{+\infty} g\left(1 - F_S(x)\right) \mathrm{d}x. \tag{3.28}$$

Choosing the function $\phi(\cdot,\cdot)$ in the equation [UE] such that $\phi(s,\pi) = s/\pi$ and using the left-hand derivative $g'_-(1-F_S(s))$ instead of $v(s)$, using integration by parts we get the desired unifying approach. The choice of $v(\cdot)$, which at first glance may look artificial, is very natural if one wants to have $\mathrm{E}[v(S)] = 1$.

This risk measure can be characterized by the following axioms:

A6.1. $\Pr[X \leq Y] = 1 \Longrightarrow \pi(X) \leq \pi(Y)$;

A6.2. If risks X and Y are comonotonic then $\pi(X+Y) = \pi(X) + \pi(Y)$;

A6.3. $\pi(1) = 1$.

Remark 3.14. *It is clear that this principle results in large upper bounds because*

$$\Pr[X \geq \pi + u] \leq \mathrm{E}\left[\frac{X \cdot g'_-(1 - F_X(X))}{\pi + u}\right] = \frac{\pi}{\pi + u}. \tag{3.29}$$

It is also clear that the set of risks for which π is finite contains all risks with finite mean.

3.7. Mixtures of Esscher principles

The mixture of Esscher principles was introduced by Gerber and Goovaerts (1981). It is defined as follows:

Definition 3.6. *For a bounded r.v. S, we say a principle $\pi = \pi(S)$ is a* mixture of Esscher principles *if it is of the form*

$$\pi_F(S) = F(-\infty)\phi(-\infty) + \int_{-\infty}^{+\infty} \phi(t)\mathrm{d}F(t) + (1 - F(+\infty))\,\phi(+\infty), \tag{3.30}$$

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where F is a non-decreasing function satisfying $0 \le F(t) \le 1$ *and* ϕ *is of the form*

$$\phi(t) = \phi_S(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \mathrm{E}\left[e^{tS}\right], \quad t \in \mathbb{R}. \tag{3.31}$$

Actually we can regard F as a possibly defective cdf with mass at both $-\infty$ and $+\infty$. Since the variable S is bounded, $\phi(-\infty) = \min[S]$ and $\phi(+\infty) = \max[S]$. In addition, $\phi_S(t)$ is the Esscher premium of S with parameter $t \in \mathbb{R}$.

In the special case where the function F is zero outside the interval $[0, \infty]$, the mixture of Esscher principles is a mixture of premiums with a non-negative safety loading coefficient. We show that in this case the mixture of Esscher premiums can also be derived from the Markov inequality. Actually,

$$\pi_F(S) = \int_0^{+\infty} \phi(t) \mathrm{d}F(t) + (1 - F(+\infty))\,\phi(+\infty) = \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t). \tag{3.32}$$

It can be shown that the mixture of Esscher principles is translation invariant. Hence in what follows, we simply assume, without loss of generality, that $\min[S] \ge 0$ because otherwise a translation on S can be used. We notice that, for any $t \in [0, +\infty]$,

$$\phi(t) = \frac{\mathrm{E}\left[Se^{tS}\right]}{\mathrm{E}\left[e^{tS}\right]} \ge \mathrm{E}[S]. \tag{3.33}$$

The inequality (3.33) can, for instance, be deduced from the fact that the variables S and e^{tS} are comonotonic, hence positively correlated. Since we have assumed that $\min[S] \ge 0$, now we choose in [GMI] the functions $v(\cdot) \equiv 1$ and $\phi(s,\pi) = s/\pi$, then we obtain that

$$\Pr[S > \pi] \le \mathrm{E}[\phi(S,\pi)] \le \frac{1}{\pi} \mathrm{E}[S] \le \frac{1}{\pi} \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t), \tag{3.34}$$

where the last step in (3.34) is due to the inequality (3.33) and the fact that $F([0, +\infty]) = 1$. Letting the r.h.s. of (3.34) be equal to 1, we immediately obtain (3.32).

We now verify another result: the tail probability $\Pr[S > \pi + u]$ decreases exponentially fast in $u \in [0, +\infty)$. The proof is not difficult. Actually, since the risk variable S is bounded, it holds for any $\alpha > 0$ that

$$\Pr[S > \pi + u] \le \exp\{-\alpha(\pi + u)\} \cdot \mathrm{E}[\exp\{\alpha S\}]. \tag{3.35}$$

Hence, in order to get the announced result, it suffices to prove that, for some $\alpha > 0$,

$$\mathrm{E}[\exp\{\alpha S\}] \le \exp\{\alpha\pi\} = \exp\left\{\alpha \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t)\right\},$$

or equivalently to prove that, for some $\alpha > 0$,

$$\log \mathrm{E}[\exp\{\alpha S\}] \leq \alpha \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t). \tag{3.36}$$

In the trivial case where the risk S is degenerate, both sides of (3.36) are equal for any $\alpha > 0$. If F is not degenerate, the Esscher premium $\phi(t)$ is strictly increasing in $t \in [0,+\infty]$, and we can find some $\alpha_0 > 0$ such that

$$\phi(\alpha) \leq \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t) \tag{3.37}$$

holds for any $\alpha \in [0, \alpha_0]$. Thus in any case we obtain that (3.36) holds for any $\alpha \in [0, \alpha_0]$. We summarize:

Remark 3.15. *For the mixture of Esscher premiums* π *defined above, if F is concentrated on* $[0,+\infty]$*, then*

$$\Pr[S > \pi + u] \leq \exp\{-\alpha_0 u\} \tag{3.38}$$

holds for any $u \geq 0$*, where the constant* $\alpha_0 > 0$ *is the solution of the equation*

$$\phi(\alpha) = \int_{[0,+\infty]} \phi(t) \mathrm{d}F(t). \tag{3.39}$$

The mixture of Esscher premiums is characterized by the following axioms; see Gerber and Goovaerts (1981):

A7.1. $\phi_{X_1}(t) \leq \phi_{X_2}(t) \quad \forall t \in \mathbb{R} \Longrightarrow \pi_F(X_1) \leq \pi_F(X_2)$;

A7.2. It holds for any two independent risks X_1 and X_2 that

$$\pi_F(X_1 + X_2) = \pi_F(X_1) + \pi_F(X_2). \tag{3.40}$$

Hence this risk measure is additive for independent risks. When the function F in (3.32) is non-zero only on the interval $[0,\infty]$, the premium contains a positive safety loading.

4. Conclusions

This paper shows how many of the usual premium calculation principles (or risk measures) can be deduced from a generalized Markov inequality. All risk measures provide information concerning the VaR, as well as the asymptotic behavior of $\Pr[S > \pi + u]$. Therefore, the effect of using a risk measure and requiring additional properties is equivalent to making a selection of admissible risks. Notice that when using a risk measure, additional requirements are usually needed about convergence of certain integrals. In this way, the set of admissible risks is restricted, e.g. the one having finite mean, finite variance, finite moment generating function and so on.

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