Canad. Math. Bull. Vol. **53** (3), 2010 pp. 394–397
doi:10.4153/CMB-2010-055-1

On Nearly Equilateral Simplices and Nearly l_∞ Spaces

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Abstract. By $d(X, Y)$ we denote the (multiplicative) Banach–Mazur distance between two normed spaces X and Y. Let X be an n-dimensional normed space with $d(X, l_\infty^n) \le 2$, where l_∞^n stands for $\mathbb{R}^n$ endowed with the norm $\|(x_1, \dots, x_n)\|_\infty := \max\{|x_1|, \dots, |x_n|\}$. Then every metric space (S, ρ) of cardinality $n + 1$ with norm ρ satisfying the condition $\max D / \min D \le 2 / d(X, l_\infty^n)$ for $D := \{\rho(a, b) : a, b \in S,\ a \ne b\}$ can be isometrically embedded into X.

1 Introduction

The theory of embeddings of finite metric spaces into normed spaces is used in various applied disciplines, *e.g.*, for qualitative analysis of large data sets (see [7, Chapter 15] and [5]). The spaces close to l_∞^n typically exhibit marginal properties in the indicated theory. More precisely, they are known to have the "richest" metric structure; cf. [5, §8.1.3] and a recent result from [1]. The theorem proved in this note provides another confirmation of the above informal statement.

The *Banach–Mazur distance* $d(X, Y)$ between two n-dimensional normed spaces X and Y, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, is the least $\alpha \ge 1$ such that for some bijective linear map T from X to Y one has $\|x\|_X \le \|Tx\|_Y \le \alpha \|x\|_X\ \forall x \in X$.

Theorem 1.1 *Let X be an n-dimensional normed space with $\alpha := d(X, l_\infty^n) \le 2$. Let S be a set of cardinality $n + 1$, and ρ be a metric satisfying*

$$\frac{\max D}{\min D} \le \frac{2}{\alpha} \tag{1.1}$$

for $D := \{\rho(a, b) : a, b \in S,\ a \ne b\}$. Then the space (S, ρ) can be isometrically embedded into X.

Theorem 1.1 is similar to [3, Theorem 1.9], providing an analogous statement with l_2^n (n-dimensional Euclidean space) in place of l_∞^n. The metric space (S, ρ) can be viewed as an abstract n-dimensional simplex, which we wish to realize in certain normed spaces. The quantity $\max D / \min D$ estimates the distance of (S, ρ) to the *equilateral metric space* (*i.e.*, the space with all non-zero distances equal). In fact, for $\alpha = 2$ the only metric space (S, ρ) satisfying (1.1) is the equilateral one. For $\alpha = 1$ the space X from Theorem 1.1 is necessarily isometric to l_∞^n, and the inequality (1.1) attains its weakest form $\max D / \min D \le 2$.

Received by the editors August 23, 2007.
Published electronically May 11, 2010.
AMS subject classification: **52A21**, 51F99, 52C99.

Theorem 1.1 generalizes the result of Swanepoel and Villa [10] saying that any n-dimensional normed space X with $\mathrm{d}(X, l_\infty^n) \le \frac{3}{2}$ contains $n+1$ points at pairwise distance one to each other. The proof of Theorem 1.1 extends the arguments from [10, Theorem B] by employing the observation that for every metric ρ on the set $S = \{s_1, \dots, s_{n+1}\}$ of cardinality $n+1$ the mapping

$$s_i \mapsto \big(\rho(s_1, s_i), \dots, \rho(s_n, s_i)\big), \qquad 1 \le i \le n+1,$$

is an isometric embedding of (S, ρ) into l_∞^n (see [9]). One of the ingredients of the proof is the Brouwer fixed point theorem (see also [2] for the use of that theorem in a similar context).

Let $\mathcal{X}^n$ be the class of all n-dimensional Banach spaces. It is known that

$$C \cdot n \le \max_{X,Y \in \mathcal{X}^n} \mathrm{d}(X, Y) \le n$$

for some universal constant $0 < C < 1$ (see [4] and [6, Section 4.1 and Theorem 5.2.1]). From these bounds it is seen that Theorem 1.1 can be applied to "rather many" n-dimensional Banach spaces if n is small, say $n = 3$ or $n = 4$, and to n-dimensional normed spaces which are "very close" to l_∞^n if n is large.

2 Proof

Let $S = \{s_1, \dots, s_{n+1}\}$. In what follows i, j, k are integer indices. For $1 \le i,\ j \le n+1$ we put $\rho_{i,j} := \rho(s_i, s_j)$. Without loss of generality let

$$\min D = 1. \tag{2.1}$$

Then (1.1) amounts to

$$\max D \le \frac{2}{\alpha}. \tag{2.2}$$

Choosing an appropriate coordinate system we may assume that

$$\|x\| \le \|x\|_\infty \le \alpha \|x\|, \tag{2.3}$$

where $\|\cdot\|$ denotes the norm of X. In what follows we shall consider vectors from $\mathbb{R}^{n(n+1)/2}$, whose coordinates will be indexed by the elements of the set

$$I := \{(i, j) : 1 \le i < j \le n+1\}.$$

Let us introduce the $n(n+1)/2$-dimensional cube

$$P := \prod_{(i,j) \in I} [\,0, 2(\alpha - 1)/\alpha] = [0, 2(\alpha - 1)/\alpha]^{n(n+1)/2}.$$

https://doi.org/10.4153/CMB-2010-055-1 Published online by Cambridge University Press

Given the variable vector

$$z := (z_{i,j})_{(i,j)\in I} \in P \tag{2.4}$$

consider the vector functions

$$\begin{aligned}
p_1(z) &:= (\rho_{1,1}, \dots, \rho_{n,1}),\\
&\;\vdots\\
p_j(z) &:= (\rho_{1,j} + z_{1,j}, \dots, \rho_{j-1,j} + z_{j-1,j}, \rho_{j,j}, \dots, \rho_{n,j}) \quad \text{for } 2 \le j \le n,\\
&\;\vdots\\
p_{n+1}(z) &:= (\rho_{1,n+1} + z_{1,n+1}, \dots, \rho_{n,n+1} + z_{n,n+1})
\end{aligned}$$

with values in $\mathbb{R}^n$. Given $1 \le i < j \le n+1$, we have

$$\|p_j(z) - p_i(z)\|_\infty = \max\{R^1_{i,j}(z), R^2_{i,j}(z), R^3_{i,j}(z), R^4_{i,j}(z)\},$$

where

$$\begin{aligned}
R^1_{i,j}(z) &:= \max\bigl\{ |\rho_{k,i} - \rho_{k,j} + z_{k,i} - z_{k,j}| \;:\; 1 \le k \le i-1 \bigr\},\\
R^2_{i,j}(z) &:= |\rho_{i,i} - \rho_{i,j} - z_{i,j}|,\\
R^3_{i,j}(z) &:= \max\bigl\{ |\rho_{k,i} - \rho_{k,j} - z_{k,j}| \;:\; i+1 \le k \le j-1 \bigr\},\\
R^4_{i,j}(z) &:= \max\bigl\{ |\rho_{k,i} - \rho_{k,j}| \;:\; j \le k \le n \bigr\}.
\end{aligned}$$

Let us estimate $R^1_{i,j}(z), \dots, R^4_{i,j}(z)$. For $1 \le i < j \le n+1$ and $1 \le k \le n+1$ with $k \notin \{i, j\}$, we get

$$\begin{aligned}
|\rho_{k,i} - \rho_{k,j} + z_{k,i} - z_{k,j}| &\le |\rho_{k,i} - \rho_{k,j}| + |z_{k,i} - z_{k,j}|\\
&\overset{(2.1),(2.2),(2.4)}{\le} \tfrac{2}{\alpha} - 1 + \tfrac{2(\alpha-1)}{\alpha} = 1 \overset{(2.1)}{\le} \rho_{i,j},\\
|\rho_{i,i} - \rho_{i,j} - z_{i,j}| &= \rho_{i,j} + z_{i,j},\\
|\rho_{k,i} - \rho_{k,j} - z_{k,j}| &\le |\rho_{k,i} - \rho_{k,j}| + |z_{k,j}|\\
&\overset{(2.1),(2.2),(2.4)}{\le} \bigl(\tfrac{2}{\alpha} - 1\bigr) + \tfrac{2(\alpha-1)}{\alpha} = 1 \overset{(2.1)}{\le} \rho_{i,j},\\
|\rho_{k,i} - \rho_{k,j}| &\le \rho_{i,j}.
\end{aligned}$$

Consequently, $R^1_{i,j}(z), R^3_{i,j}(z), R^4_{i,j}(z)$ are not greater than $\rho_{i,j}$ and $R^2_{i,j}(z) = \rho_{i,j} + z_{i,j}$. Hence

$$\|p_i(z) - p_j(z)\|_\infty = \rho_{i,j} + z_{i,j}. \tag{2.5}$$

We define mapping $F(z) := (F_{i,j}(z))_{(i,j)\in I}$ from P to $\Pi_{(i,j)\in I}\mathbb{R} = \mathbb{R}^{n(n+1)/2}$ by

$$F_{i,j}(z) := \rho_{i,j} + z_{i,j} - \|p_i(z) - p_j(z)\|.$$

The mapping $F(z)$ is continuous. The range of $F_{i,j}(z)$ can be found as follows:

$$\begin{aligned} F_{i,j}(z) &\overset{(2.3)}{\geq} \rho_{i,j} + z_{i,j} - \|p_i(z) - p_j(z)\|_\infty \overset{(2.5)}{=} 0, \\ F_{i,j}(z) &\overset{(2.3)}{\leq} \rho_{i,j} + z_{i,j} - \frac{1}{\alpha}\|p_i(z) - p_j(z)\|_\infty \overset{(2.5)}{=} \frac{\alpha-1}{\alpha}(\rho_{i,j} + z_{i,j}) \\ &\overset{(2.2),(2.4)}{\leq} \frac{\alpha-1}{\alpha}\Big(\frac{2}{\alpha} + \frac{2(\alpha-1)}{\alpha}\Big) = \frac{2(\alpha-1)}{\alpha}. \end{aligned}$$

The above inequalities can be reformulated as the inclusion $F(P) \subseteq P$. Thus, the *Brouwer fixed point theorem* (see [8, p. 107]) yields the existence of $z' \in P$ with $F(z') = z'$. This implies the equality $\|p_i(z') - p_j(z')\| = \rho_{i,j}$ for $1 \leq i < j \leq n+1$, *i.e.*, the mapping $s_i \mapsto p_i(z')$ is an isometric embedding of S into X.

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