# BERNSTEIN POWER SERIES

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**1. Introduction.** In Bernstein's proof of the Weierstrass Approximation Theorem, the polynomials

$$B_n(f, x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right)\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu} \tag{1}$$

are constructed in correspondence with a function $f \in C\,[0, 1]$ and are shown to converge uniformly to $f$. These Bernstein polynomials have been the starting point of many investigations, and a number of generalizations of them have appeared. It is our purpose here to consider several generalizations in the form of infinite series and to establish some of their properties. The series to be discussed have their origins in the works of Szász **(8)** and of Meyer-König and Zeller **(3)**.

The succeeding sections are devoted to the following topics: convergence of the sequences, variation-diminishing properties, convergence questions in the complex domain, and an application to differential equations.

**2. The operators and their convergence properties.** In the same way that the Bernstein polynomials originate in the identity

$$1 = \sum_{\nu=0}^{n} \binom{n}{\nu}x^{\nu}(1-x)^{n-\nu} \tag{2}$$

so another sequence of polynomials may be based upon the following identity from **(9**, eq. 5.1.9**)**

$$1 = (1-x)^{\alpha+1}\exp\left(\frac{tx}{1-x}\right)\sum_{\nu=0}^{\infty} L_{\nu}^{(\alpha)}(t)x^{\nu} \qquad (\alpha > -1) \tag{3}$$

in which $L_{\nu}^{(\alpha)}$ denotes the Laguerre polynomial of degree $\nu$. We define an operator by the equation

$$P_n(f, x) = (1-x)^{n+1}\exp\left(\frac{tx}{1-x}\right)\sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right)L_{\nu}^{(n)}(t)x^{\nu} \tag{4}$$

in which $t$ is a parameter assumed $\leqslant 0$. The properties of this operator will be studied subsequently. The case $t = 0$ was the subject of study by Meyer-König and Zeller. We reserve for it a special notation:

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$$M_n(f, x) = (1-x)^{n+1} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right)\binom{\nu+n}{\nu} x^\nu. \tag{5}$$

Actually, the operator of Meyer-König and Zeller had $f(\nu/(\nu+n+1))$ in place of $f(\nu/(\nu+n))$. Our slight modification enables us to establish in the next section the monotonic convergence of $M_n f$ to $f$ whenever $f$ is convex.

In a similar manner, starting from the identity

$$1 = e^{-nx} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (nx)^\nu \tag{6}$$

we are led to the operator introduced and studied by Szász,

$$S_n(f, x) = e^{-nx} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right) \frac{1}{\nu!} (nx)^\nu. \tag{7}$$

The convergence of $S_n(f, x)$ to $f$ uniformly in any interval $[0, a]$ in which $f$ is continuous was established by Szász. For the operator $P_n$ the convergence theorem is as follows.

THEOREM. *If $f$ is continuous on $[0, 1]$, if $a < 1$, and if $t/n \to 0$, then $P_n(f, x)$ converges to $f(x)$ uniformly on $[0, a]$.*

*Proof.* By a theorem of Korovkin (**4**, p. 14) it will suffice to prove that $P_n$ is a positive linear operator and that the desired convergence occurs whenever $f$ is a quadratic function. The linearity is obvious, and the fact that $P_n f \geqslant 0$ when $f \geqslant 0$ comes about because $L_\nu^{(n)}(t) \geqslant 0$ when $t \leqslant 0$. From (3) we see, for the function $f(s) \equiv 1$, that $P_n(1, x) \equiv 1$. Consider then the function $f(s) \equiv s$. For it we have

$$P_n(s, x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} \frac{\nu}{\nu+n} L_\nu^{(n)}(t) x^\nu. \tag{8}$$

Using the known recurrence formula (**9**, eq. 5.1.14),

$$tL_\nu^{(n+1)}(t) = (\nu+n)L_{\nu-1}^{(n)}(t) - \nu L_\nu^{(n)}(t), \tag{9}$$

we get from (8), using then (3),

$$\begin{aligned} P_n(s, x) &= (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} x^\nu \left[L_{\nu-1}^{(n)}(t) - \frac{t}{\nu+n} L_{\nu-1}^{(n+1)}(t)\right] \\ &= (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right)\left[x \sum_{\nu=0}^{\infty} x^\nu L_\nu^{(n)}(t) \right. \\ &\qquad\qquad \left. - tx \sum_{\nu=0}^{\infty} \frac{x^\nu}{\nu+n+1} L_\nu^{(n+1)}(t)\right] \\ &= x - \frac{tx}{(n+1)(1-x)}. \end{aligned} \tag{10}$$

Thus $P_n(s, x) \to x$ uniformly in any interval $[0, a]$ with $a < 1$ provided that $t/n \to 0$.

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We proceed, then, to a consideration of the function $f(s) \equiv s^2$. We have

$$(11) \qquad P_n(s^2, x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} \left(\frac{\nu}{\nu+n}\right)^2 L_\nu^{(n)}(t) x^\nu.$$

Using the recurrence formula (9) twice, we can prove that

$$(12) \quad \left(\frac{\nu}{\nu+n}\right)^2 L_\nu^{(n)}(t) = \frac{\nu+n-1}{\nu+n} L_{\nu-2}^{(n)}(t) - \frac{t}{\nu+n} L_{\nu-2}^{(n+1)}(t) - \frac{\nu t}{(\nu+n)^2} L_{\nu-1}^{(n+1)}(t).$$

Thus the right member of (11) splits naturally into three parts, which we analyse separately below. Since $(\nu+1)/(\nu+n+1)^2 \leqslant 1/n$, we have, using (3),

$$(13) \quad \left|(1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} \frac{\nu t}{(\nu+n)^2} L_{\nu-1}^{(n+1)}(t) x^\nu\right|$$
$$= |t| x (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=0}^{\infty} \frac{\nu+1}{(\nu+n+1)^2} L_\nu^{(n+1)}(t) x^\nu$$
$$\leqslant \frac{|t|x}{n(1-x)} (1-x)^{n+2} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=0}^{\infty} L_\nu^{(n+1)}(t) x^\nu = \frac{|t|x}{n(1-x)}.$$

Similarly, since $1/(\nu+n) \leqslant 1/n$, we have, using (3),

$$(14) \quad \left|(1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} \frac{t}{\nu+n} L_{\nu-2}^{(n+1)}(t) x^\nu\right|$$
$$\leqslant \frac{|t|x^2}{n(1-x)} (1-x)^{n+2} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=0}^{\infty} L_\nu^{(n+1)}(t) x^\nu = \frac{|t|x^2}{n(1-x)}.$$

Finally, since

$$1 - \frac{1}{n} < \frac{\nu+n-1}{\nu+n} < 1,$$

we have, using (3),

$$(15) \qquad x^2\left(1 - \frac{1}{n}\right) < (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{\nu=1}^{\infty} \frac{\nu+n-1}{\nu+n} L_{\nu-2}^{(n)}(t) x^\nu < x^2.$$

Thus if $t/n \to 0$, $P_n(t^2, x) \to x^2$ uniformly in $[0, a]$.

Before leaving the topic of convergence we wish to answer the natural question of whether other operators such as $P_n$ can possibly preserve all linear functions. The operator $P_n$ is a special case of the general class of positive operators $T_n$ defined as follows:

$$(16) \qquad T_n(f, x) = \frac{1}{h_n(t, x)} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{\nu+n}\right) c_{n\nu}(t) x^\nu \qquad (c_{n\nu}(t) > 0).$$

If we are to have $T_n(1, x) \equiv 1$, then

$$h_n(t, x) = \sum_{\nu=0}^{\infty} c_{n\nu}(t)x^\nu. \tag{17}$$

Let us assume that this series (for fixed $t$) defines an analytic function of $x$ so that we may apply the theorem on uniqueness of power series. If $T_n(t, x) \equiv x$, then we must have

$$x = \frac{1}{h_n(t, x)} \sum_{\nu=1}^{\infty} \frac{\nu}{\nu + n} c_{n\nu}(t)x^\nu,$$

whence

$$h_n(t, x) = \sum_{\nu=0}^{\infty} \frac{\nu + 1}{\nu + 1 + n} c_{n,\nu+1}(t)x^\nu. \tag{18}$$

Comparing (17) and (18) we see that

$$c_{n,\nu+1}(t) = \frac{\nu + 1 + n}{\nu + 1} c_{n\nu}(t),$$

whence

$$c_{n\nu}(t) = \binom{n + \nu}{\nu} c_{n0}(t)$$

and

$$h_n(t, x) = c_{n0}(t)(1 - x)^{-n-1}.$$

Thus, $M_n$ as defined in (5) is essentially the only operator of the form (16) which preserves all linear functions.

By precisely the same reasoning applied to the family of operators

$$T_n(f, x) = \frac{1}{h_n(t, x)} \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right) c_{n\nu}(t)x^\nu, \tag{19}$$

which includes as a special case the Szász operator (7), we may conclude that only the Szász operator preserves linear functions. We remark that operators arising from sequences of analytic functions such as (16) and (19) have been studied by Baskakov in **(2)**.

**3. Variation-diminishing properties.** I. J. Schoenberg has introduced in **(7**, p. 250**)** the concept of a variation-diminishing operator. Such an operator has the property

$$V[L_n f] \leqslant V[f],$$

where $V[f]$ is the variation of $f$, defined as the number of changes of sign of the function as $x$ varies across its domain. The relation between variation-diminishing and positivity is indicated in the following result.

THEOREM. *A variation-diminishing one-to-one linear operator on $C[a, b]$ is positive, or its negative is positive.*

*Proof.* If $L$ is not positive, there exist two functions $g, h \geqslant 0$, which are not identically zero, such that $Lg = g_1 \geqslant 0$, $Lh = h_1 \leqslant 0$. Taking $0 \leqslant \lambda \leqslant 1$, we see that $f = \lambda g + (1 - \lambda)h \geqslant 0$ and is not identically zero and that $Lf = \lambda g_1 + (1 - \lambda)h_1$. Each of the functions $Lf$ must be of constant sign. It follows that for some $\lambda$, $Lf = 0$; hence $L$ is not one-to-one.

A direct corollary of Korovkin's theorem (**4**, p. 14) is that no positive linear operator on $C[a, b]$ except the identity can preserve quadratic functions. Indeed, if such a linear operator $L$ exists, then the sequence $L_n = L$ has the property that $L_n f \to f$ for all $f$ in $C[a, b]$, whence $Lf = f$. Another theorem of this type was given by Schoenberg (**7**, p. 254) and has the following generalization.

THEOREM. *Let $\{f_1, f_2, f_3\}$ be any Tchebycheff system of order 3 on the interval $[a, b]$. Then the identity operator is the only variation-diminishing linear operator $L$ such that $Lf_i = f_i$ for $i = 1, 2, 3$.*

*Proof.* (According to Schoenberg for $f_i(x) \equiv x^{i-1}$.) Suppose $L \neq I$. Then for some $g$, $h \equiv Lg \neq g$. Thus for some $x_0$, $h(x_0) \neq g(x_0)$. Suppose for definiteness that $g(x_0) < c < h(x_0)$. If $L$ is variation-diminishing and satisfies $Lf_i = f_i$, then

$$V[h + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3] \leqslant V[g + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3]. \tag{20}$$

By (**4**, p. 44, Lemma 7), there exists a choice of $\beta$'s such that

$$\sum_{i=1}^{3} \beta_i f_i(x) > 0$$

when $x \neq x_0$, while

$$\sum_{i=1}^{3} \beta_i f_i(x_0) = 0.$$

From the characteristic property of Tchebycheff systems, there exist constants $\gamma_i$ such that

$$\sum_{i=1}^{3} \gamma_i f_i(x_0) = c.$$

Then for large $\lambda$, the function

$$\sum_{i=1}^{3} (\gamma_i + \lambda\beta_i) f_i$$

crosses $h$ at least twice and $g$ not at all, contradicting (20).

Before proving that the operator $P_n$ is variation-diminishing we require another general result.

LEMMA. *Let*

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu x^\nu$$

*be uniformly convergent in* $[0, A]$. *Then* $V[f] \leqslant V[\{a_n\}]$.

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*Proof.* If the sequence $\{a_n\}$ presents an infinite number of variations in sign, there is nothing to prove. Suppose then that this sequence has $m$ variations in sign, and suppose that $V[f] > m$. Then there are points

$$0 \leqslant x_1 < x_2 < \ldots < x_{m+2} \leqslant A$$

such that $f(x_\nu)$ alternates in sign. Put $\epsilon = \min|f(x_\nu)|$. Choose $N$ so that

$$\left| \sum_{\nu=N+1}^{\infty} a_\nu x^\nu \right| < \epsilon.$$

Then the polynomial

$$p(x) = \sum_{\nu=0}^{N} a_\nu x^\nu$$

has at the points $x_\nu$ the same sign as $f(x)$, and must therefore have at least $m + 1$ positive zeros, in contradiction with Descartes' rule of signs.

THEOREM. *The operators $P_n$ and $S_n$ defined in* (4) *and* (7) *are variation-diminishing.*

*Proof.* Since

$$(1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) L_\nu^{(n)}(t) > 0,$$

$$V[P_n f] \leqslant V\left[\left\{f\left(\frac{\nu}{\nu+n}\right)\right\}\right] \leqslant V[f],$$

by the preceding theorem. The proof for $S_n$ is similar.

Results analogous to one proved by B. Averbach for Bernstein polynomials (**7**, p. 253) are proved next for the operators $M_n$ and $S_n$ defined in (5) and (7).

THEOREM. *If $f$ is convex, then $M_n f$ is decreasing in $n$, unless $f$ is linear* (*in which case $M_n f = M_{n+1} f$ for all $n$*).

*Proof.*

$$\begin{aligned} M_n(f,x) - M_{n+1}(f,x) &= (1-x)^{n+1} \sum_{\nu=0}^{\infty} \left[ f\left(\frac{\nu}{\nu+n}\right)\binom{\nu+n}{\nu} \right. \\ &\qquad \left. - (1-x) f\left(\frac{\nu}{\nu+n+1}\right)\binom{\nu+n+1}{\nu}\right] x^\nu \\ &= (1-x)^{n+1} \sum_{\nu=1}^{\infty} \left[ \binom{\nu+n}{\nu} f\left(\frac{\nu}{\nu+n}\right) - \binom{\nu+n+1}{\nu} f\left(\frac{\nu}{\nu+n+1}\right) \right. \\ &\qquad \left. + \binom{\nu+n}{\nu-1} f\left(\frac{\nu-1}{\nu+n}\right)\right] x^\nu. \end{aligned}$$

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This last will surely be non-negative if, for all $\nu$,

$$f\left(\frac{\nu}{\nu+n+1}\right) \leqslant \alpha_1 f(x_1) + \alpha_2 f(x_2), \tag{21}$$

where

$$\alpha_1 = \binom{\nu+n+1}{\nu}^{-1}\binom{\nu+n}{\nu-1}, \qquad \alpha_2 = \binom{\nu+n+1}{\nu}^{-1}\binom{\nu+n}{\nu},$$

$$x_1 = \frac{\nu-1}{\nu+n}, \qquad x_2 = \frac{\nu}{\nu+n}.$$

But (21) is a direct consequence of convexity of $f$ since $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, and

$$\alpha_1 x_1 + \alpha_2 x_2 = \nu/(\nu+n+1).$$

The equality of $M_n(f, x)$ and $M_{n+1}(f, x)$ can occur only if

$$f\left(\frac{\nu}{\nu+n+1}\right) = \alpha_1 f(x_1) + \alpha_2 f(x_2) \tag{22}$$

for all $\nu$. But if $f$ is convex, (22) implies that its graph is linear in each of the intervals

$$[x_1, x_2]: \left[0, \frac{1}{n+1}\right], \left[\frac{1}{n+2}, \frac{2}{n+2}\right], \ldots .$$

Since these intervals overlap, $f$ must be linear.

THEOREM. *If $f$ is convex, then $S_n f$ is decreasing in $n$, unless $f$ is linear (in which case $S_n f = S_{n+1} f$ for all $n$).*

*Proof.*

$$S_n(f, x) - S_{n+1}(f, x)$$
$$= e^{-nx-x}\left[e^x \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n}\right)\frac{1}{\nu!}(nx)^\nu - \sum_{\nu=0}^{\infty} f\left(\frac{\nu}{n+1}\right)\frac{1}{\nu!}(n+1)^\nu x^\nu\right].$$

Inserting the Taylor series for $e^x$ and carrying out the multiplication by Cauchy's rule, we obtain, for the expression in brackets,

$$\sum_{\nu=0}^{\infty}\left[\sum_{k=0}^{\nu} f\left(\frac{k}{n}\right)\frac{1}{k!}n^k\frac{1}{(n-k)!} - f\left(\frac{\nu}{n+1}\right)\frac{1}{\nu!}(n+1)^\nu\right]x^\nu.$$

It is enough then to establish that

$$f\left(\frac{\nu}{n+1}\right) \leqslant \frac{\nu!}{(n+1)^\nu}\sum_{k=0}^{\nu} f\left(\frac{k}{n}\right)\frac{1}{k!}n^k\frac{1}{(\nu-k)!}.$$

As in the preceding theorem this is a direct consequence of convexity; we omit the calculations. Equality can occur only if $f$ is linear in $[0, \nu/n]$, for all $\nu$, which implies that $f$ is linear in $[0, \infty)$.

LEMMA. *If* $|f''| \leqslant k$, $0 \leqslant \alpha$, $0 \leqslant \beta$, *and* $\alpha + \beta = 1$, *then*

$$|\alpha f(x) + \beta f(y) - f(\alpha x + \beta y)| \leqslant \tfrac{1}{4}k(x-y)^2.$$

THEOREM. *If* $|f''| \leqslant k$, *then the following estimates are valid:*

$$|M_n(f,x) - M_{n+1}(f,x)| \leqslant \tfrac{1}{4}k(1-x)n^{-2},$$
$$|M_n(f,x) - f(x)| \leqslant \tfrac{1}{3}k(1-x)n^{-1}.$$

*Proof.* Referring to the theorem on monotonicity of $M_n f$ when $f$ is convex, we have

$$M_n(f,x) - M_{n+1}(f,x)$$
$$= (1-x)^{n+1} \sum_{\nu=1}^{\infty} \binom{\nu+n+1}{\nu}[\alpha_1 f(x_1) + \alpha_2 f(x_2) - f(\alpha_1 x_1 + \alpha_2 x_2)]x^\nu.$$

Applying the lemma, and inserting the expressions for $x_1$ and $x_2$, we obtain

$$\begin{aligned}|M_n(f,x) - M_{n+1}(f,x)| &\leqslant (1-x)^{n+1} \sum_{\nu=1}^{\infty} \binom{\nu+n+1}{\nu} \frac{k}{4}(x_1-x_2)^2 x^\nu \\ &\leqslant (1-x)^{n+1} \frac{k}{4} \sum_{\nu=1}^{\infty} \frac{(\nu+n+1)x^\nu}{(\nu+n)(n^2+n)} \binom{\nu+n-1}{\nu} \\ &\leqslant \frac{k}{4n^2}(1-x).\end{aligned}$$

The second estimate is obtained from the first as follows:

$$\begin{aligned}|f(x) - M_n(f,x)| &\leqslant \sum_{\nu=n}^{\infty} |M_{\nu+1}(f,x) - M_\nu(f,x)| \\ &\leqslant \frac{k}{4}(1-x) \sum_{\nu=n}^{\infty} \frac{1}{\nu^2} \leqslant \frac{k}{3n}(1-x).\end{aligned}$$

**4. Convergence of the Szász operator for complex values.** In order to analyse the behaviour of $S_n(f,z)$ for complex $z$ we require the following result.

LEMMA. *If $f$ is a polynomial of degree $\leqslant m$, then so is $S_n(f,x)$ for all $n$.*

*Proof* (by induction). Suppose the theorem is true for all polynomials of degree $<m$. To prove it for polynomials of degree $m$ it suffices, because of the linearity of $S_n$, to prove it for $x^m$. We compute

$$\begin{aligned}S_n(t^m,x) &= e^{-nx} \sum_{\nu=0}^{\infty} \left(\frac{\nu}{n}\right)^m \frac{1}{\nu!}(nx)^\nu \\ &= e^{-nx} n^{1-m} x \sum_{\nu=0}^{\infty} (\nu+1)^{m-1} \frac{1}{\nu!}(nx)^\nu \\ &= x \sum_{k=0}^{m-1} \binom{m-1}{k} n^{1-m+k}\left[e^{-nx} \sum_{\nu=0}^{\infty} \left(\frac{\nu}{n}\right)^k \frac{1}{\nu!}(nx)^\nu\right].\end{aligned}$$

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Since the term in square brackets is $S_n(t^k, x)$ and $k < m$, the induction hypothesis may be applied to conclude that $S_n(t^m, x)$ is a polynomial of degree $m$. To complete the proof we observe that $S_n(1, x) \equiv 1$.

With the aid of the above theorem we can establish exactly as is done for Bernstein polynomials (**6**. p, 90) the analogue of Kantorovitch's theorem.

THEOREM. *Suppose that $f$ is analytic on the interior of an ellipse having foci* 0 *and* 1. *Then $S_n(f, z) \to f(z)$ uniformly on any closed set interior to the ellipse.*

**5. An application to differential equations.** The following theorem is analogous to one given by Arama (**1**) for Bernstein polynomials.

THEOREM. *The functions $y_n(x)$ defined recursively by*

$$y_0(x) = y_0, \qquad y_n(x) = y_0 + \int_0^x M_n\{f[t, y_{n-1}(t)], s\}ds,$$

*converge uniformly to a solution of the initial value problem*

$$y' = f(x, y), \qquad y(0) = y_0,$$

*for $x \in [0, 1)$, provided that $f$ and its first two partial derivatives are bounded in the strip $0 \leqslant x < 1$, $-\infty < y < \infty$ and that $f$ satisfies*

$$|f(x, y_1) - f(x, y_2)| \leqslant \lambda\, |y_1 - y_2|, \quad \textit{with } \lambda < 1.$$

*Proof.* We shall show that the series

$$y_0 + \sum_{n=0}^{\infty} [y_{n+1}(x) - y_n(x)]$$

is uniformly convergent in [0, 1). The general term of this series in absolute value is given by

$$\begin{aligned} |\epsilon_n(x)| &= |y_{n+1}(x) - y_n(x)| \\ &\leqslant \int_0^x |M_{n+1}\{f[t, y_n(t)], s\} - M_n\{f[t, y_{n-1}(t)], s\}|ds \qquad (23) \\ &\leqslant I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_0^x |M_{n+1}\{f[t, y_n(t)], s\} - M_n\{f[t, y_n(t)], s\}|ds$$

and

$$I_2 = \int_0^x |M_n\{f[t, y_n(t)], s\} - M_n\{f[t, y_{n-1}(t)], s\}|ds.$$

We may estimate the integral $I_1$ with the aid of a theorem in Section 3 as follows:

$$I_1 \leqslant \int_0^x \frac{k(1-s)}{4n^2}\,ds = \frac{k}{4n^2}\left(x - \frac{x^2}{2}\right) \leqslant \frac{k}{8n^2} \qquad (0 \leqslant x < 1),$$

where

$$k = \sup_{0 \leqslant x < 1} \left| \frac{d^2}{dx^2} f(x, y_n(x)) \right| .$$

We must verify next that our hypotheses are sufficient to guarantee that $k < \infty$. Let $K$ denote a common bound for the absolute value of $f$ and its first two partial derivatives in the infinite strip $0 \leqslant x < 1$. From the definition of $M_n$ it is clear that for any $g$

$$(24) \qquad |M_n(g, x)| \leqslant \sup_{0 \leqslant s < 1} |g(s)| \qquad (0 \leqslant x < 1).$$

Thus from the definition of $y_n(x)$, we have

$$|y_n'(x)| = |M_n\{f[t, y_{n-1}(t)], x\}| \leqslant K \qquad (0 \leqslant x < 1)$$

as well as

$$y_n''(x) = \frac{d}{dx} M_n\{f[t, y_{n-1}(t)], x\} = \frac{d}{dx}\left\{(1-x)^{n+1} \sum_{\nu=0}^{\infty} F\left(\frac{\nu}{\nu+n}\right)\binom{\nu+n}{\nu} x^\nu\right\}$$
$$= (1-x)^n \sum_{\nu=0}^{\infty} (\nu+n+1)\binom{\nu+n}{\nu}\left[F\left(\frac{\nu+1}{\nu+n}\right) - F\left(\frac{\nu}{\nu+n}\right)\right] x^\nu,$$

where we have written $F(x) = f(x, y_{n-1}(x))$. Thus, using the mean-value theorem,

$$\left| F\left(\frac{\nu+1}{\nu+n+1}\right) - F\left(\frac{\nu}{\nu+n}\right)\right| = |F'(\xi)| \frac{n}{(\nu+n)(\nu+n+1)} .$$

Since $F'(x) = f_1[x, y_{n-1}(x)] + f_2[x, y_{n-1}(x)]y'_{n-1}(x)$, we have

$$|F'(\xi)| \leqslant K + K^2.$$

Consequently

$$|y_n''(x)| \leqslant (K+K^2)(1-x)^n \sum_{\nu=0}^{\infty} (\nu+n+1)\binom{\nu+n}{\nu} \frac{n}{(n+\nu)(\nu+n+1)} x^\nu$$
$$\leqslant K + K^2.$$

Since

$$\frac{d^2}{dx^2} f[x, y_n(x)] = f_{n} + (2f_{12} + f_{22}y_n')y_n' + f_2 y_n'',$$
$$k \leqslant K + (2K + K^2)K + K(K+K^2).$$

We have, therefore, established that $I_1 = O(n^{-2})$, uniformly for $0 \leqslant x < 1$. Turning to $I_2$, we have

$$I_2 \leqslant \int_0^x |M_n\{f[t, y_n(t)] - f[t, y_{n-1}(t)], s\}| ds.$$

Now $|f[t, y_n(t)] - f[t, y_{n-1}(t)]| \leqslant \lambda\, |y_n(t) - y_{n-1}(t)| = \lambda\, |\epsilon_{n-1}(t)|$. Thus, by (24),

$$I_2 \leqslant \lambda \sup_{0 \leqslant t < 1} |\epsilon_{n-1}(t)|.$$

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It is easy to see that

$$\sup_{0 \leqslant x < 1} |\epsilon_n(x)| < 2(|y_0| + K).$$

Indeed,

$$|y_n(x)| \leqslant |y_0| + \int_0^x |M_n\{f[t, y_{n-1}(t)], s\}| ds \leqslant |y_0| + K$$

by (24), and then the inequality for $\epsilon_n$ follows from (23).

Returning to (23) we have

$$|\epsilon_n(x)| \leqslant \lambda \sup_{0 \leqslant t < 1} |\epsilon_{n-1}(t)| + O(n^{-2}),$$

which is enough to show that $y_n(x)$ converges uniformly in [0, 1) to a function $y(x)$, which is of course continuous on [0, 1).

To show that $y(x)$ is a solution of the initial value problem, we first observe that the series $y_0(x) + \sum[y_{n+1}(x) - y_n(x)]$ may be differentiated term by term because the derived series converges uniformly. Indeed,

$$\begin{aligned}|y'_{n+1}(x) - y_n'(x)| \leqslant |M_{n+1}\{f[t, y_n(t)], x\} &- M_n\{f[t, y_n(t)], x\}| \\ &+ |M_n\{f[t, y_n(t)], x\} - M_n\{f[t, y_{n-1}(t)], x\}|\end{aligned}$$

and this is not greater than

$$\frac{k(1-x)}{4n^2} + \lambda \sup_{0 \leqslant t < 1} |\epsilon_{n-1}(t)|$$

as we have already seen. Hence from

$$y(x) = \lim_n y_n(x)$$

we may conclude that

$$y'(x) = \lim_n y_n'(x) = \lim_n M_n\{f[t, y_{n-1}(t)], x\}.$$

Finally, we have

$$\begin{aligned}|M_n\{f[t, y_n(t)], x\} - f[x, y(x)]| \leqslant |M_n\{f[t, y_n(t)], x\} &- M_n\{f[t, y(t)], x\}| \\ &+ |M_n\{f[t, y(t)], x\} - f[x, y(x)]|.\end{aligned}$$

The second term clearly converges to zero, and the first, by (24), does not exceed

$$\sup_{0 \leqslant t < 1} |f[t, y_n(t)] - f[t, y(t)]| \leqslant \lambda \sup_{0 \leqslant t < 1} |y_n(t) - y(t)|.$$

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