MAXIMAL TOPOLOGIES

ASIT BARAN RAHA

(Received 16 June 1971)

Communicated by J. B. Miller

Abstract

This article is devoted to studying maximal π spaces where π = Lindelöf, countably compact, connected, lightly compact or pseudocompact. Necessary and sufficient conditions for Lindelöf or countably compact spaces to be maximal Lindelöf or maximal countably compact have been obtained. On the other hand only necessary conditions for maximal π spaces have been deduced where π = connected, lightly compact or pseudocompact.

Introduction

A topological space $(X, \mathscr{T})$ with property π is said to be maximal π if there is no strictly larger (or stronger) π topology on X. Vaidyanathaswamy [9] showed that every compact Hausdorff space is maximal compact. Maximal compact spaces have been characterised in an excellent paper of Smythe and Wilkins [6]. A non-Hausdorff maximal compact space has also been exhibited in the same paper. Necessary and sufficient conditions for an absolutely closed (or H-closed) space to become maximal absolutely closed have been obtained by Mioduszewski and Rudolf [4]. It has been demonstrated that every absolutely closed space admits a maximal absolutely closed topology finer than the given topology. Recently Thomas [8] has studied maximal connected spaces and has raised the question whether there exist maximal connected Hausdorff spaces.

In this article we shall investigate maximal π-spaces where π = Lindelöf, countably compact, pseudocompact, lightly compact or connected. As a matter of fact, we shall obtain only necessary conditions for maximal π-spaces where π = connected, lightly compact or pseudocompact while maximal Lindelöf and maximal countably compact spaces will be completely characterised. Our terminology will be after Bourbaki [1], the only difference is that our topological spaces need not always be Hausdorff. The term 'space' stands for a topological space.

1. Lindelöf spaces

DEFINITION. A Lindelöf space $(X, \mathscr{T})$ is *maximal Lindelöf* if any *Lindelöf topology on X stronger than $\mathscr{T}$, necessarily, equals $\mathscr{T}$.*

Before we start studying maximal Lindelöf spaces let us first recall that any one-one continuous map from a compact space onto a Hausdorff space (i.e., a continuous bijection from a compact space to a T_2 space) is a homeomorphism. Here, the Hausdorff property of the range space is merely a sufficient condition and a close look into the proof of the above fact will reveal that the closedness of every compact subset of the range space is precisely what is needed. The fact has been observed by several authors. But the characterisation of maximal compact spaces, due to Smythe and Wilkins, provides an interesting connection between maximal compact spaces and those topological spaces onto which any continuous bijection of a compact space is a homeomorphism as follows:

THEOREM 1. *The following are equivalent*:

(a) *X is maximal compact.*

(b) *The set of all closed subsets of X = the set of all compact subsets of X.*

(c) *Any continuous bijection f from a compact space Y onto X is a homeomorphism.*

The equivalence of (a) and (b) is due to Smythe and Wilkins. The purpose of the above paragraph is to motivate our results on maximal Lindelöf spaces. We prove

THEOREM 2. *The following are equivalent*:

(i) *$(X,\mathscr{T})$ is maximal Lindelöf.*

(ii) *The set of all closed subsets of X coincides with the set of all Lindelöf subspaces of X.*

(iii) *If Y is a Lindelöf space and f is any continuous bijection from Y onto X, then f is a homeomorphism.*

PROOF. (i) $\Rightarrow$ (ii): Suppose there exists a Lindelöf subspace A of $(X,\mathscr{T})$ which is not closed. Obviously, $A^c(= X-A)\notin\mathscr{T}$. Let $\mathscr{T}'$ be the topology generated by $\mathscr{T}\cup\{A^c\}$. Then

$$\mathscr{T}' = \{(A^c\cap U)\cup V: U, V\in\mathscr{T}\}$$

and is strictly stronger than $\mathscr{T}$. We shall now show that $(X,\mathscr{T}')$ is Lindelöf. Let $\{W_i: i\in I\}$ be an open cover of $(X,\mathscr{T}')$. Let

$$W_i = (A^c\cap U_i)\cup V_i.$$

Obviously $\cup\{V_i: i\in I\}\supset A$ and A is Lindelöf. So there exists a countable subset I_1 of I such that

$$\cup\{V_i: i\in I_1\}\supset A.$$

Put $V = \cup\{V_i: i\in I_1\}$. Then $V\in\mathscr{T}$. Consequently V^c is closed in $\mathscr{T}$ and is, therefore, Lindelöf. Again $V^c\subset A^c$. Consider

https://doi.org/10.1017/S1446788700013197 Published online by Cambridge University Press

$$W_i \cap V^c = (A^c \cap U_i \cap V^c) \cup (V^c \cap V_i) = V^c \cap (U_i \cup V_i).$$

Thus $\mathscr{T}' | V^c = \mathscr{T} | V^c$. Inasmuch as V^c is Lindelöf when $\mathscr{T}$ is relativised to it, there exists a countable subset $I_2 \subset I$ such that

$$\cup \{W_i : i \in I_2\} \supset V^c$$

and thence $\cup \{W_i : i \in I_1 \cup I_2\} \supset V^c \cup V = X$. But $I_1 \cup I_2$ is a countable subset of I. Hence, $(X, \mathscr{T}')$ is Lindelöf, a contradiction to the fact that $(X, \mathscr{T})$ is maximal Lindelöf. Thus (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Since f is a continuous bijection onto X, the inverse f^{-1} is well-defined from X onto Y. We need only to show f^{-1} is continuous. It suffices to show that for each closed subset F of Y, $(f^{-1})^{-1}(F) = f(F)$ is closed in X. Now F is closed in $Y \Rightarrow F$ is a Lindelöf subset of $Y \Rightarrow f(F)$ is a Lindelöf subset of X i.e. $f(F)$ is a closed subset of X.

(iii) $\Rightarrow$ (i): If $\mathscr{T}'$ is any Lindelöf topology on X such that $\mathscr{T}$ is contained in $\mathscr{T}'$, the identity map $i: (X, \mathscr{T}') \to (X, \mathscr{T})$ satisfies the conditions of (iii). So $\mathscr{T}' = \mathscr{T}$, i.e., $(X, \mathscr{T})$ is maximal Lindelöf.

It is easy to see from theorems 1 and 2 that both maximal compact and maximal Lindelöf spaces are T_1 and, a fortiori, T_0. If X is a countable set, the maximal Lindelöf space is nothing but the discrete space. If we restrict our attention to Lindelöf spaces we shall presently obtain another characterisation for such spaces to be maximal.

Let us first note that in any space an open set is a G-delta. Let us rather consider those topological spaces where every G-delta set is open. Such spaces are called P-spaces by Gillman and Jerison [3]. Every discrete space is trivially a P-space, but Hausdorff non-discrete topological spaces can be constructed in which, at every point x, the intersection of any countable family of neighbourhoods of x is again a neighbourhood of x (cf. Dieudonné [2]; an easy example is an uncountable set where each point is isolated but for one whose neighbourhoods are complements of countable subsets).

The relationship between maximal Lindelöf T_2 spaces and Lindelöf Hausdorff P-spaces is brought out through the following theorem.

THEOREM 3. *The following are equivalent*
(i) *X is maximal Lindelöf and T_2.*
(ii) *X is a Lindelöf Hausdorff P-space.*

PROOF. (i) $\Rightarrow$ (ii). By theorem 2 we know that a subset of X is Lindelöf if and only if it is closed. Let $G = \bigcap_{n=1}^{\infty} G_n$ be a G-delta subset of X where each G_n is open in X. Now

$$X - G = \bigcup_{n=1}^{\infty} (X - G_n)$$

and $G_n^c = X - G_n$ is closed and so Lindelöf for each n. So $X - G$, being a countable union of Lindelöf subspaces, is Lindelöf and hence closed. Thus, G is open in X. (We have, in fact, proved that any maximal Lindelöf space is a P-space).

(ii) $\Rightarrow$ (i): Since X is Lindelöf, every closed subset of X is Lindelöf. We need to show only that every Lindelöf subspace is closed, then we are done in virtue of theorem 2. Suppose, A is a Lindelöf subspace of X. Let $x \in \bar{A}$, the closure of A in X. It suffices to prove that $A = \bar{A}$. Suppose $x \notin A$. Let $\mathscr{N}(x)$ denote the filter base of open neighbourhoods of x. Since $x \in \bar{A}$,

$$\mathscr{F} = \{V \cap A: V \in \mathscr{N}(x)\}$$

is a filter base of open subsets of A. If $\{V_n \cap A\}$ is a countable collection from $\mathscr{F}$

$$\bigcap_{n=1}^{\infty} (V_n \cap A) \neq \varnothing$$

because

$$\bigcap_{n=1}^{\infty} (V_n \cap A) = (\cap V_n) \cap A = V \cap A \neq \varnothing$$

as $V \in \mathscr{N}(x)$ by (ii). Since $X$is T_2,

$$\{x\} = \cap \{\bar{V}: V \in \mathscr{N}(x)\} \text{ and } x \notin A \Rightarrow X - \{x\} \supset A$$

i.e., $\cup \{(\bar{V})^c: V \in \mathscr{N}(x)\} \supset A$. But A is Lindelöf; therefore there exist $V_n \in \mathscr{N}(x)$, $n \geqq 1$ such that

$$\bigcup_{n=1}^{\infty} (\bar{V}_n)^c \supset A$$

i.e.

$$\bigcap_{n=1}^{\infty} V_n \subset \bigcap_{n=1}^{\infty} \bar{V}_n \subset A^c.$$

But $\bigcap_{n=1}^{\infty} V_n \in \mathscr{N}(x)$ so $A \cap (\bigcap_{n=1}^{\infty} V_n) \neq \varnothing$, a contradiction. Thus A is closed.

A consequence of theorem 3 is that every Hausdorff maximal Lindelöf space is regular and, hence paracompact.

2. Countably compact spaces

DEFINITION. A topological space is said to be countably compact if every countable open covering of X contains a finite open covering of X. A countably compact space $(X, \mathscr{T})$ is called *maximal countably compact* if X cannot support any strictly stronger countably compact topology.

Before we mention the theorem characterising maximal countably compact space let us list some important properties of countably compact spaces without proof.

2(a). *Every closed subspace of a countably compact space is countably compact.*

2(b). *Let f be a continuous mapping of a countably compact space X into a topological space Y. Then f(X) is a countably compact subset of Y.*

2(c). *If X is Hausdorff and is first countable, then every countably compact subspace of X is closed in X.*

Theorem 4, below, is also motivated by Theorem 1. The proof is, however, omitted as it is similar to that of Theorem 2.

THEOREM 4. *For a topological space* $(X,\mathscr{T})$ *the following are equivalent:*

(i) $(X,\mathscr{T})$ *is maximal countably compact.*

(ii) *The set of all closed subsets of* X = *the set of all countably compact subspaces of* X.

(iii) *Any continuous bijection f from a countably compact space Y onto X is a homeomorphism.*

COROLLARY 5. *Any first countable, countably compact Hausdorff space is maximal countably compact.*

PROOF. Suppose X satisfies the conditions of the theorem. By property 2(a) all closed subsets of X are countably compact. Since X is first countable and T_2, property 2(c) yields that every countably compact subset is closed. Thus a subset of X is closed when and only when it is countably compact. Theorem 4 then helps us to conclude that X is maximal countably compact.

DEFINITION. A topological space $(X,\mathscr{T})$ is called *minimal first countable Hausdorff* if (i) $(X,\mathscr{T})$ is T_2 and first countable and (ii) $\mathscr{T}$ does not contain any strictly smaller first countable Hausdorff topology.

COROLLARY 6. *Any first countable, countably compact Hausdorff space is minimal first countable Hausdorff.*

PROOF. Suppose $(X,\mathscr{T})$ is a first countable, countably compact T_2 space. Suppose $\mathscr{T}'$ is a first countable Hausdorff topology strictly smaller than $\mathscr{T}$. Then $\mathscr{T}'$ satisfies all the conditions of Corollary 5 and is, thus, maximal countably compact. But by Corollary 5, $\mathscr{T}$ is also maximal countably compact, a contradiction as $\mathscr{T}'$ is strictly smaller than $\mathscr{T}$.

Corollary 6 easily yields the following known result regarding metric spaces.

THEOREM 7. *A countably compact metric space is compact.*

PROOF. If $(X,\mathscr{T})$ is a countably compact metric space, it is minimal first countable Hausdorff. Naturally $\mathscr{T}$ cannot contain any strictly weaker metric topology. So $(X,\mathscr{T})$ is a minimal metric space and Scarborough and Stephenson, Jr. [5] have shown that a minimal metric space is compact.

REMARK. Combination of Corollaries 5 and 6 derives that a first countable, countably compact Hausdorff space is maximal countably compact and minimal first countable Hausdorff. This is in the spirit of the exact analogue of the following well-known topological fact: A compact T_2 space is maximal compact and minimal Hausdorff.

Theorem 4, incidentally shows that every maximal countably compact space is T_1 and is, hence, T_0.

3. Connected Spaces

DEFINITION. A connected space $(X,\mathscr{T})$ is called *maximal connected* if any connected topology stronger than $\mathscr{T}$ necessarily coincides with $\mathscr{T}$.

As mentioned earlier, we shall present only a necessary condition for a connected space to be maximal connected. But this calls for the following prerequisites.

In a topological space $(Y,\mathscr{S})$, $V \in \mathscr{S}$ is called regular-open if $V = (\bar{V})^0$ where for any subset A of Y, $\bar{A}$ and A^0 denote respectively the closure and interior of A in the topology $\mathscr{S}$. Now, a topological space is called semi-regular if its regular-open sets form a base for the given topology. Given any space $(Y,\mathscr{S})$ the regular-open sets in $\mathscr{S}$ form a base for a unique semi-regular topology, called the semi-regular topology on Y associated with $\mathscr{S}$. We shall make the convention to denote the semi-regular topology associated with an arbitrary topology $\mathscr{S}$ by $\mathscr{S}_0$. Trivially, a topology $\mathscr{S}$ is semi-regular if and only if $\mathscr{S} = \mathscr{S}_0$.

Let $(Y,\mathscr{S}_0)$ be a semi-regular space. Set $E(\mathscr{S}_0) = \{\mathscr{T} : \mathscr{T}$ a topology on Y and $\mathscr{T}_0 = \mathscr{S}_0\}$. It has been demonstrated in Bourbaki [1] that $E(\mathscr{S}_0)$ has a maximal element with respect to the relation "$\mathscr{T}$ is weaker than $\mathscr{T}'$". A maximal element of $E(\mathscr{S}_0)$ is called a *submaximal topology* and Y endowed with such a topology is referred to as a *submaximal space*. Now, our necessary condition for a connected space to be maximal connected runs as follows:

THEOREM 8. *Suppose $(X,\mathscr{T})$ is a maximal connected space. Then every dense subset of X is open in $\mathscr{T}$.*

The proof will be accomplished with the aid of the following string of lemmas.

LEMMA 9. *A space $(X,\mathscr{T})$ is connected if and only if $(X,\mathscr{T}_0)$ is connected.*

PROOF. By definition $\mathscr{T}_0$ is coarser than $\mathscr{T}$, thus connectedness of $\mathscr{T}$ will force $\mathscr{T}_0$ to be connected. Conversely, if $\mathscr{T}$ is not connected, there would exist non-void disjoint open sets G and V in $\mathscr{T}$ such that $G \cup V = X$. Now, G and V

are both open and closed subsets in $\mathscr{T}$, so G and V are regular-open i.e., they are in $\mathscr{T}_0$. Therefore $\mathscr{T}_0$ is not a connected topology.

LEMMA 10. *A maximal connected space* $(X,\mathscr{T})$ *is submaximal.*

PROOF. $\mathscr{T}$ is a member of $E(\mathscr{T}_0)$. Lemma 9 together with maximal connectedness of $\mathscr{T}$ implies that $\mathscr{T}$ is submaximal.

The following result characterises submaximal topologies and can be found in Bourbaki [1].

LEMMA 11. *A topology* $\mathscr{T}$ *on* X *is submaximal if and only if every subset of* X *which is dense in the topology* $\mathscr{T}$ *is open in* $\mathscr{T}$.

The proof of theorem 8 immediately follows from lemmas 10 and 11. An easy consequence of lemma 11 is that every submaximal topology is T_0. As a result, every maximal connected space is T_0. Theorem 8 states that a maximal connected space is necessarily submaximal. But submaximality of a connected space is not a sufficient condition for its being maximal connected. We shall substantiate this by means of the following example.

EXAMPLE 1. $X = \{1,2,3,4\}$

$$\mathscr{T} = \{X, \varnothing, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$$

$(X,\mathscr{T})$ is easily seen to be connected and submaximal (dense sets are sets containing both 1 and 2 and they are all open in $\mathscr{T}$). Let us look at the topology

$$\mathscr{T}_1 = \{X, \varnothing, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,4\}\}$$

$\mathscr{T}_1$ is connected and strictly bigger than $\mathscr{T}$. So $\mathscr{T}$ is not maximal connected.

REMARKS. It is still an open question whether there exist maximal connected Hausdorff spaces. Nevertheless, such spaces, if at all they exist, cannot be locally compact, since a locally compact connected T_2 space cannot even be submaximal (Bourbaki [1]).

4. Lightly compact spaces

DEFINITION. A space X is said to be *lightly compact* if every locally finite family of non-void open sets in X is necessarily finite. A lightly compact space $(X,\mathscr{T})$ is called *maximal lightly compact* provided X does not admit any strictly stronger lightly compact topology.

To start with we shall present 2 characterisations of a lightly compact space.

PROPOSITION 12. (Stephenson [7], page 439). *On a space* X *the following are equivalent.*

https://doi.org/10.1017/S1446788700013197 Published online by Cambridge University Press

(a) *X is lightly compact.*

(b) *If $\mathscr{U}$ is a countable open cover of X, then there exists a finite sub-collection of $\mathscr{U}$ whose closures cover X.*

(c) *Every countable open filter base on X has an adherent point.*

LEMMA 13. *A space $(X,\mathscr{T})$ is lightly compact if and only if $(X,\mathscr{T}_0)$ is lightly compact, where $\mathscr{T}_0$ is the semi-regular topological structure associated with $\mathscr{T}$.*

PROOF. Since $\mathscr{T}_0 \subset \mathscr{T}$, from the definition it follows that $(X,\mathscr{T}_0)$ is lightly compact whenever $(X,\mathscr{T})$ is lightly compact. Suppose, conversely, $(X,\mathscr{T}_0)$ is lightly compact. Let $\mathscr{U}$ be a countable cover of X consisting of sets from $\mathscr{T}$. If $\mathscr{U} = \{V_n : n \geqq 1\}$ put

$$G_n = (\bar{V}_n)^0 = \mathscr{T}\text{-interior of } \mathscr{T}\text{-closure of } V_n.$$

Then $G_n \in \mathscr{T}_0$, and $V_n \subset G_n$, so that $\{G_n : n \geqq 1\}$ is an open cover of $(X,\mathscr{T}_0)$ and hence admits a finite subfamily $\{G_{n_i} : 1 \leqq i \leqq k\}$ such that

$$\cup \{\mathscr{T}_0\text{-closure of } G_{n_i} : i = 1,2,\cdots,k\} = X.$$

Obviously, now, $G_{n_i} \in \mathscr{T}$ for $i = 1,2,\cdots,k$ and since $U \in \mathscr{T} \Rightarrow$

$$\mathscr{T}_0\text{-closure of } U = \mathscr{T}\text{-closure of } U$$

we have $\bigcup_{i=1}^{k} \bar{G}_{n_i} = X$ i.e., $\bigcup_{i=1}^{k} (\overline{\bar{V}_{n_i}})^0 = X$ i.e., $\bigcup_{i=1}^{k} \bar{V}_{n_i} = X$. This shows that $(X,\mathscr{T})$ is slightly compact.

Now results analogous to Lemmas 10 and 11 lead to the following theorem.

THEOREM 14. *Suppose $(X,\mathscr{T})$ is a maximal lightly compact space. Then every dense subset of X is open in $\mathscr{T}$.*

Next we shall show that theorem 14 provides us only a necessary condition. That mere submaximality does not guarantee maximality of a lightly compact space is brought out by the following example.

EXAMPLE 2. $X =$ Any infinite set.

x_0 is a fixed point in X.

$$\mathscr{T} = \{V \subset X : x_0 \in V\} \cup \{\varnothing\}.$$

Then $(X,\mathscr{T})$ is a submaximal topological space. We shall first show that $(X,\mathscr{T})$ is lightly compact. If $\{G_n\}$ is any countable open cover of X, then $x_0 \in G_n$ for each n. So $\bar{G}_n = X$ for each n so that by proposition 12(b) X is lightly compact.

Let us now fix $x_1 \in X$ such that $x_1 \neq x_0$. Look at the topology $\mathscr{T}'$ on X determined by the following base of open sets:

$$x \notin \{x_0, x_1\} \Rightarrow \{x, x_0\} \text{ is open}$$

https://doi.org/10.1017/S1446788700013197 Published online by Cambridge University Press

$$\{x_0\} \text{ and } \{x_1\} \text{ are isolated.}$$

$\mathscr{T}'$ is strictly stronger than $\mathscr{T}$. We shall show that $\mathscr{T}'$ is lightly compact. Let $\{G_n\}$ be a countable open cover of X. There exist n_1 and n_2 such that $x_0 \in G_{n_1}$ and $x_1 \in G_{n_2}$ and obviously $\mathscr{T}'$-closure of $(G_{n_1} \cap G_{n_2}) = X$. Then $(X, \mathscr{T})$ is submaximal and lightly compact but not maximal lightly compact.

5. Pseudocompact spaces

Definition. A pseudocompact space $(X, \mathscr{T})$ is said to be *maximal pseudocompact* if $\mathscr{T}' = \mathscr{T}$ whenever $\mathscr{T}'$ is pseudocompact and is stronger than $\mathscr{T}$. We first remark that every lightly compact space is pseudocompact. Analogous to theorems 8 and 14 we prove the following.

Theorem 15. *Every maximal pseudocompact space is submaximal.*

In order to prove above theorem the first step is the following lemma:

Lemma 16. *A space $(X, \mathscr{T})$ is pseudocompact if and only if $(X, \mathscr{T}_0)$ is pseudocompact.*

Proof. The only nontrivial part is to show the 'if' part of the assertion. Denote by $C(X; \mathscr{T}_0)$ the space of all continuous real-valued function on $(X, \mathscr{T}_0)$ and by $C(X; \mathscr{T})$ the space of real-valued continuous functions on $(X, \mathscr{T})$. We shall show that $C(X; \mathscr{T}) = C(X; \mathscr{T}_0)$. We only need to prove that $C(X; \mathscr{T}) \subset C(X; \mathscr{T}_0)$. Let $f \in C(X; \mathscr{T})$, and U be any open subset of the real line. Then $f^{-1}(U) \in \mathscr{T}$. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. By using regularity of the real line we have $V \in \mathscr{T}$ such that $x \in V \subset \bar{V} \subset f^{-1}(U)$ and this implies $x \in V \subset (\bar{V})^0 \subset f^{-1}(U)$. But $(\bar{V})^0 \in \mathscr{T}_0$. So $f^{-1}(U) \in \mathscr{T}_0$. f thus belongs to $C(X; \mathscr{T}_0)$.

Proof of theorem 15. As usual let $E(\mathscr{T}_0) = \{\mathscr{S} : \mathscr{S}$ a topology on X with $\mathscr{S}_0 = \mathscr{T}_0\}$. By maximality of $(X, \mathscr{T})$ and lemma 16 and since $\mathscr{T} \in E(\mathscr{T}_0)$ we conclude that $\mathscr{T}$ is submaximal.

Since lightly compact $\Rightarrow$ pseudocompact, Example 2 offers even an example that a submaximal pseudocompact space need not be maximal pseudocompact.

Next we shall give an example of a compact Hausdorff space which acts as an omnibus example for a maximal compact, maximal countably compact, maximal lightly compact or a maximal pseudo-compact space.

Example 3. Let N stand for the natural numbers with discrete topology.

Denote by $N_1 = N \cup \{\omega\}$, the one-point compactification of N. N_1 is compact T_2 and so is maximal compact. Everything will be shown in one stroke if we prove that N_1 is maximal pseudocompact (N_1 is obviously pseudocompact). Suppose not. Then there must exist a strictly bigger topology on N_1 which is still pseudo-

https://doi.org/10.1017/S1446788700013197 Published online by Cambridge University Press

compact. In that topology there should be an open neighbourhood of ω, say V, such that $N_1 - V$ is infinite. $N_1 - V$ is contained in N and is, of course, open in the latter topology. Define the function $f\colon N_1 \to R$ as follows:

$$f(n) = n \text{ if } n \in N_1 - V$$
$$= -1 \text{ if } n \in V$$

f is a continuous real-valued function on N_1, no doubt, but f is not bounded. There lies the contradiction.

6. Products and subspaces

It has been already mentioned that Hausdorff maximal compact spaces are nothing except the compact Hausdorff spaces. So Hausdorff maximal compact spaces are indeed closed under product (in fact, arbitrary product). But if X is a non-Hausdorff maximal compact space, we intend to demonstrate that $X \times X$ is not going to be maximal compact under the product topology. Let $D = \{(x,x)\colon x \in X\}$ denote the diagonal of $X \times X$. D is easily seen to be compact. Since X is not T_2, D cannot be closed. The existence of a non-closed compact set, viz. D, shows that $X \times X$ is not maximal compact.

If we turn our attention to maximal absolutely closed (or H-closed) spaces, the product of a maximal absolutely closed space with itself is going to be absolutely closed (since H-closed spaces are closed under products) but the product may fail to be maximal absolutely closed. We shall substabtiate this claim by means of an example.

It was established in [4] that an absolutely closed space is maximal absolutely closed if and only if it is submaximal. Let N_1 be same as in Example 3. N_1 is compact T_2 and hence H-closed. Since N and N_1 are the only two dense subsets and both are open, N_1 is submaximal (Lemma 11)· It follows from the first line of this paragraph that N_1 is maximal absolutely closed. Consider, now, $N_1 \times N_1$ with product topology. $N_1 \times N_1$ is, obviously, absolutely closed. We shall prove that $N_1 \times N_1$ is not submaximal and that will prove our claim. Consider the subset $A = N \times N \cup \{(\omega,\omega)\}$. Since $N \times N$ is already dense in $N_1 \times N_1$, A is dense in $N_1 \times N_1$. But A does not contain any open neighbourhood of (ω,ω) and so fails to be open in $N_1 \times N_1$. Consequently $N_1 \times N_1$ is not maximal absolutely closed.

Thomas [8] has constructed an example of a maximal connected space X such that $X \times X$ with the product topology is not maximal connected. The study of the product of maximal π spaces where $\pi =$ Lindelöf, countably compact, lightly compact or pseudocompact is, of course, rendered uninteresting by the mere fact that these topological properties are not, in general, productive. Still we shall deal with some very special cases.

Suppose we start with a non-Hausdorff maximal Lindelöf space X such that $X \times X$ is Lindelöf when endowed with the product topoloy. Still $X \times X$ cannot be maximal Lindelöf due to the fact that the diagonal of $X \times X$ is not closed but Lindelöf. However, in the Hausdorff case we can prove the following assertion.

THEOREM 17. *$X \times X$ is a maximal Lindelöf space provided $X \times X$ is Lindelöf and X is maximal Lindelöf as well as T_2.*

PROOF. By invoking theorem 3 we can assert that X is a P-space. By hypothesis $X \times X$ is a Lindelöf Hausdorff space. If we can show that $X \times X$ is a P-space theorem 3 will imply that it is maximal Lindelöf. Let $A \subset X \times X$ be any G-delta i.e. $A = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open in $X \times X$. Let $(x, y) \in G_n$ for each n. There exist U_n and V_n open in X such that

$$(x, y) \in U_n \times V_n \subset G_n$$

for each n, so

$$(x, y) \in (\cap U_n) \times (\cap V_n) \subset A.$$

By hypothesis $\cap U_n$ and $\cap V_n$ are open subsets of X. So A is open in $X \times X$.

A trivial consequence of Corollary 5 is that every first countable compact T_2 space is maximal countably compact. Hence if we start with a first countable compact Hausdorff space X, the product space $X \times X$ is also a space of the same type and, a fortiori, maximal countably compact.

Next we shall try to determine which subspaces of maximal π spaces are maximal π where π = compact, Lindelöf, countably compact, connected, lightly compact or pseudocompact. For π = lightly compact or pseudocompact no satisfactory answers could be obtained. Thomas [8] has proved that connected open subsets of a maximal connected space are maximal connected. But all maximal connected subspaces need not, however, be open (Examples can be easily constructed). In rest of the cases we really possess characterisations of maximal π subspaces. When π = compact, Lindelöf or countably compact, *closed subspaces* of maximal π spaces are the only *maximal π subspaces*. This follows easily from theorems 1, 2 and 4.

7. Concluding remarks

We are able to obtain only necessary conditions for a π-space to be maximal π when π = connected, lightly compact or pseudo-compact. But sufficient conditions are lacking. Our theorems reveal this fact that a definite relationship between π subspaces and closed subspaces has played the key-role in characterising the maximal π-spaces in case π = compact, Lindelöf or countably compact. Absence of such a relationship, in the former cases, turns out to be a real snag. For instance, we cannot in general assert that closed subspaces of a pseudocom-

https://doi.org/10.1017/S1446788700013197 Published online by Cambridge University Press

pact space are pseudocompact. As a matter of fact, a non-countably compact pseudocompact T_1 space must have a proper closed subspace that is not pseudocompact, the reason being that a T_1 topological space is countably compact if and only if every closed subspace is pseudocompact. The topological properties considered in this article are compactness, Lindelöfness, countably compactness, connectedness, lightly compactness and pseudocompactness. Another interesting feature to be noted is that the first three properties for a space are not usually determined by the associated semi-regular space (i.e., for π = compact, Lindelöf or countably compact it is not, in general, true that a space has property π if the associated semi-regular space has property π) while the last three properties are always so determined (Lemmas 9, 13 and 16). And, strangely, in the first three cases we have got complete characterisations and we failed to get any in the latter three cases. As topological properties connectedness and lightly compactness are different, no doubt, but they have at least one interesting property in common (as mentioned earlier): both are determined by the associated semi-regular topologies. And, it is this very property that forces two seemingly different kinds of spaces like maximal connected spaces and maximal lightly compact spaces to satisfy the condition: every dense set is open (i.e., the condition of submaximality by lemma 11). We conclude with the observation that all the maximal π spaces considered in this article are T_0.

8. Acknowledgements

The author thanks Dr. Ashok Maitra for suggesting some modifications of an earlier version of this manuscript.

References

[1] N. Bourbaki, *General Topology* (*Part* 1) (Addison-Wesley Publishing Company, Reading, Massachussetts, 1966).

[2] J. Dieudonné, 'Notes de Teratopologie (I)', *Revue scientifique*, 1939, p. 39.

[3] Leonard Gillman and M. Jerisen, *Rings of continuous functions*, (D. Van Nostrand and Co., Princeton, New Jersey, (1960)).

[4] J. Mioduszewski and L. Rudolf, '*H*-closed and extremally disconnected Hausdorff spaces', *Dissertations Mathematicase* LXVI (1969).

[5] C. T. Scarborough and R. M. Stephenson, Jr., 'Minimal topologies,' *Colloqium. Math.* 19 (1968), 215–219.

[6] N. Smythe and and C. A. Wilkins, 'Minimal Hausdorff and maximal compact spaces', *J. Austral. Math. Soc.* 3 (1963), 167–171.

[7] R. M. Stephenson, Jr., 'Pseudocompact spaces', *Trans. Amer. Math. Soc.* 134 (1968), 437–448.

[8] J. P. Thomas, 'Maximal connected topologies,' *Jour. Austral. Math. Soc.* 8 (1968), 700–705.

[9] R. Vaidynathaswamy, *Treatise on set topology* (Ind. Math. Soc. Madras (1947)).

Indian Statistical Institute
203, Barrackpore Trunk Rd
Calcutta — 35 India