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Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian

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We study the existence of positive solutions for the nonlinear Schrödinger equation with the fractional Laplacian

$$\begin{gathered}(-\Delta)^\alpha u + u = f(x,u) \text{ in } \mathbb{R}^N,\\ u > 0 \text{ in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0.\end{gathered}$$

Furthermore, we analyse the regularity, decay and symmetry properties of these solutions.

1. Introduction

We study nonlinear Schrödinger equations with fractional diffusion. More precisely, we are concerned with solutions to the following problem:

$$\left.\begin{gathered}(-\Delta)^\alpha u + u = f(x,u) \text{ in } \mathbb{R}^N,\\ u > 0 \text{ in } \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0,\end{gathered}\right\} \tag{1.1}$$

where $0 < \alpha < 1$, $N \geqslant 2$ and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is superlinear and has subcritical growth with respect to u. Here, the fractional Laplacian can be characterized as $\mathcal{F}((-\Delta)^\alpha \phi)(\xi) = |\xi|^{2\alpha}\mathcal{F}(\phi)(\xi)$, where $\mathcal{F}$ is the Fourier transform.

Equation (1.1) arises in the study of the fractional Schrödinger equation

$$\left.\begin{aligned}\mathrm{i}\partial_t \Psi + (-\Delta)^\alpha \Psi = F(x,\Psi) \quad & \text{in } \mathbb{R}^N,\\ \lim_{|x|\to\infty} |\Psi(x,t)| = 0 \quad & \text{for all } t > 0,\end{aligned}\right\} \tag{1.2}$$

when looking for standing waves, that is, solutions with the form $\Psi(x,t) = \mathrm{e}^{-\mathrm{i}ct}u(x)$, where c is a constant. This equation is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes.

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A path integral over the Lévy flights paths and a fractional Schrödinger equation of fractional quantum mechanics are formulated by Laskin [24] from the idea of Feynman and Hibbs's path integrals (see also [25]).

The Lévy processes occur widely in physics, chemistry and biology. The stable Lévy processes that give rise to equations with the fractional Laplacians have recently attracted much research interest. Nonlinear boundary-value problems in various settings, including phase transition and free boundary, have recently been studied by Cabré and Solà Morales [8], Cabré and Roquejoffre [6], Silvestre [32], Capella *et al*. [11], Cabré and Sire [7], Cabré and Tan [9], Sire and Valdinoci [33], Frank and Lenzmann [20] and Brändle *et al*. [5].

A one-dimensional version of (1.1) has been studied in the context of solitary waves by Weinstein [37], Bona and Li [4] and de Bouard and Saut [15]. More recently, Frank and Lenzmann [20] studied the uniqueness of the positive solution to (1.1) in the one-dimensional autonomous case, and the existence and symmetry of solutions is sketched in terms of previous works (see also the work by Kenig *et al*. [23] and Maris [28] on the Benjamin–Ono equation when $\alpha = \frac{1}{2}$ and $N \geqslant 2$).

When $\alpha = 1$ we have the classical nonlinear Schrödinger equation

$$-\Delta u + u = f(x, u) \quad \text{in } \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N),$$

which has been extensively studied in the last 20 years by many authors. We mention here the earlier work by Floer and Weinstein [19], Rabinowitz [31], Wang [36] and del Pino and Felmer [16] without attempting to review these references here.

Our goal is to study the existence, regularity and qualitative properties of ground states of (1.1) in the case where $0 < \alpha < 1$. Before continuing, we make precise definitions of the notion of solutions for the equation

$$(-\Delta)^\alpha u + u = g \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Definition 1.1. Given $g \in L^2(\mathbb{R}^N)$, we say that $u \in H^\alpha(\mathbb{R}^N)$ is a weak solution of (1.3) if

$$\int_{\mathbb{R}^N} |\xi|^{2\alpha} \hat{u}\hat{v} \, \mathrm{d}\xi = \int_{\mathbb{R}^N} gv \, \mathrm{d}x \quad \text{for all } v \in H^\alpha(\mathbb{R}^N).$$

Here, $\hat{}$ denotes the Fourier transform and $H^\alpha(\mathbb{R}^N)$ denotes the fractional Sobolev space (see § 2).

When u has sufficient regularity, it is possible to have a pointwise expression of the fractional Laplacian as follows:

$$(-\Delta)^\alpha u = -\int_{\mathbb{R}^N} \frac{\delta(u)(x, y)}{|y|^{N+2\alpha}} \, \mathrm{d}y, \tag{1.4}$$

where $\delta(u)(x, y) = u(x + y) + u(x - y) - 2u(x)$ (see, for example, [35]).

Definition 1.2. Given $g \in C(\mathbb{R}^N)$, we say that a function $u \in C(\mathbb{R}^N)$ is a classical solution of (1.3) if $(-\Delta)^\alpha u$ can be written as (1.4) and equation (1.3) is satisfied pointwise in all $\mathbb{R}^N$.

Now we state our main assumptions. In order to find solutions of (1.1), we will assume the following general hypotheses.

(f_0) $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is such that $\xi \to f(x,\xi)$ is continuous for almost every (a.e.) $x \in \mathbb{R}^N$, and $x \to f(x,\xi)$ is Lebesgue measurable for all $\xi \in \mathbb{R}$.

(f_1) $f(x,\xi) \geqslant 0$ if $\xi \geqslant 0$ and $f(x,\xi) \equiv 0$ if $\xi \leqslant 0$, for a.e. $x \in \mathbb{R}^N$.

(f_2) The function

$$\xi \to \frac{f(x,\xi)}{\xi} \quad \text{is increasing for } \xi > 0 \text{ and a.e. } x \in \mathbb{R}^N.$$

(f_3) $\lim_{\xi \to 0} f(x,\xi)/\xi = 0$ uniformly in x.

(f_4) There exists $\theta > 2$ such that, for all $\xi > 0$ and a.e. $x \in \mathbb{R}^N$,

$$0 < \theta F(x,\xi) \leqslant \xi f(x,\xi),$$

where $F(x,\xi) = \int_0^\xi f(x,\tau)\,\mathrm{d}\tau$.

(f_5) There exists $p > 1$ such that $p < (N+2\alpha)/(N-2\alpha)$, so that

$$f(x,\xi) \leqslant C(1+|\xi|)^p \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N.$$

(f_6) The function $f(x,u)$ is Hölder continuous in both variables.

At this point we state our existence theorem for the autonomous equation, that is, when the nonlinearity f does not depend on x. This theorem will serve as a basis for the proof of the main existence theorem for the case where f depends on x.

Theorem 1.3. *Assume that $0 < \alpha < 1$, $N \geqslant 2$ and that $\bar{f}\colon \mathbb{R} \to \mathbb{R}$ is a function. Then we have the following.*

(i) *If $\bar{f}$ satisfies* (f_1)–(f_5)*, then*

$$(-\Delta)^\alpha u + u = \bar{f}(u) \quad \text{in } \mathbb{R}^N$$

has a weak solution, which satisfies $u \geqslant 0$ almost everywhere in $\mathbb{R}^N$.

(ii) *If we further assume that $\bar{f}$ satisfies* (f_6)*, then u is a classical solution that satisfies $u > 0$ in $\mathbb{R}^N$.*

The simplest case of a function $\bar{f}$ satisfying the hypotheses (f_1)–(f_6) is $\bar{f}(s) = s_+^p$, where p is as in (f_5) and $s_+ = \max\{s, 0\}$. Naturally, the class of functions satisfying these hypotheses is much ampler than this homogeneous case.

In the x-dependent case, we have to consider the behaviour of the nonlinearity for large values of x in order to obtain proper compactness conditions. In the simplest model case, we may consider the x-dependent nonlinearity $f(x,s) = b(x)s_+^p$, where $b(x) \geqslant 1$. If this inequality is strict somewhere and $\lim_{|x| \to \infty} b(x) = 1$, then we will prove that a solution of (1.1) exists. However, we could consider a more general class of x-dependent nonlinearities. We consider the following hypothesis.

(f_7) There exist continuous functions $\bar{f}$ and a, defined in $\mathbb{R}$ and $\mathbb{R}^N$, respectively, such that $\bar{f}$ satisfies (f_1)–(f_5) and

$$0 \leqslant f(x,\xi) - \bar{f}(\xi) \leqslant a(x)(|\xi| + |\xi|^p) \quad \text{for all } \xi \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N,$$
$$\lim_{|x|\to\infty} a(x) = 0$$

and

$$|\{x \in \mathbb{R}^N \mid f(x,\xi) > \bar{f}(\xi) \text{ for all } \xi > 0\}| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure.

Now we state our main existence theorem.

THEOREM 1.4. *Assume that $0 < \alpha < 1$, $N \geqslant 2$. Then we have the following.*

(i) *If f satisfies* (f_1)–(f_5) *and* (f_7), *then equation* (1.1) *possesses at least one weak solution, which satisfies $u \geqslant 0$ almost everywhere in $\mathbb{R}^N$.*

(ii) *If we further assume that f satisfies* (f_6), *then equation* (1.1) *possesses at least one classical solution that satisfies $u > 0$ in $\mathbb{R}^N$.*

The nonlinear problem (1.1) involves the fractional Laplacian $(-\Delta)^\alpha$, $0 < \alpha < 1$, which is a non-local operator. A common approach for dealing with this problem was proposed in [10] (see also [32]), allowing (1.1) to be transformed into a local problem via the Dirichlet–Neumann map. For $u \in H^\alpha(\mathbb{R}^N)$, we consider the problem

$$\left.\begin{aligned} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 \quad &\text{in } \mathbb{R}^{N+1}_+, \\ v(x,0) = u \quad &\text{on } \mathbb{R}^N, \end{aligned}\right\} \tag{1.5}$$

from where the fractional Laplacian is obtained as

$$(-\Delta)^\alpha u(x) = -b_\alpha \lim_{y\to 0^+} y^{1-2\alpha} v_y,$$

where b_α is an appropriate constant.

However, in this paper we prefer to analyse the problem directly in $H^\alpha(\mathbb{R}^N)$. This allows us to prove the existence of a weak solution of (1.1), resembling the case where $\alpha = 1$ in some ways. This approach could extend many other problems, known for $\alpha = 1$, to the general case $\alpha \in (0,1)$.

The proof of theorem 1.3 is done in several steps. First, we prove the existence of weak solutions of (1.1) by applying the mountain-pass theorem [2] to the functional I defined on $H^\alpha(\mathbb{R}^N)$ as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\xi|^{2\alpha}|\hat{u}|^2 + |\hat{u}|^2)\,\mathrm{d}\xi - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \tag{1.6}$$

where $\hat{u}$ denotes the Fourier transform of u. However, the direct application of the mountain-pass theorem is not sufficient, since the Palais–Smale sequences might lose compactness in the whole space $\mathbb{R}^N$. To overcome this difficulty, we use a comparison argument devised in [31] for $\alpha = 1$, based on the energy functional

$$\bar{I}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\xi|^{2\alpha}|\hat{u}|^2 + |\hat{u}|^2)\,\mathrm{d}\xi - \int_{\mathbb{R}^N} \bar{F}(u)\,\mathrm{d}x. \tag{1.7}$$

The non-negativity of weak solutions is proved by a version of the weak maximum principle suitable for our setting.

The next step is to prove regularity of weak solutions. Here we use the usual iteration technique, based on L^p theory for the Laplacian, together with a localization trick, inspired by ideas in [32]. We believe that the argument may be useful for other problems, as an alternative to regularity theory for degenerate elliptic equations [18] that has been used in previous works. Finally, we prove the positivity of classical solutions by direct use of the integral representation of the fractional Laplacian (1.4).

Our approach takes advantage of the representation formula

$$u = \mathcal{K} * f = \int_{\mathbb{R}^n} \mathcal{K}(x-\xi) f(\xi)\, \mathrm{d}\xi$$

for solutions to the equation

$$(-\Delta)^{\alpha} u + u = f \quad \text{in } \mathbb{R}^N,$$

where $\mathcal{K}$ is the Bessel kernel

$$\mathcal{K}(x) = \mathcal{F}^{-1}\biggl(\frac{1}{1+|\xi|^{2\alpha}}\biggr). \tag{1.8}$$

Rather than knowing one reference for all the basic properties of the Bessel kernel, we instead know various different sources. Based on [1, 3, 30, 34], we sketch the analysis in the appendix for the reader's convenience. We emphasize that many properties that we need in what follows are obtained using the kernel, such as the Rellich–Kondrachov theorem for $H^{\alpha}(\mathbb{R}^N)$ and the basic properties for the fractional L^p Sobolev spaces. We have attempted to be self-contained.

Our second main theorem concerns the decay of classical solutions for (1.1). We find suitable comparison functions based on the Bessel kernel $\mathcal{K}$ to find out that solutions of (1.1) have a power-type decay at infinity. More precisely, we have the following.

Theorem 1.5. *Assume that* $0 < \alpha < 1$, $N \geqslant 2$, f *satisfies* (f_3) *and that* u *is a positive classical solution of* (1.1). *Then there exist constants* $0 < C_1 \leqslant C_2$ *such that*

$$C_1|x|^{-(N+2\alpha)} \leqslant u(x) \leqslant C_2|x|^{-(N+2\alpha)} \quad \textit{for all } |x| \geqslant 1.$$

We see that the solution is bounded below by the power $N+2\alpha$, in great contrast with the case where $\alpha = 1$. As mentioned above, we construct a suitable subsolution and supersolution and we use a simple comparison argument to obtain the decay inequalities.

Our third goal is to prove that, when f does not depend on x, the positive solutions are radial. We have the following theorem.

Theorem 1.6. *Assume that* $0 < \alpha < 1$, $N \geqslant 2$, f *does not depend on* x *and that it satisfies* (f_1)–(f_5). *Moreover, assume that*

(f_9) $f \in C^1(\mathbb{R})$, *increasing, and there exists* $\tau > 0$ *such that*

$$\lim_{\xi \to 0^+} \frac{f'(\xi)}{\xi^{\tau}} = 0.$$

Then all positive solutions of (1.1) *are radially symmetric.*

We give a proof based on the moving planes method as developed recently in [12, 13, 26]. Their ideas provide an integral approach that is suitable for equations involving the fractional Laplacian, where the radial symmetry and monotonicity properties of the kernel $\mathcal{K}$ plays a key role. The approach here is different from the usual moving planes technique originated in [22] for the case where $\alpha = 1$.

The rest of the paper is organized as follows. In § 2 we prove the existence of a weak solution for equation (1.1) and obtain part (i) of theorem 1.4. In § 3 we study the regularity of weak solutions and we complete the proof of theorem 1.4. In § 4 we find appropriate supersolutions and subsolutions and prove theorem 1.5. In § 5 we apply the moving planes technique to prove that the solutions of the autonomous problem are radially symmetric. Finally, in the appendix we sketch some properties of the kernel and the function spaces that are required for our approach. Although these results are known, we provide the sketched proofs for the reader's convenience.

2. The ground state

We seek for solutions of equations (1.1) using a variational approach, based on the mountain-pass theorem. We consider the Sobolev space

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \Big/ \int_{\mathbb{R}^N} (|\xi|^{2\alpha}|\hat{u}|^2 + |\hat{u}|^2)\,\mathrm{d}\xi < \infty \right\},$$

whose norm is defined as

$$\|u\|_\alpha^2 = \int_{\mathbb{R}^N} (|\xi|^{2\alpha}|\hat{u}|^2 + |\hat{u}|^2)\,\mathrm{d}\xi.$$

On the space $H^\alpha(\mathbb{R}^N)$, we consider the functional I defined in (1.6) whose critical points correspond to the weak solutions of (1.1).

The two basic properties of the Sobolev space $H^\alpha(\mathbb{R}^N)$ that we need are summarized in the following lemma.

Lemma 2.1. *Let $2 \leqslant q \leqslant 2^\star_\alpha \equiv 2N/(N-2\alpha)$. Then we have*

$$\|u\|_{L^q(\mathbb{R}^N)} \leqslant C\|u\|_\alpha, \quad \forall u \in H^\alpha(\mathbb{R}^N). \tag{2.1}$$

If, furthermore, $2 \leqslant q < 2^\star_\alpha$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then every bounded sequence $\{u_k\} \subset H^\alpha(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.

Proof. Property (2.1) is the classical Sobolev embedding, which is a particular case of part (i) of theorem 3.2. The second part is also well known, but we do not know of a reference of a proof that does not require interpolation machinery. We provide a sketch of the proof for the reader's convenience, using standard arguments as in [17].

The idea is to apply the Arzelà–Ascoli theorem to the mollified sequence $u_k^\varepsilon = \eta_\varepsilon * u_k$. Here $\eta_\epsilon(x) = \varepsilon^{-N}u(x/\varepsilon)$, where η is non-negative with support in the ball $B_1(0)$ and with $\int_{B_1(0)} \eta(x)\,\mathrm{d}x = 1$. If we look at the proof of the Rellich–Kondrachov theorem in [17], we observe that the main point is to prove that $u_k^\varepsilon \to u_k$ in $L^q(\Omega)$ as $\varepsilon \to 0$, uniformly in k. To do this we consider the sequence $f_k \in L^2(\mathbb{R}^N)$ defined as

$$f_k = (-\Delta)^{\alpha/2}u_k + u_k,$$

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which is bounded. Thus, $u_k = \mathcal{K}_{\alpha/2} * f_k$, where $\frac{1}{2}\mathcal{K}\alpha$ is the Bessel kernel, and whose properties are summarized in theorem 3.3. Then we have

$$\begin{aligned}\|u_k^{\varepsilon} - u_k\|_{L^2} &= \|\eta_\varepsilon * u_k - u_k\|_{L^2} \\ &= \|(\eta_\varepsilon * \mathcal{K}_{\alpha/2} - \mathcal{K}_{\alpha/2}) * f_k\|_{L^2}.\end{aligned}$$

Since $\mathcal{K} \in L^1(\mathbb{R}^N)$, we have $\|\eta_\varepsilon * \mathcal{K}_{\alpha/2} - \mathcal{K}_{\alpha/2}\|_{L^1} \to 0$ as $\varepsilon \to 0$, and then $\|u_k^{\varepsilon} - u_k\|_{L^2} \to 0$ as $\varepsilon \to 0$.

Finally, using the Hölder inequality and (2.1), we obtain

$$\|u_k^{\varepsilon} - u_k\|_{L^q} \leqslant 2^{\lambda} \|u_k^{\varepsilon} - u_k\|_{L^2}^{1-\lambda} \|u_k\|_{H^\alpha}^{\lambda},$$

with

$$\tfrac{1}{2}(1-\lambda) + \frac{\lambda}{2_\alpha^\star} = \frac{1}{q},$$

from which the desired convergence follows. □

The following lemma is a version of the concentration compactness principle proved in [14].

Lemma 2.2. *Let $N \geqslant 2$. Assume that $\{u_k\}$ is bounded in $H^\alpha(\mathbb{R}^N)$, and that it satisfies*

$$\lim_{k\to\infty} \sup_{\xi\in\mathbb{R}^N} \int_{B_R(\xi)} |u_k(x)|^2 \,\mathrm{d}x = 0,$$

where $R > 0$. Then $u_k \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2_\alpha^\star$.

Proof. Let $2 < q < 2_\alpha^\star$. Given $R > 0$ and $\xi \in \mathbb{R}^N$, by using the Hölder inequality, we obtain, for every k, that

$$\|u_k\|_{L^q(B_R(\xi))} \leqslant \|u_k\|_{L^2(B_R(\xi))}^{1-\lambda} \|u_k\|_{L^{2_\alpha^\star}(B_R(\xi))}^{\lambda},$$

where

$$\tfrac{1}{2}(1-\lambda) + \frac{\lambda}{2_\alpha^\star} = \frac{1}{q}.$$

Now, covering $\mathbb{R}^N$ with balls of radius R in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we deduce that

$$\int_{\mathbb{R}^N} |u_k|^q \,\mathrm{d}x \leqslant (N+1) \|u_k\|_{L^2(B_R(\xi))}^{(1-\lambda)q} \|u_k\|_{L^{2_\alpha^\star}(\mathbb{R}^N)}^{\lambda q}.$$

Using lemma 2.1 and the assumption, we have $u_k \to 0$ in $L^q(\mathbb{R}^N)$. □

Using the properties of the Nemytskii operator and the embedding given in lemma 2.1, it can be proved that the functional I is of class C^1. In the search for critical values, it is convenient to consider the Nehari manifold

$$\varLambda = \{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} / I'(u)u = 0\}.$$

We observe that if u in $\varLambda$, then $u_+ \neq 0$. Thanks to assumptions (f_1)–(f_4), given $u \in H^\alpha(\mathbb{R}^N)$ with $u_+ \neq 0$, the function $t \in \mathbb{R}_+ \to I(tu)$ has a unique maximum $t(u) > 0$ and $t(u)u \in \varLambda$. We define

$$c^* = \inf_{u\in\varLambda} I(u) \tag{2.2}$$

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and we observe that

$$c^* = \inf_{u \in H^\alpha(\mathbb{R}^N)\setminus\{0\}} \sup_{\theta \geqslant 0} I(\theta u). \tag{2.3}$$

On the other hand, we consider the set of functions

$$\Gamma = \{g \in C([0,1],\ H^\alpha(\mathbb{R}^N))/g(0) = 0,\ I(g(1)) < 0\}$$

and define

$$c = \inf_{g \in \Gamma} \sup_{t \in [0,1]} I(g(t)). \tag{2.4}$$

Under our assumptions, Γ is certainly not empty, and $c > 0$. The following lemma is crucial and it uses (f_4).

Lemma 2.3. $c = c^*$.

Proof. Given any $u \in \Lambda$, we may define a path g_u as $g_u(t) = tTu$, where $I(Tu) < 0$ and obtain that $g_u \in \Gamma$. Thus, $c \leqslant c^*$.

The other inequality follows from the fact that, for any $g \in \Gamma$, there exists $t \in (0,1)$ such that $g(t) \in \Lambda$. To prove this fact, we see that if $I'(u)u \geqslant 0$, then, by (f_4),

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\int_{\mathbb{R}^N} f(x,u)u\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\geqslant (\tfrac{1}{2}\theta - 1)\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\geqslant 0. \end{aligned}$$

Thus, if we assume that $I'(g(t))g(t) > 0$ for all $t \in (0,1]$, then $I(g(t)) \geqslant 0$ for all $t \in (0,1]$, contradicting $I(g(1)) < 0$. $\square$

We define $\bar{\Lambda}$, $\bar{\Gamma}$ and $\bar{c}$, replacing f by $\bar{f}$. The following theorem gives the existence of a solution in part (i) in theorem 1.3, and it is a crucial step to prove theorem 1.4.

Theorem 2.4. *$\bar{I}$ has at least one critical point with critical value $\bar{c}$.*

Proof. By the Ekeland variational principle (see [29]), there is a sequence u_n such that

$$\bar{I}(u_n) \to \bar{c} \quad \text{and} \quad \bar{I}'(u_n) \to 0.$$

Using (f_4), it is standard to check that, given $\varepsilon > 0$, for sufficiently large n,

$$\bar{c} + \varepsilon + \|u_n\|_\alpha \geqslant \bar{I}(u_n) - \frac{1}{\theta}I'(u_n)u_n \geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_\alpha^2,$$

so that (u_n) is a bounded sequence. Then, using lemma 2.1, there is a subsequence of (u_n) converging weakly in $H^\alpha(\mathbb{R}^N)$ and $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ to $u \in H^\alpha(\mathbb{R}^N)$. Thus, for such a subsequence and any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n \to \infty} \bar{I}'(u_n)\varphi = \bar{I}'(u)\varphi = 0.$$

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If we show that $u \neq 0$, then $\bar{I}'(u) = 0$, and then $\bar{I}(u) \geqslant \bar{c}$. On the other hand, using (f_4) again, we see that, for every $R > 0$,

$$\bar{I}(u_n) - \tfrac{1}{2}\bar{I}'(u_n)u_n = \int_{\mathbb{R}^N} (\tfrac{1}{2}\bar{f}(u_n)u_n - \bar{F}(u_n))\,\mathrm{d}x$$
$$\geqslant \int_{B_R} (\tfrac{1}{2}\bar{f}(u_n)u_n - \bar{F}(u_n))\,\mathrm{d}x.$$

Taking the limit as $n \to \infty$, we obtain

$$\bar{c} \geqslant \int_{B_R} (\tfrac{1}{2}\bar{f}(u)u - \bar{F}(u))\,\mathrm{d}x.$$

Then, observing that the inequality holds when the integral is taken on $\mathbb{R}^N$, since it holds for all R, and as $I'(u) = 0$, it follows that $\bar{I}(u) \leqslant \bar{c}$.

In order to complete the proof, we just need to show that u is non-trivial. For this purpose, by lemma 2.2 it is possible to find a sequence $y_n \in \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that

$$\int_{B_R(y_n)} u_n^2\,\mathrm{d}x > \beta, \quad \forall n.$$

In fact, assuming the contrary, we have $u_n \to 0$ in $L^{p+1}(\mathbb{R}^N)$. But then, for large n and some constants $a > 0$ and $A > 0$, we have

$$\tfrac{1}{2}\bar{c} \leqslant \bar{I}(u_n) - \tfrac{1}{2}\bar{I}'(u_n)u_n = \int_{\mathbb{R}^N} (\tfrac{1}{2}\bar{f}(u_n)u_n - \bar{F}(u_n))\,\mathrm{d}x$$
$$\leqslant \int_{\mathbb{R}^N} a|u_n|^2 + A|u_n|^{p+1}\,\mathrm{d}x,$$

providing a contradiction, since $\bar{c} > 0$. Here we have used (f_3) and (f_5).

Now we define $\tilde{u}_n(x) = u_n(x + y_n)$, and we use the discussion given above to find that $u = \text{w-lim}\,\tilde{u}_n$, is a non-trivial critical point of $\bar{I}$. □

Now we prove the following.

THEOREM 2.5. *I has at least one critical point with critical value $c < \bar{c}$.*

Proof. There exists a sequence $u_n \in \Lambda$ such that

$$c = \lim_{n\to\infty} I(u_n).$$

If $g_n = g_{u_n}$, as defined in the proof of lemma 2.3, using the Ekeland variational principle we find sequences $t_n \in [0, 1]$ and $w_n \in H^\alpha(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} I(w_n) = c, \quad \lim_{n\to\infty} I'(w_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|w_n - g_n(t_n)\|_\alpha = 0. \tag{2.5}$$

Proceeding as in the proof of theorem 2.4, we find a subsequence of w_n (that we keep calling w_n), that converges weakly to w and, in order to show that $w \neq 0$, we find $R, \beta > 0$ and a sequence $y_n \in \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} w_n^2\,\mathrm{d}x > \beta, \quad \forall n. \tag{2.6}$$

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In y_n has a bounded subsequence, then (2.6) guarantees that $w \neq 0$ and the result follows. Let us assume, then, that y_n is unbounded. We may assume that, for given $R > 0$,

$$\lim_{n\to\infty} \int_{B_R(0)} |w_n|^2 \, \mathrm{d}x = 0, \tag{2.7}$$

since the contrary implies that $w \neq 0$ following the same arguments as above. In order to complete the proof, we first obtain that

$$c < \bar{c}. \tag{2.8}$$

To see this, we use the characterization of c and $\bar{c}$ as in lemma 2.3. Let $\bar{w}$ be a non-trivial critical point of $\bar{I}$ given by theorem 2.4 and let

$$A = \{x \in \mathbb{R}^N \mid f(x,\xi) > \bar{f}(\xi) \text{ for all } \xi > 0\}.$$

Then, by (f_7) and the fact that $\bar{w}$ is non-zero, there exists $y \in \mathbb{R}^N$ such that the function w_y, defined as $w_y(x) = w(x+y)$, satisfies

$$|\{x \in \mathbb{R}^N / |w_y(x)| > 0\} \cap A| > 0,$$

where $|\cdot|$ denotes the Lebesgue measure. But then

$$\bar{c} = \bar{I}(w_y) \geqslant \bar{I}(\theta w_y) > I(\theta w_y) \quad \text{for all } \theta > 0.$$

Choosing $\theta = \theta^* > 0$ such that $I(\theta^* w_y) = \sup_{\theta>0} I(\theta w_y)$, we find $\theta^* w_y \in \Lambda$ and we conclude that

$$\bar{c} > I(\theta^* w_y) \geqslant \inf_{w\in\Lambda} I(w) = c,$$

proving (2.8). Now we see that, for $\theta \geqslant 0$, from (f_7) we have

$$\begin{aligned} I(\theta u_n) &= \bar{I}(\theta u_n) + \int_{\mathbb{R}^N} (\bar{F}(\theta u_n) - F(x,\theta u_n)) \, \mathrm{d}x \\ &\geqslant \bar{I}(\theta u_n) - \int_{\mathbb{R}^N} Ca(x)(|\theta u_n|^2 + |\theta u_n(x)|)^{p+1} \, \mathrm{d}x. \end{aligned}$$

Let $\varepsilon > 0$. Then, using (2.5) and (f_7) again, there exists $R > 0$ such that

$$\int_{B_R(0)^c} Ca(x)(|\theta u_n|^2 + |\theta u_n(x)|)^{p+1} \, \mathrm{d}x \leqslant \varepsilon,$$

for θ bounded. Then, by (2.7) and (2.5),

$$\lim_{n\to\infty} \int_{B_R(0)} Ca(x)(|\theta u_n|^2 + |\theta u_n(x)|)^{p+1} \, \mathrm{d}x = 0.$$

Choosing $\theta = \bar{\theta}$ such that $\bar{I}(\bar{\theta} u_n) = \max_{\theta\geqslant 0} \bar{I}(\theta u_n)$, we see that $c \geqslant \bar{c} - \varepsilon$. If $\varepsilon > 0$ is chosen sufficiently small, this contradicts (2.8). □

Proof of part (i) of theorem 1.3 and part (i) of theorem 1.4. We only need to prove that the weak solution found in theorems 2.4 and 2.5 is non-negative. We use the same argument in both cases. First we observe that

$$\int_{\mathbb{R}^N} (-\Delta^\alpha u)\varphi \, \mathrm{d}x = C \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \tag{2.9}$$

for all $\varphi \in H^\alpha(\mathbb{R}^N)$, which follows from the identity

$$\int_{\mathbb{R}^N} (-\Delta^\alpha u)u \, \mathrm{d}x = \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}|^2 \, \mathrm{d}\xi$$
$$= C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y,$$

proved, for example, in [21, lemma 3.1]. Now we claim that

$$\|u_+\|_{H^\alpha(\mathbb{R}^N)} \leqslant \|u\|_{H^\alpha(\mathbb{R}^N)}, \tag{2.10}$$

where $u_+ = \max\{u, 0\}$. In fact, given $u \in H^1(\mathbb{R}^N)$, it is known that $u_+ \in H^1(\mathbb{R}^N)$. Hence,

$$\begin{aligned}\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_+(x) - u_+(y))^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y &= \int_{\{u>0\}\times\{u<0\}} \frac{u(x)^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \int_{\{u<0\}\times\{u>0\}} \frac{u(y)^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \iint_{\{u>0\}\times\{u>0\}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y \\ &\leqslant \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y.\end{aligned}$$

Therefore, by (2.9), we obtain (2.10) for all $u \in H^1(\mathbb{R}^N)$, and the claim follows by density. Using $u_- = u^+ - u$ as a test function, by the positivity of $f(x, u(x))$ we obtain

$$\int_{\mathbb{R}^N} (-\Delta^\alpha u)u_- \, \mathrm{d}x = \int_{\mathbb{R}^N} u_-^2,$$

completing the proof, since the left-hand side is non-positive. In fact, the function $(u(x)-u(y))(u_-(x)-u_-(y)) = (u(x)-u(y))(u_-(x))$ is negative on $\{u<0\}\times\{u>0\}$ and $(u(x) - u(y))(u_-(x) - u_-(y)) = (u(x) - u(y))(-u_-(y))$ is also negative on $\{u > 0\} \times \{u < 0\}$. Thus,

$$\int_{\mathbb{R}^N} (-\Delta^\alpha u)u_- = C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(u_-(x) - u_-(y))}{|x-y|^{N+2\alpha}} \, \mathrm{d}x \, \mathrm{d}y,$$

is non-positive. □

3. Regularity of weak solutions

In this section we prove that weak solutions of (1.1) are of class $C^{0,\mu}$, for certain $\mu \in (0, 1)$. Actually, we obtain estimates of the norm in $C^{0,\mu}(\mathbb{R}^N)$. These estimates will be the basis for the qualitative analysis we make in the next section, particularly to obtain positivity and asymptotic decay of the solutions.

We start the analysis by recalling the definition of the fractional Sobolev spaces for $p \geqslant 1$ and $\beta > 0$:

$$\mathcal{L}^{\beta,p} = \{u \in L^p(\mathbb{R}^N)/\mathcal{F}^{-1}[(1 + |\xi|^2)^{\beta/2}\hat{u}] \in L^p(\mathbb{R}^N)\}$$

and, associated to the fractional Laplacian, the space (see [34])

$$\mathcal{W}^{\beta,p} = \{u \in L^p(\mathbb{R}^N) / \mathcal{F}^{-1}[(1 + |\xi|^\beta)\hat{u}] \in L^p(\mathbb{R}^N)\}.$$

The following two theorems are basic results for these spaces that we use later. The first theorem is on the definition of these spaces and the role of $(-\Delta)^\alpha$ as an operator between them.

THEOREM 3.1. *Assuming that $p \geqslant 1$ and $\beta > 0$, the following hold.*

(i) *$\mathcal{L}^{\beta,p} = \mathcal{W}^{\beta,p}$ and $\mathcal{L}^{n,p} = W^{n,p}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, where $W^{n,p}(\mathbb{R}^N)$ is the usual Sobolev space.*

(ii) *For $\alpha \in (0,1)$ and $2\alpha < \beta$, we have $(-\Delta)^\alpha \colon \mathcal{W}^{\beta,p} \to \mathcal{W}^{\beta-2\alpha,p}$.*

(iii) *For $\alpha, \gamma \in (0,1)$ and $0 < \mu \leqslant \gamma - 2\alpha$, we have*

$$(-\Delta)^\alpha \colon C^{0,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N) \quad \text{if } 2\alpha < \gamma,$$

and, for $0 \leqslant \mu \leqslant 1 + \gamma - 2\alpha$,

$$(-\Delta)^\alpha \colon C^{1,\gamma}(\mathbb{R}^N) \to C^{0,\mu}(\mathbb{R}^N) \quad \text{if } 2\alpha > \gamma.$$

The second theorem is about embeddings.

THEOREM 3.2.

(i) *If $0 \leqslant s$, and either $1 < p \leqslant q \leqslant Np/(N - sp) < \infty$ or $p = 1$ and $1 \leqslant q < N/(N - s)$, then $\mathcal{L}^{s,p}$ is continuously embedded in $L^q(\mathbb{R}^N)$.*

(ii) *Assume that $0 \leqslant s \leqslant 2$ and $s > N/p$. If $s - N/p > 1$ and $0 < \mu \leqslant s - N/p - 1$, then $\mathcal{L}^{s,p}$ is continuously embedded in $C^{1,\mu}(\mathbb{R}^N)$. If $s - N/p < 1$ and $0 < \mu \leqslant s - N/p$, then $\mathcal{L}^{s,p}$ is continuously embedded in $C^{0,\mu}(\mathbb{R}^N)$.*

Next we recall the main properties of the kernel $\mathcal{K}$, which are useful in what follows and also for proving some parts of theorem 3.1.

THEOREM 3.3. *Let $N \geqslant 2$ and $\alpha \in (0,1)$. Then we have the following.*

(i) *$\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$. Moreover, it is nonincreasing as a function of $r = |x|$.*

(ii) *For appropriate constants C_1 and C_2,*

$$\mathcal{K}(x) \leqslant \frac{C_1}{|x|^{N+2\alpha}} \text{ if } |x| \geqslant 1 \quad \text{and} \quad \mathcal{K}(x) \leqslant \frac{C_2}{|x|^{N-2\alpha}} \text{ if } |x| \leqslant 1. \tag{3.1}$$

(iii) *There is a constant C such that*

$$|\nabla \mathcal{K}(x)| \leqslant \frac{C}{|x|^{N+1+2\alpha}}, \quad |D^2 \mathcal{K}(x)| \leqslant \frac{C}{|x|^{N+2+2\alpha}} \quad \text{if } |x| \geqslant 1. \tag{3.2}$$

(iv) *If $q \geqslant 1$ and $s \in (N - 2\alpha - N/q, N + 2\alpha - N/q)$, then $|x|^s \mathcal{K}(x) \in L^q(\mathbb{R}^N)$.*

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(v) *If* $q \in [1, N/(N-2\alpha))$, *then* $\mathcal{K} \in L^q(\mathbb{R}^N)$.

(vi) $|x|^{N+2\alpha}\mathcal{K}(x) \in L^\infty(\mathbb{R}^N)$.

For the reader's convenience, in the appendix we provide a sketch of the proof of the last three basic theorems.

Now we state our main regularity result. The proof of this theorem is based on the classical L^p theory for second-order elliptic equations together with a localization technique inspired on an idea in [32].

Theorem 3.4. *Suppose that* $u \in H^\alpha(\mathbb{R}^N)$ *is a weak solution of* (1.1) *and* f *satisfies conditions* (f_0)–(f_5). *Then* $u \in L^{q_0}(\mathbb{R}^N)$ *for some* $q_0 \in [2, \infty)$ *and* $u \in C^{0,\mu}(\mathbb{R}^N)$ *for some* $\mu \in (0,1)$. *Moreover,* $|u(x)| \to 0$ *as* $|x| \to \infty$.

Proof. We are given $u \in H^\alpha = \mathcal{W}^{\alpha,2}$, which satisfies (1.1) in the weak sense, then it satisfies

$$(-\Delta)^\alpha u + u = f(x,u) \tag{3.3}$$

in the sense of distributions. Let $1 = r_0 > r_1 > r_2 > \cdots$, and consider $B_i = B(0, r_i)$, the ball of radius r_i and centred at the origin. We define $h(x) = f(x, u(x))$ and $g(x) = -u(x) + f(x, u(x))$. Then, by ($f_3$) and ($f_5$), we have

$$|g| \leqslant C(|u| + |u|^p) \quad \text{and} \quad |h| \leqslant C(|u| + |u|^p). \tag{3.4}$$

Since $u \in H^\alpha$, by Sobolev embedding we have $u \in L^{q_0}(\mathbb{R}^N)$, where $q_0 = 2N/(N-2\alpha)$. Now we let $\eta_1 \in C^\infty$ with $0 \leqslant \eta_1 \leqslant 1$, with support in B_0 and such that $\eta_1 \equiv 1$ in $B_{1/2}$, where $B_{1/2} = B(0, r_{1/2})$ with $r_1 < r_{1/2} < r_0$. Let u_1 be the solution of the equation

$$(-\Delta)^\alpha u_1 + u_1 = \eta_1 h \qquad \text{in } \mathbb{R}^N, \tag{3.5}$$

then

$$(-\Delta)^\alpha (u - u_1) + (u - u_1) = (1-\eta_1)h \quad \text{in } \mathbb{R}^N, \tag{3.6}$$

so that

$$-u_1 = \mathcal{K} * \{(1-\eta_1)h\}. \tag{3.7}$$

Since $u \in L^{q_0}(\mathbb{R}^N)$, using the Hölder inequality and part (ii) of theorem 3.3, we have, for all $x \in B_1$,

$$|u(x) - u_1(x)| \leqslant C\{\|\mathcal{K}\|_{L^{s_0}(B^c_{1/2})}\|(1-\eta_1)u\|_{L^{q_0}} + \|\mathcal{K}\|_{L^{s_1}(B^c_{1/2})}\|(1-\eta_1)u\|^p_{L^{q_0}}\}, \tag{3.8}$$

where $s_0 = q_0/(q_0 - 1)$ and $s_1 = q_0/(q_0 - p)$. In view of this inequality, we have to concentrate our attention in u_1. We have that u_1 satisfies

$$(-\Delta)^\alpha u_1 = g_1,$$

where $g_1 = -u_1 + \eta_1 h$. Since $u \in L^{q_0}(\mathbb{R}^N)$, by (3.4), and since B_0 is bounded, we obtain that $\eta_1 h \in L^{p_1}(\mathbb{R}^N)$, where $p_1 = q_0/p$. Then, since u_1 satisfies (3.5), we have, by definition of the space $\mathcal{W}^{2\alpha,p_1}$, that $u_1 \in \mathcal{W}^{2\alpha,p_1}$. We note that $\|u_1\|_{\mathcal{W}^{2\alpha,p_1}}$ depends on N, α and $\|u\|_{H^\alpha}$. At this point we have three cases: $p_1 < N/(2\alpha)$, $p_1 = N/(2\alpha)$ and $p_1 > N/(2\alpha)$.

Let us assume that the first case holds. Then we use Sobolev embedding and (3.8) to see that $u \in L^{q_1}(B_1)$, where $q_1 = p_1 N/(N - 2\alpha p_1)$. Now we repeat the procedure, but considering a smooth function η_2 such that $0 \leqslant \eta_2 \leqslant 1$, with support in B_1 and $\eta_2 \equiv 1$ in $B_{3/2}$, where $B_{3/2} = B(0, r_{3/2})$ with $r_2 < r_{3/2} < r_1$. Proceeding as above, with the obvious changes we obtain that

$$u_2 = \mathcal{K} * (\eta_2 h)$$

satisfies $u_2 \in \mathcal{W}^{2\alpha,p_2}$, where $p_2 = q_1/p$. Again, at this point we have three cases: $p_2 < N/(2\alpha)$, $p_2 = N/(2\alpha)$ and $p_2 > N/(2\alpha)$.

While the first case holds, we have that $u \in L^{q_2}(B_2)$, where $q_2 = p_2 N/(N - 2\alpha p_2)$. Repeating this argument, we will define a sequence q_j such that

$$\frac{1}{q_{j+1}} = \sum_{i=1}^{j} p^i \left(\frac{1}{q_1} - \frac{1}{q_0} \right) + \frac{1}{q_1}.$$

We note that, since $1 < p < (N + 2\alpha)/(N - 2\alpha)$, $q_1 > q_0$, the right-hand side of the above equation becomes negative for large j. Let j be the smallest natural such that the sum is non-positive. Then $p_{j+1} = N/(2\alpha)$ or $p_{j+1} > N/(2\alpha)$.

In the case where $p_{j+1} > N/(2\alpha)$, we have that $u_{j+1} \in \mathcal{W}^{2\alpha,p_{j+1}}$ so that, by Sobolev embedding, we may choose

$$0 < \mu < \min\{2\alpha - N/p_{j+1}, 1\}$$

so that $u_{j+1} \in C^{0,\mu}(\mathbb{R}^N)$. From (3.7) and (3.8), for u_{j+1} and η_{j+1} instead of u_1 and η_1, we obtain the L^∞ estimate for $u - u_{j+1}$. Using the smoothness of $\mathcal{K}$ away from the origin, we have

$$\begin{aligned} |\nabla(u - u_{j+1})(x)| &\leqslant \int_{\mathbb{R}^N} |\nabla \mathcal{K}(x - y)| \, |(1 - \eta_{j+1}(y)) h(y)| \, \mathrm{d}y \\ &\leqslant C \int_{\mathbb{R}^N \setminus B_{j+1/2}} |\nabla \mathcal{K}(x - y)| \, |(1 - \eta_{j+1}(y))(|u(y)| + |u(y)|^p) \, \mathrm{d}y, \end{aligned} \tag{3.9}$$

for all $x \in B_{j+1}$. Here we observe that, for $x \in B_{j+1}$, $|x - y| \geqslant r_{j+1/2} - r_{j+1} > 0$ over the integral. Recalling that $u \in L^{q_0}(\mathbb{R}^N)$, using part (iii) of theorem 3.3 and the Hölder inequality we find that

$$|\nabla(u - u_{j+1})(x)| \leqslant C(N, \alpha, \|u\|_{H^\alpha}) \quad \text{for all } x \in B_{j+1}. \tag{3.10}$$

Thus, $u_{j+1} \in C^{0,\mu}(\mathbb{R}^N)$, and then $u \in C^{0,\mu}(B_{j+1})$. The $C^{0,\mu}$ norm of u in B_{j+1} depends only on N, α, $\|u\|_{H^\alpha}$ and the finite sequence $r_0, r_1, \dots, r_{j+1}$.

In the case where $p_{j+1} = N/(2\alpha)$, we consider the fact that $u_{j+1} \in \mathcal{W}^{2\tilde{\alpha},p_{j+1}}$, for $\tilde{\alpha} < \alpha$. Then we have $p_{j+1} < N/(2\tilde{\alpha})$ and we can make another iteration of the procedure. If $\tilde{\alpha}$ is sufficiently close to α, we obtain $p_{j+2} > N/(2\tilde{\alpha})$, and we complete the argument.

The ball $\bar{B} = B_{j+1}$ or $\bar{B} = B_{j+2}$ is centred at the origin, but we may arbitrarily move it around $\mathbb{R}^N$. Covering $\mathbb{R}^N$ with these balls, we obtain that $u \in C^{0,\mu}(\mathbb{R}^N)$. Finally, the fact that $u \in L^{q_0}(\mathbb{R}^N) \cap C^{0,\mu}(\mathbb{R}^N)$ implies that $u(x) \to 0$ as $x \to \infty$, completing the proof. □

Now we are able to complete the proof of theorem 1.4.

Proof of theorems 1.3 and 1.4 (continued). From theorem 3.4, we have that weak solutions of (1.1) are of the class $C^{0,\mu}$. Since we are assuming (f_6), we see that the function $h(x) = f(x, u(x))$ is in $C^{0,\sigma}$ for certain $\sigma > 0$. Let η_1 be a non-negative, smooth function with support in $B_1(0)$ such that $\eta_1 = 1$ in $B_{1/2}(0)$. Let $u_1 \in H^\alpha(\mathbb{R}^N)$ be the solution of (3.5). Then, as proved above, $u_1 \in L^q(\mathbb{R}^N)$ for all $q \geqslant 2$, then $u_1 \in W^{2\alpha,q}(\mathbb{R}^N)$ and thus $u_1 \in C^{0,\sigma_0}$ for some $\sigma_0 \in (0, \sigma)$.

Now we look at the equation

$$-\Delta w = -u_1 + \eta_1 h \in C^{0,\sigma_0}.$$

By Hölder regularity theory for the Laplacian, we find $w \in C^{2,\sigma_0}$, so that if $2\alpha+\sigma_0 > 1$, then $(-\Delta)^{1-\alpha} w \in C^{1,2\alpha+\sigma_0-1}$, while if $2\alpha+\sigma_0 \leqslant 1$, then $(-\Delta)^{1-\alpha} w \in C^{0,2\alpha+\sigma_0}$. Then, since

$$(-\Delta)^\alpha (u_1 - (-\Delta)^{1-\alpha} w) = 0,$$

the function $u_1 - (-\Delta)^{1-\alpha} w$ is harmonic, we find that u_1 has the same regularity as $(-\Delta)^{1-\alpha} w$. To conclude, we look at (3.9) for $2\alpha + \sigma_0 \leqslant 1$, and to a corresponding inequality for the second derivative in the case where $2\alpha+\sigma_0 > 1$. See property (iii) of the kernel $\mathcal{K}$ in theorem 3.3.

Thus, we conclude that $u \in C^{1,2\alpha+\sigma_0-1}$ if $2\alpha + \sigma_0 > 1$, while $u \in C^{0,2\alpha+\sigma_0}$ if $2\alpha+\sigma_0 \leqslant 1$. Note that these conclusions hold locally, but the corresponding Hölder norms depend only on η_1, α, N and $\|u\|_{H^\alpha}$, so these estimates are global in $\mathbb{R}^N$.

In any case, the regularity obtained above implies that representation (1.4) of $(-\Delta)^\alpha u$ holds, so that

$$\int_{\mathbb{R}^N} \frac{\delta(u)(x,y)}{|y|^{N+2\alpha}} \, \mathrm{d}y = u(x) - f(x, u(x)).$$

Assuming that u is non-trivial, and knowing that $u \geqslant 0$ in $\mathbb{R}^N$, we assume that there is a global minimum point $x_0 \in \mathbb{R}^N$. Then the right-hand side vanishes at x_0, while the left-hand side is positive there, providing a contradiction. □

4. Qualitative properties of positive solutions

In this section we study the asymptotic behaviour of positive solutions of (1.1) and we prove theorem 1.5. In the following lemma we prove a lower bound on the behaviour of $\mathcal{K}$ complementing part (ii) of theorem 3.3. We have the following.

Lemma 4.1. *There is a positive constant c such that*

$$\mathcal{K}(x) \geqslant \frac{c}{|x|^{N+2\alpha}} \quad \text{for all } |x| \geqslant 1.$$

Proof. The inequality is a direct consequence of (A 4). In fact, if $|x| \geqslant 1$, and using (A 4), we have

$$\mathcal{K}(x) \geqslant c_1 \int_0^{|x|^\alpha} \mathrm{e}^{-t} \frac{t}{|x|^{N+2\alpha}} \, \mathrm{d}t \geqslant \left(c_1 \int_0^1 \mathrm{e}^{-t} t \, \mathrm{d}t \right) \frac{1}{|x|^{N+2\alpha}} = \frac{c}{|x|^{N+2\alpha}},$$

for the appropriate c. □

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We remark that, from (A 4), we may also prove that

$$\lim_{|x|\to\infty} \mathcal{K}(x)|x|^{N+2\alpha} = C,$$

for a certain constant $C > 0$, but we cannot take advantage of this more precise property.

Now we see two lemmas from which we obtain our subsolution and supersolution. We start with the following.

Lemma 4.2. *There is a continuous functions w in R^N satisfying*

$$(-\Delta)^\alpha w(x) + w(x) = 0 \quad \textit{if } |x| > 1 \tag{4.1}$$

in the classical sense, and

$$w(x) \geqslant \frac{c_1}{|x|^{N+2\alpha}} \tag{4.2}$$

for an appropriate $c_1 > 0$.

Proof. We just consider the function $w = \mathcal{K} * \chi_{B_1}$, where χ_{B_1} is the characteristic function of the unit ball B_1. This function clearly satisfies the equation outside B_1 and the decaying estimate thanks to lemma 4.1. □

Similarly, we have the following.

Lemma 4.3. *There is a continuous functions w in R^N satisfying*

$$(-\Delta)^\alpha w(x) + \tfrac{1}{2} w(x) = 0 \quad \textit{if } |x| > 1 \tag{4.3}$$

in the classical sense, and

$$0 < w(x) \leqslant \frac{c_2}{|x|^{N+2\alpha}} \tag{4.4}$$

for an appropriate $c_2 > 0$.

Proof. In this case we consider the function $w = \mathcal{K} * \chi_{B_a}$, where B_a is the ball of radius $a = 2^{-1/(2\alpha)}$. Then, by scaling, $w_a(x) = w(ax)$ satisfies equation (4.3) and, using part (ii) of theorem 3.3, we obtain (4.4). □

Proof of theorem 1.5. We consider the function w given by lemma 4.2. By continuity of u and w, there exists $C_1 > 0$ such that $W(x) = u(x) - C_1 w(x) \geqslant 0$ in $\bar{B}_1$. Moreover, $W(x) \to 0$ as $|x| \to \infty$, and $(-\Delta)^\alpha W \geqslant -W$ in B_1^c. Then, assuming that $W(x) \ngeqslant 0$ in $B_{R_0}^c$ implies that there exists a global negative minimum point $x_0 \in B_{R_0}^c$, but this is impossible, since then $(-\Delta)^\alpha W(x_0) < 0$. This completes the proof of the first inequality.

Now we use again that $u(x) \to 0$ as $|x| \to \infty$ to find that u satisfies

$$(-\Delta)^\alpha u + \tfrac{1}{2} u \leqslant 0 \quad \text{in } B_{R_1}^c,$$

for some large $R_1 > 1$. Then we consider the function w found in lemma 4.3, which satisfies (4.3) in B_1^c and then in $B_{R_1}^c$. Using similar comparison arguments, we conclude the second inequality. □

5. Symmetry of positive solutions

We prove the radial symmetry of positive solutions found in theorem 1.4 by using the moving planes method recently developed in the context of integral operator in [12, 13, 26] (see also [27] for the integral equation involving the Bessel kernel related to $(-\Delta + \mathrm{id})^{\alpha}$, $0 < \alpha < 1$).

Let us consider planes parallel to $x_1 = 0$ and define

$$
\begin{aligned}
{}_{\lambda} &= \{x \in \mathbb{R}^N \mid x_1 > \lambda\}, & T_\lambda &= \{x \in \mathbb{R}^N \mid x_1 = \lambda\}, \\
x^\lambda &= (2\lambda - x_1, x_2, \dots, x_N), & u_\lambda(x) &= u(x^\lambda).
\end{aligned}
$$

Define also

$$\lambda_0 = \sup\{\lambda \mid u_\lambda(x) < u(x) \text{ for all } \Sigma_\lambda\}.$$

Proposition 5.1. *We have $u_{\lambda_0}(x) = u(x)$ for all $x \in \Sigma_{\lambda_0}$.*

For the proof, we will use the following auxiliary lemmas.

Lemma 5.2. *If $q > r > N/(2\alpha)$, then there exists a constant $C > 0$ such that*

$$\left\| \int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right\|_{L^q(\Omega)} \leqslant C \|g\|_{L^r(\Omega)}. \tag{5.1}$$

This result is valid for any measurable set $\Omega \subset \mathbb{R}^N$.

Proof. Since $\mathcal{K} \in L^s(\mathbb{R}^N)$ for $r' < N/(N-2\alpha)$, by the Hölder inequality with $1/r + 1/(r') = 1$, we obtain

$$\left\| \int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right\|_{L^\infty(\Omega)} \leqslant C \|g\|_{L^r(\Omega)}.$$

Since $\mathcal{K} \in L^1(\mathbb{R}^N)$, by using the integral Minkowski inequality we also obtain

$$\left\| \int_\Omega g(y-x) \mathcal{K}(y)\,\mathrm{d}y \right\|_{L^r(\Omega)} \leqslant \int_\Omega \left(\int_\Omega (g(x-y))^r\,\mathrm{d}x \right)^{1/r} \mathcal{K}(y)\,\mathrm{d}y \leqslant C \|g\|_{L^r(\Omega)}.$$

On the other hand,

$$
\begin{aligned}
&\left\| \int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right\|_{L^q(\Omega)} \\
&\qquad = \left[\int_\Omega \left(\int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right)^{(q-r)} \left(\int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right)^{r} \right]^{1/q} \\
&\qquad \leqslant \left\| \int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right\|_{L^\infty(\Omega)}^{(q-r)/q} \left\| \int_\Omega \mathcal{K}(x-y) g(y)\,\mathrm{d}y \right\|_{L^r(\Omega)}^{r/q}.
\end{aligned}
$$

Thus, we conclude using the above inequality. □

Lemma 5.3. *We have*

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} (\mathcal{K}(x-\xi) - \mathcal{K}(x_\lambda - \xi))(f(u_\lambda(\xi)) - f(u(\xi)))\,\mathrm{d}\xi.$$

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Proof. It is easy to check that

$$u(x) = \int_{\Sigma_\lambda} \mathcal{K}(x-\xi) f(u(\xi))\,\mathrm{d}\xi + \int_{\Sigma_\lambda} \mathcal{K}(x_\lambda - \xi) f(u(\xi))\,\mathrm{d}\xi$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} \mathcal{K}(x-\xi) f(u(\xi))\,\mathrm{d}\xi + \int_{\Sigma_\lambda} \mathcal{K}(x - \xi_\lambda) f(u(\xi))\,\mathrm{d}\xi.$$

The fact that $|x-\xi^\lambda| = |x^\lambda - \xi|$ implies the desired result. □

Proof of proposition 5.1. First we will see that λ_0 is finite. Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid u(x) < u(x^\lambda)\}.$$

By using the fact that $|\xi - x^\lambda| \geqslant |\xi - x|$ in $\xi \in \Sigma_\lambda^-$, $\mathcal{K}$ is decreasing, f is increasing and lemma 5.3, we have

$$u_\lambda(x) - u(x) \leqslant \int_{\Sigma_\lambda^-} \mathcal{K}(x-\xi)(f(u_\lambda(\xi)) - f(u(\xi)))\,\mathrm{d}\xi.$$

Note now that, by (f_9) for $M := \sup u$, there exists C such that $|f'(x)| \leqslant C|x|^\tau$ for all $0 \leqslant x \leqslant M$. Therefore, by the positivity of $\mathcal{K}$ and the mean value theorem, we obtain

$$|u_\lambda(x) - u(x)| \leqslant C \int_{\Sigma_\lambda^-} \mathcal{K}(x-\xi)|(u_\lambda^\tau(\xi))(u_\lambda(\xi) - u(\xi))|\,\mathrm{d}\xi.$$

Thus, by lemma 5.2 for $q = m$ and $r = \frac{1}{2}m$ with m large, such that $m > N/\alpha$ and $m\tau \geqslant 2$, we obtain

$$\begin{aligned} \|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^-)} &\leqslant C \left\| \int_{\Sigma_\lambda^-} \mathcal{K}(x-\xi) u_\lambda^\tau(\xi)(u_\lambda(\xi) - u(\xi))\,\mathrm{d}\xi \right\|_{L^m(\Sigma_\lambda^-)} \\ &\leqslant C \|u_\lambda^\tau(u_\lambda - u)\|_{L^{m/2}(\Sigma_\lambda^-)}. \end{aligned}$$

Now using the Hölder inequality, we obtain

$$\|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^-)} \leqslant \|u_\lambda\|^\tau_{L^{m\tau}(\Sigma_\lambda^-)} \|u_\lambda(x) - u(x)\|_{L^m(\Sigma_\lambda^-)}.$$

Then, by choosing λ large (negative), we obtain $\|u_\lambda\|_{L^{m\tau}(\Sigma_\lambda^-)} \leqslant \frac{1}{2}$, since $m\tau \geqslant 2$. This implies that, for λ large enough (negative), $|\Sigma_\lambda^-| = 0$, hence, Σ_λ^- is empty. That is, there exists λ_L such that $\lambda < \lambda_L$,

$$u(x) \geqslant u_\lambda(x), \quad \forall x \in \Sigma_\lambda.$$

On the other hand, since u decays at infinity, it is clear that there exists λ_+ such that $u(x) < u_{\lambda_+}(x)$ for some $x \in \Sigma_{\lambda_+}$. From here, we obtain that λ_0 is finite.

Suppose now that $u(x) \geqslant u_{\lambda_0}(x)$, but $u(x) \not\equiv u_{\lambda_0}(x)$ in Σ_{λ_0}. Using the monotonicity of f and lemma 5.3, we see that, in fact, $u(x) > u_{\lambda_0}(x)$ in Σ_{λ_0}. Then we can move the plane further to the right. More precisely, there is an ε depending on N, α and the solution u, satisfying $u(x) \geqslant u_\lambda(x)$, on Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$. In fact, if $u(x) < u_\lambda(x)$ in Σ_λ^- for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, then a similar approach gives

$$\|u_\lambda - u\|_{L^m(\Sigma_\lambda^-)} \leqslant C \|u_\lambda\|^\tau_{L^{m\tau}(\Sigma_\lambda^-)} \|u_\lambda - u\|_{L^m(\Sigma_\lambda^-)}. \tag{5.2}$$

Since $m\tau \geqslant 2$, there exist $\varepsilon_1 > 0$ and R large such that, for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1)$, we have $C\|u_\lambda\|^{\tau}_{L^{m\tau}(B^c_R(0))} \leqslant \frac{1}{4}$. Now, using the continuity of u and the strict positivity of $u - u_{\lambda_0}$ in Σ_{λ_0}, we see that $|\Sigma^-_\lambda \cap B_R(0)|$ is small for ε_2 sufficiently small, and we can obtain

$$C\|u_\lambda\|^{\tau}_{L^{m\tau}(\Sigma^-_\lambda \cap B_R(0))} \leqslant \tfrac{1}{4}.$$

Then we deduce from (5.2) that, for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$ with $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, Σ^-_λ has measure zero. □

Proof of theorem 1.6. By translation, we may say that $\lambda_0 = 0$. Thus, we have that u is symmetric about the x_1-axis, i.e. $u(x_1, x') = u(-x_1, x')$. Using the same approach in any arbitrary direction implies that u is radially symmetric. □

Appendix A.

We devote this appendix to proving some properties of the kernel $\mathcal{K}$, and some properties of Sobolev spaces and embeddings among them. All of these properties are known, but we provide some proofs for the reader's convenience.

The proof of the properties of $\mathcal{K}$ is based on [1, 3]. We start defining the heat kernel for $0 < \alpha < 1$, $t > 0$ and $x \in \mathbb{R}^N$ as

$$\mathcal{H}(x,t) = \int_{\mathbb{R}^N} \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi - t|\xi|^{2\alpha}}\,\mathrm{d}\xi, \tag{A 1}$$

which satisfies the following rescaling property:

$$\mathcal{H}(x,t) = t^{-N/2\alpha}\mathcal{H}\biggl(\frac{x}{t^{1/2\alpha}}, 1\biggr),$$

which can be easily seen by changing variables in (A 1). Then we define the kernel $\mathcal{K}$ as

$$\mathcal{K}(x) = \int_0^\infty \mathrm{e}^{-t}\mathcal{H}(t,x)\,\mathrm{d}t. \tag{A 2}$$

This is the kernel $\mathcal{K}_{2\alpha}$ defined in (A 5). In fact, for $\phi \in \mathcal{S}$, we have

$$\begin{aligned}
\langle \mathcal{K}, \phi\rangle &= \int_{\mathbb{R}^N}\int_0^\infty\int_{\mathbb{R}^N} \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\mathrm{e}^{-t(1+|\xi|^{2\alpha})}\phi(x)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N}\int_0^\infty \mathrm{e}^{-t(1+|\xi|^{2\alpha})}\int_{\mathbb{R}^N}\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\phi(x)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^N}\frac{1}{1+|\xi|^{2\alpha}}\int_{\mathbb{R}^N}\mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\phi(x)\,\mathrm{d}x\,\mathrm{d}\xi \\
&= \biggl\langle \frac{1}{1+|\xi|^{2\alpha}}, \mathcal{F}\phi\biggr\rangle.
\end{aligned}$$

Lemma A.1. *The kernel $\mathcal{K}$ is radially symmetric, non-negative and non-increasing in $r = |x|$.*

Proof. Being the Fourier transform of a radially symmetric function, $\mathcal{H}$ is radially symmetric in x and so is $\mathcal{K}$. In order to prove the other two properties, we extend

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the arguments in [1], with some details for completeness. We define the radially symmetric non-negative function f as

$$f(x) = A\left(\frac{1}{|x|^{N+2\alpha}}\chi_{\mathbb{R}^N\setminus B_1(0)}(x) + \chi_{B_1(0)}(x)\right),$$

where A is such that $\int_{\mathbb{R}^N} f(x)\,\mathrm{d}x = 1$. Here and in what follows, χ_B denotes the characteristic function of B. We have that

$$\begin{aligned}
\mathcal{F}(f)(\xi) &= \int_{\mathbb{R}^N} \cos(2\pi\xi\cdot x)f(x)\,\mathrm{d}x \\
&= 1 + A\left(\int_{B_1(0)^c} \frac{\cos(2\pi\xi\cdot x) - 1}{|x|^{N+2\alpha}}\,\mathrm{d}x + \int_{B_1(0)} (\cos(2\pi\xi\cdot x) - 1)\,\mathrm{d}x\right) \\
&= 1 + A|\xi|^{2\alpha}\int_{|y|\geqslant|\xi|} \frac{\cos(2\pi\hat{\xi}\cdot y) - 1}{|y|^{N+2\alpha}}\,\mathrm{d}y \\
&\quad\quad + A|\xi|^{-N}\int_{|y|\leqslant|\xi|} (\cos(2\pi\hat{\xi}\cdot y) - 1)\,\mathrm{d}y.
\end{aligned}$$

If we define

$$c = -A\int_{\mathbb{R}^N} \frac{\cos(2\pi\hat{\xi}\cdot y) - 1}{|y|^{N+2\alpha}}\,\mathrm{d}y,$$

then we see that $\mathcal{F}(f)(\xi) = 1 - c|\xi|^{2\alpha}(1 + \omega(\xi))$, where $\omega(\xi) \to 0$ if $\xi \to 0$. Now we define, for $n \in \mathbb{N}$,

$$f_n(x) = n^{N/(2\alpha)}(f * f * \cdots * f)(n^{1/(2\alpha)}x),$$

then

$$\begin{aligned}
\mathcal{F}(f_n)(\xi) &= \left(\mathcal{F}(f)\left(\frac{\xi}{n^{1/(2\alpha)}}\right)\right)^n \\
&= \left(1 - \frac{c|\xi|^{2\alpha}}{n}\left(1 + \omega\left(\frac{\xi}{n^{1/(2\alpha)}}\right)\right)\right)^n.
\end{aligned}$$

The right-hand side converges to $\mathrm{e}^{-c|\xi|^{2\alpha}}$ pointwise. Moreover, since $\|f_n\|_{L^1} = 1$, $|\mathcal{F}(f_n)(\xi)| \leqslant 1$ for all $\xi \in \mathbb{R}^N$, and then

$$\mathcal{F}(f_n)(\xi) \to \mathrm{e}^{-c|\xi|^{2\alpha}} \quad \text{in } \mathcal{S}'.$$

From here, it follows that f_n converges in $\mathcal{S}'$ to $\mathcal{H}(x, c)$. Since f_n is non-negative for all n, we conclude with the non-negativity of $\mathcal{H}(x, c)$ and, by scaling, of $\mathcal{H}(x, t)$ for all $t > 0$.

The monotonicity is obtained by the fact that f is non-increasing in $r = |x|$ and the following property of convolution. □

LEMMA A.2. *If f and g are $L^1(\mathbb{R}^N)$, radially symmetric functions, non-negative and decreasing in $r = |x|$, then $f * g$ is radially symmetric and decreasing in $r = |x|$.*

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Proof. If it is clear that $f * g$ is radially symmetric, then we study the monotonicity in $r > 0$ of

$$(f * g)(re_1) = \int_{\mathbb{R}^N} f(re_1 - y)g(y)\,\mathrm{d}y.$$

Next we assume that f and g are of class C^1 and have compact support. Then

$$(f * g)(re_1) = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} f(r - y_1, y')g(y_1, y')\,\mathrm{d}y_1\,\mathrm{d}y',$$

and we only need to look at the monotonicity (in $r \in \mathbb{R}_+$) of the one-dimensional convolution of the functions $F(y_1) = f(y_1, y')$ and $G(y_1) = g(y_1, y')$, with $y' \in \mathbb{R}^{N-1}$ fixed, that is,

$$(F * G)(r) = \int_{\mathbb{R}} F(r - z)G(z)\,\mathrm{d}z$$

and we may use the arguments in [1] that we repeat for completeness. We observe that F and G are even, non-negative and decreasing in $\mathbb{R}_+$. Since G is even, G' is odd and we have

$$\begin{aligned}(F * G)'(r) = F * G'(r) &= \int_{\mathbb{R}} F(r - z)G'(z)\,\mathrm{d}z \\ &= \int_0^\infty G'(z)(F(x - y) - F(x + y))\,\mathrm{d}z. \qquad \text{(A 3)}\end{aligned}$$

Given $r > 0$, we have two cases. First, if $0 \leqslant z \leqslant r$, then $0 \leqslant r - z \leqslant r + z$, and then $F(r - z) \geqslant F(r + z)$. Second, if $z \geqslant x$, then $0 \geqslant r - z \geqslant -z$, and so $F(r - z) \geqslant F(-z) = F(z)$. But we also have $0 \leqslant z \leqslant r + z$, and so $F(z) \leqslant F(r + z)$. Consequently, $F(r - z) \geqslant F(r + z)$.

In any case $F(r - z) \geqslant F(r + z)$, and since $G' \leqslant 0$ in $\mathbb{R}_+$, we obtain from (A 3) that $(F * G)'(r) \leqslant 0$ for all $r \leqslant 0$.

By approximation, we extend the property for every f and g. □

The decay properties of the kernel are obtained in [3] using the basic idea of [30]. Specifically, it is proved in [3] that

$$\lim_{|x| \to \infty} |x|^{N+2\alpha}\mathcal{H}(x, 1) = C$$

for a positive constant whose value is computed in [3]. Using this property and the scaling property of $\mathcal{H}$, we easily obtain that

$$c_1 \min\{t^{-N/2\alpha}, t|x|^{-N-2\alpha}\} \leqslant \mathcal{H}(x, t) \leqslant c_2 \min\{t^{-N/2\alpha}, t|x|^{-N-2\alpha}\}. \qquad \text{(A 4)}$$

From here, we can prove (3.1) and (3.2).

Proof of (3.1). Using (A 4), we have that, for $|x| \geqslant 1$,

$$\mathcal{K}(x) = \int_0^\infty \mathrm{e}^{-t}\mathcal{H}(x, t)\,\mathrm{d}t \leqslant \frac{C_1}{|x|^{N+2\alpha}},$$

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and, for $0 < |x| \leqslant 1$,

$$\mathcal{K}(x) \leqslant \int_0^{|x|^{2\alpha}} \mathrm{e}^{-t} \frac{t}{|x|^{N+2\alpha}} \,\mathrm{d}t + \int_{|x|^{2\alpha}}^{\infty} \mathrm{e}^{-t} t^{-N/(2\alpha)} \,\mathrm{d}t \leqslant \frac{C_2}{|x|^{N-2\alpha}}.$$

□

Proof of (3.2). In order to prove (3.2), we consider the definition of $\mathcal{K}$ using radial symmetry. We write $\mathcal{K}(x) = \mathcal{K}(r)$, so we have

$$\mathcal{K}(r) = \frac{2\pi}{r^{(N-2)/2}} \int_0^\infty \int_0^\infty \mathrm{e}^{-t(1+s^{2\alpha})} J_{(N-2)/2}(2\pi rs) s^{N/2} \,\mathrm{d}s \,\mathrm{d}t,$$

where $J_{(N-2)/2}$ is the Bessel function of order $(N-2)/2$. Differentiating $\mathcal{K}$, we find that

$$\mathcal{K}'(r) = -\frac{N-2}{2r}\mathcal{K} + \frac{2\pi}{r^{N/2}} I,$$

where I is defined as

$$I = \int_0^\infty \int_0^\infty \mathrm{e}^{-t(1+s^{2\alpha})} J'_{(N-2)/2}(2\pi rs) 2\pi r s^{(N+2)/2} \,\mathrm{d}s \,\mathrm{d}t.$$

Integrating by parts in the variable s, we obtain

$$I = -\tfrac{1}{2}(N+2)\mathcal{K}(r) + \int_0^\infty \int_0^\infty J(2\pi rs) \mathrm{e}^{-t(1+s^{2\alpha})} (2\alpha t s^{2\alpha}) s^{N/2} \,\mathrm{d}s \,\mathrm{d}t$$

and integrating by parts in t, we obtain

$$I = (-\tfrac{1}{2}(N+2) + 2\alpha)\mathcal{K}(r) - 2\alpha \int_0^\infty t\mathrm{e}^{-t} \mathcal{H}(r,t) \,\mathrm{d}t.$$

We estimate the last term using (A 4):

$$\begin{aligned}\int_0^\infty t\mathrm{e}^{-t} \mathcal{H}(r,t) \,\mathrm{d}t &\leqslant \int_0^{|x|^{2\alpha}} t^2 \mathrm{e}^{-t} r^{-N-2\alpha} \,\mathrm{d}t + \int_{|x|^{2\alpha}}^\infty t\mathrm{e}^{-t} t^{-N/2\alpha} \,\mathrm{d}t \\ &\leqslant \frac{C}{r^{N+2\alpha}}.\end{aligned}$$

Putting the pieces together, we obtain (3.2). □

In order to prove theorems 3.1 and 3.2, we define two kernels associated to the spaces $\mathcal{L}^{\beta,p}$ and $\mathcal{W}^{\beta,p}$, for a given $\beta > 0$ and $p \geqslant 1$. First we consider

$$\mathcal{K}_\beta(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^\beta}\right) \tag{A 5}$$

associated to $\mathcal{W}^{\beta,p}$ and

$$G_\beta(x) = \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^{\beta/2}}\right), \tag{A 6}$$

associated to $\mathcal{L}^{\beta,p}$. Then we define two other kernels associated to the composition of elliptic operators

$$\mu_\beta(x) = \mathcal{F}^{-1}\bigg(\frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}\bigg) \quad \text{and} \quad \Phi_\beta(x) = \mathcal{F}^{-1}\bigg(\frac{(1+|\xi|^2)^{\beta/2}}{1+|\xi|^\beta} - 1\bigg).$$

It was shown in [34, pp. 133–134] that μ_β is a finite measure and that $\Phi_\beta \in L^1$.

Proof of theorem 3.1. Given $u \in \mathcal{W}^{\beta,p}$, there exists $f \in L^p(\mathbb{R}^N)$ such that

$$(1+|\xi|^\beta)\hat{u} = \hat{f}.$$

Then we have

$$(1+|\xi|^2)^{\beta/2}\hat{u} = \frac{(1+|\xi|^2)^{\beta/2}}{(1+|\xi|^\beta)}\hat{f} = \hat{g},$$

where

$$g = \mathcal{F}^{-1}\bigg(\frac{(1+|\xi|^2)^{\beta/2}}{(1+|\xi|^\beta)}\bigg) * f = (\Phi_\beta + \delta) * f.$$

Since $\Phi_\beta \in L^1(\mathbb{R}^N)$, we find that $g \in L^p(\mathbb{R}^N)$, proving that $u \in \mathcal{L}^{\beta,p}$. To prove the reciprocal statement, we proceed similarly, but considering

$$\mathcal{F}^{-1}\bigg(\frac{(1+|\xi|^\beta)}{(1+|\xi|^2)^{\beta/2}}\bigg) = \mu_\beta(x) + G_\beta(x).$$

Since μ_β is a finite measure and G_β is in $L^1(\mathbb{R}^N)$, the result follows. This completes the proof of part (i).

In order to prove part (ii), we consider $u \in \mathcal{W}^{\beta,p}$ and $f \in L^p(\mathbb{R}^N)$ such that $(1+|\xi|^2)^{\beta/2}\hat{u} = \hat{f}$. Then we have

$$(1+|\xi|^2)^{\beta/2-\alpha}\mathcal{F}((-\Delta)^\alpha u) = \frac{|\xi|^{2\alpha}}{(1+|\xi|^2)^\alpha}\hat{f} = \mathcal{F}(\mu_{2\alpha} * f).$$

Since $\mu_{2\alpha}$ is a finite measure, $\mu_{2\alpha} * f$ is in $L^p(\mathbb{R}^N)$, and we conclude that $(-\Delta)^\alpha u \in \mathcal{W}^{\beta-2\alpha,p}$.

For part (iii), given $u \in C^{0,\gamma}(\mathbb{R}^N)$, we have that

$$w(x) = (-\Delta)^\alpha u(x) = \int_{\mathbb{R}^N} \frac{\delta(x,z;u)}{|z|^{N+2\alpha}}\,\mathrm{d}z,$$

where $\delta(x,z;u) = u(x+z) + u(x-z) - 2u(x)$. Since $u \in C^{0,\gamma}(\mathbb{R}^N)$, we have

$$|\delta(x,z;u) - \delta(y,z;u)| \leqslant C|x-y|^\gamma$$

and

$$|\delta(x,z;u) - \delta(y,z;u)| \leqslant C|z|^\gamma.$$

Then we have

$$|w(x) - w(y)| \leqslant \int_{B_r} \frac{C|z|^\gamma}{|z|^{N+2\alpha}}\,\mathrm{d}z + \int_{B_r^c} \frac{C|x-y|^\gamma}{|z|^{N+2\alpha}}\,\mathrm{d}z \leqslant C|x-y|^\mu$$

for an appropriate constant C and with $r = |x-y|$.

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For part (iv), we use part (iii) and the commutation of Δ^α with differentiation.

For part (v) we assume first that $\alpha < \frac{1}{2}$, and we use that if $u \in C^{1,\gamma}(\mathbb{R}^N)$, we have

$$|\delta(x,z;u) - \delta(y,z;u)| \leqslant C|x-y|^\gamma |z|$$

and

$$|\delta(x,z;u) - \delta(y,z;u)| \leqslant C|z|^{\gamma+1}.$$

Then we use the same argument as in part (iii), and we obtain the result. If $\alpha > \frac{1}{2}$, we use two times the last properties with $\Delta^{\alpha_1}(\Delta^{\alpha_2})$, where $\Delta^\alpha = \Delta^{\alpha_1}(\Delta^{\alpha_2})$ and $\alpha_i < \frac{1}{2}$.

For the proof of property (vi) we refer the interested reader to [34]. □

Proof of theorem 3.2. The first embedding is a consequence of the Sobolev inequality valid for $0 < \beta < N$, $1 < p < q < \infty$ and $1/q = 1/p - \beta/N$. Then

$$\|u\|_{L^q} \leqslant C\|(-\Delta)^{\beta/2}u\|_{L^p}, \quad \forall u \in \mathcal{L}^{\beta,p},$$

which is proved in [34].

In order to obtain the second embedding, we only need to prove that $\Delta^{s/2}u = f$ with $f \in L^p(\mathbb{R}^N)$, then $u \in C^{0,\mu}(\mathbb{R}^N)$. For that, we observe that $\Delta(\Delta^{s/2-1}u) = f$, and thus $\Delta^{s/2-1}u \in W^{2,p}(\mathbb{R}^N)$ by the regularity of Δ. Then, by the Morrey inequality, $u = \Delta^{1-s/2}g$ with $g \in C^{1,\gamma}(\mathbb{R}^N)$ with $\gamma = 1 - N/p$. Thus, in the case where $2(1-s/2) > 1 - N/p$, we use part (v) of theorem 3.1 to obtain $u \in u \in C^{0,\mu}(\mathbb{R}^N)$ with $\mu = s - N/p$. The case where $2(1-s/2) > 1 - N/p$ is similar and we have more regularity, since we use part (iv) of theorem 3.1. □

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