# NORMAL-CONVEX EMBEDDINGS OF INVERSE SEMIGROUPS

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Normal-convex embeddings are introduced for inverse semigroups, generalizing the group-theoretic concept, due to Papakyriakopoulos [4]. It is shown that every $E$-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group, a stronger version of a result by O'Carroll [3]. A general embedding result for inverse semigroups is also obtained.

**1. Preliminaries.** The general terminology and notation are those of Petrich [5].

Let $S$ be an inverse semigroup and let $R \subseteq S \times S$ be a relation on $S$. We denote by $R^{\#}$ the congruence on $S$ generated by $R$, that is, the transitive closure of $\{(aub, avb) : a, b \in S^1 \text{ and } (u, v) \in R \cup R^{-1}\}$. The natural projection $S \to S/R^{\#}$ is denoted by $(R^{\#})^{\natural}$.

Let $\varphi : S \to T$ be a homomorphism of inverse semigroups and let $R$ be a relation on $S$. The relation

$$R\varphi = \{(u\varphi, v\varphi) : (u, v) \in R\}$$

is said to be the relation on $T$ induced by $R$ and $\varphi$. It follows easily that

$$R^{\#}\varphi \subseteq (R\varphi)^{\#}. \tag{1.1}$$

If $\varphi$ is injective, we say that $\varphi$ is an embedding of inverse semigroups.

Now let $\varphi : S \to T$ be an embedding of inverse semigroups. We say that $\varphi$ is *normal-convex* if and only if, for every relation $R$ on $S$,

$$(R\varphi)^{\#} \cap (S \times S)\varphi \subseteq R^{\#}\varphi.$$

Note that, by (1.1), the inclusion $R^{\#}\varphi \subseteq (R\varphi)^{\#} \cap (S \times S)\varphi$ is always true. Also by (1.1), we know that $\varphi$ induces a unique homomorphism $\varphi_R : S/R^{\#} \to T/(R\varphi)^{\#}$ such that the canonical diagram

$$\begin{array}{ccc}
S & \xrightarrow{\varphi} & T \\
{\scriptstyle (R^{\#})^{\natural}} \downarrow & & \downarrow {\scriptstyle |(R\varphi)^{\#}|^{\natural}} \\
S/R^{\#} & \xrightarrow[\varphi_R]{} & T/(R\varphi)^{\#}
\end{array} \tag{1.2}$$

commutes. Now we have

LEMMA 1.1. *Let* $\varphi : S \to T$ *be an embedding of inverse semigroups. Then* $\varphi$ *is normal-convex if and only if* $\varphi_R$ *is injective for every relation* $R$ *on* $S$.

*Proof.* Suppose that $\varphi$ is normal-convex and let $R$ be a relation on $S$. Let $a, b \in S$ be such that $(aR^{\#})_{\varphi_R} = (bR^{\#})_{\varphi_R}$. Since (1.2) commutes, we have $(a\varphi)(R\varphi)^{\#} = (b\varphi)(R\varphi)^{\#}$. Hence $(a\varphi, b\varphi) \in (R\varphi)^{\#} \cap (S \times S)\varphi$. Since $\varphi$ is normal-convex, this yields $(a\varphi, b\varphi) \in R^{\#}\varphi$. Thus $aR^{\#} = bR^{\#}$ and so $\varphi_R$ is injective.

Conversely, suppose that $\varphi_R$ is injective for every relation $R$ on $S$. Suppose that $(a\varphi, b\varphi) \in (R\varphi)^{\#}$ for some $a, b \in S$. Since (1.2) commutes, we have $(aR^{\#})_{\varphi_R} = (bR^{\#})_{\varphi_R}$,

and since $\varphi_R$ is injective, $aR^\# = bR^\#$. Therefore $(a\varphi, b\varphi) \in R^\# \varphi$ and so $\varphi$ is normal-convex.

The following result shows that the class of normal-convex embeddings is closed under composition.

LEMMA 1.2. *Let $\varphi : S \to T$ and $\psi : T \to U$ be normal-convex embeddings of inverse semigroups. Then $\varphi\psi$ is a normal-convex embedding.*

*Proof.* It is trivial that $\varphi\psi$ is an embedding. Now let $R$ be a relation on $S$. Since $(\varphi\psi)_R$ is uniquely defined, we certainly have $(\varphi\psi)_R = \varphi_R \psi_{R\varphi}$ and so $(\varphi\psi)_R$ is injective. Thus, by Lemma 1.1, $\varphi\psi$ is normal-convex.

The next result shows an application of the concept of normal-convex embedding.

Given a semigroup $S$ and a relation $R$ on $S$, the *word problem* for $R$ consists in finding an algorithm which determines, for every $a, b \in S$, whether or not $(a, b) \in R^\#$.

THEOREM 1.3. *Let $\varphi : S \to T$ be a normal-convex embedding of inverse semigroups and let $R$ be a relation on $S$. Then the word problem for $R$ is solvable if the word problem for $R\varphi$ is solvable.*

*Proof.* Suppose that the word problem for $R\varphi$ is solvable. Let $a, b \in S$. By Lemma 1.1, $\varphi_R$ is injective and so $aR^\# = bR^\# \Leftrightarrow (aR^\#)\varphi_R = (bR^\#)\varphi_R$. Since (1.2) commutes, we have $(aR^\#)_{\varphi_R} = (bR^\#)_{\varphi_R} \Leftrightarrow (a\varphi)(R\varphi)^\# = (b\varphi)(R\varphi)^\#$. Since the word problem for $R\varphi$ is solvable, we can determine whether or not this latter equality holds, hence the word problem for $R$ is solvable and the theorem is proved.

Now let $S$ be an inverse semigroup with semilattice of idempotents $E(S)$. The *least group congruence* on $S$ is defined by

$$(a, b) \in \sigma \Leftrightarrow \exists e \in E(S) : ae = be.$$

We say that $S$ is *E-unitary* if

$$\forall a \in S, \qquad a\sigma = 1 \Rightarrow a \in E(S).$$

Let $M$ denote an inverse monoid with least group congruence $\sigma$. Then $M$ is said to be *F-inverse* if every $\sigma$-class of $M$ has a maximal element under the natural partial order. It is well-known that every $F$-inverse monoid is $E$-unitary [5, §VII.5].

Let $G$ be a group and let $K$ be a semilattice. An *action* of $G$ on $K$ by left automorphisms is a map $G \times K \to K : (g, A) \mapsto gA$ such that, for every $g, h \in G$ and $A, B \in K$,

$$g(hA) = (gh)A,$$
$$g(AB) = (gA)(gB),$$
$$1A = A.$$

It follows easily that, for every $g \in G$ and $A, B \in K$, we have

$$A \leqslant B \Rightarrow gA \leqslant gB.$$

The *semidirect product* of $K$ by $G$ induced by this action is the inverse semigroup $K \times G$ with the operation given by $(A, g)(B, h) = (A(gB), gh)$. When no ambiguity arises about the action, we shall denote this semigroup by $K \overline{\times} G$.

Now suppose that $L$ is an ideal of $K$ such that $GL = K$. Then we say that $(G, K, L)$ is a *strong McAlister triple* and

$$P(G, K, L) = \{(A, g) \in L \times G : g^{-1}A \in L\}$$

is an inverse subsemigroup of $K \bar{\times} G$ [1].

LEMMA 1.4 [1]. *Let $M$ be an inverse monoid. Then $M$ is F-inverse if and only if $M \simeq P(G, K, L)$ for some strong McAlister triple $(G, K, L)$ such that $L$ has a unity.*

Let $S$ be an inverse semigroup and let $\tau$ be a congruence on $S$. We say that $\tau$ is *idempotent-pure* if, for every $(a, b) \in \tau$,

$$a \in E(S) \Rightarrow b \in E(S).$$

We say that $\tau$ is *idempotent-separating* if, for every $(a, b) \in \tau$,

$$a \in E(S) \Rightarrow b \notin E(S).$$

Finally, an inverse semigroup $S$ is said to be *quasi-free* if $T \simeq F/\tau$ for some free inverse semigroup $F$ and some idempotent-pure congruence $\tau$ on $F$.

LEMMA 1.5 [2]. *Let $S$ be a quasi-free inverse semigroup. Then $S \simeq P(G, K, L)$ for some strong McAlister triple $(G, K, L)$ with $G$ free.*

**2. Strong McAlister triples.** In this section we show that, for every strong McAlister triple $(G, K, L)$, there exists a canonical embedding of $P(G, K, L)$ into a semidirect product of a semilattice by a group.

THEOREM 2.1. *Let $(G, K, L)$ be a strong McAlister triple. Then the inclusion map $\varphi : P(G, K, L) \to K \bar{\times} G$ is normal-convex.*

*Proof.* Let $S = P(G, K, L)$ and let $T = K \bar{\times} G$. Let $R$ be a relation on $S$, say $R = \{((A_i, g_i), (B_i, h_i)) : i \in I\}$. Without loss of generality, we can assume that $R$ is symmetric. Let $(U, u), (V, v) \in S$ be such that $(U, u)(R\varphi)^{\#} = (V, v)(R\varphi)^{\#}$. We want to prove that $(U, u)R^{\#} = (V, v)R^{\#}$. Since $R$ is symmetric, we know that there exist $(W_0, w_0), \ldots, (W_n, w_n) \in T$ such that

$$(W_0, w_0) = (U, u)$$

$$(W_n, w_n) = (V, v)$$

$$\forall j \in \{1, \ldots, n\} \ \exists (P_j, p_j), (Q_j, q_j) \in T \ \exists i_j \in I :$$

$$(W_{j-1}, w_{j-1}) = (P_j, p_j)(A_{i_j}, g_{i_j})(Q_j, q_j)$$

and

$$(W_j, w_j) = (P_j, p_j)(B_{i_j}, h_{i_j})(Q_j, q_j).$$

Now we show that, for every $m \in \{0, \ldots, n\}$,

$$\exists P'_m, Q'_m, W'_m \in L :$$
$$(W'_m, w_m) \in S,$$
$$(W'_m, w_m)R^{\#} = (U, u)R^{\#},$$
$$(W'_m, w_m) = (P'_m, 1)(W_m, w_m)(Q'_m, 1). \tag{2.1}$$

We use induction on $m$. Defining $P_0' = U$, $Q_0' = u^{-1}U$ and $W_0' = U$, we see that (2.1) holds for $m = 0$.

Now suppose that (2.1) holds for $m = j - 1$, with $j \in \{1, \ldots, n\}$. Then

$$
\begin{aligned}
(W_{j-1}', w_{j-1}) &= (W_{j-1}', 1)(W_{j-1}', w_{j-1})(w_{j-1}^{-1}W_{j-1}', 1) \\
&= (W_{j-1}', 1)(P_{j-1}', 1)(W_{j-1}', w_{j-1})(Q_{j-1}', 1)(w_{j-1}^{-1}W_{j-1}', 1) \\
&= (W_{j-1}', 1)(P_{j-1}', 1)(P_j, p_j)(A_{i_j}, g_{i_j})(Q_j, q_j)(Q_{j-1}', 1)(w_{j-1}^{-1}W_{j-1}', 1).
\end{aligned}
$$

It is clear that

$$
W_{j-1}' \leqslant P_{j-1}' P_j \tag{2.2}
$$

and so

$$
(W_{j-1}', 1)(P_{j-1}', 1)(P_j, p_j) = (W_{j-1}', p_j).
$$

Similarly,

$$
W_{j-1}' \leqslant (p_j g_{i_j} Q_j)(p_j g_{i_j} q_j Q_{j-1}')
$$

and so

$$
g_{i_j}^{-1} p_j^{-1} W_{j-1}' \leqslant Q_j(q_j Q_{j-1}'). \tag{2.3}
$$

Hence

$$
(Q_j, q_j)(Q_{j-1}', 1)(w_{j-1}^{-1}W_{j-1}', 1) = (g_{i_j}^{-1} p_j^{-1} W_{j-1}', q_j).
$$

Thus

$$
(W_{j-1}', w_{j-1}) = (W_{j-1}', p_j)(A_{i_j}, g_{i_j})(g_{i_j}^{-1} p_j^{-1} W_{j-1}', q_j).
$$

Since $W_{j-1}' \leqslant p_j A_{i_j}$, we have $p_j^{-1}W_{j-1}' \leqslant A_{i_j} \in L$. But $L$ is an ideal of $K$ and so $p_j^{-1}W_{j-1}' \in L$. Since $W_{j-1}' \in L$, we obtain $(W_{j-1}', p_j) \in S$. Similarly, we have $g_{i_j}^{-1}p_j^{-1}W_{j-1}' \leqslant g_{i_j}^{-1}p_j^{-1}(p_j A_{i_j}) = g_{i_j}^{-1}A_{i_j} \in L$, and $q_j^{-1}g_{i_j}^{-1}p_j^{-1}W_{j-1}' = w_{j-1}^{-1}W_{j-1}' \in L$. Hence

$$
(g_{i_j}^{-1} p_j^{-1} W_{j-1}', q_j) \in S.
$$

Let $P_j' = W_{j-1}'$, $Q_j' = w_{j-1}^{-1}W_{j-1}'$ and $W_j' = W_{j-1}'(p_j B_{i_j})(w_j w_{j-1}^{-1}W_{j-1}')$. Obviously, $P_j', Q_j' \in L$ and since $L$ is an ideal of $K$, we have $W_j' \in L$ as well. We have $(W_j', w_j) = (W_{j-1}', p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W_{j-1}', q_j)$, that is, $(W_j', w_j)$ is a product of elements of $S$. Therefore $(W_j', w_j) \in S$. Moreover,

$$
\begin{aligned}
(W_j', w_j)R^{\#} &= [(W_{j-1}', p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W_{j-1}', q_j)]R^{\#} \\
&= [(W_{j-1}', p_j)(A_{i_j}, g_{i_j})(g_{i_j}^{-1}p_j^{-1}W_{j-1}', q_j)]R^{\#} = (W_{j-1}', w_{j-1})R^{\#} = (U, u)R^{\#}.
\end{aligned}
$$

It follows from (2.2) that $(W_{j-1}', p_j) = (W_{j-1}', 1)(P_j, p_j)$. Similarly, (2.3) yields $(g_{i_j}^{-1}p_j^{-1}W_{j-1}', q_j) = (Q_j, q_j)(w_{j-1}^{-1}W_{j-1}', 1)$. Hence

$$
\begin{aligned}
(W_j', w_j) &= (W_{j-1}', p_j)(B_{i_j}, h_{i_j})(g_{i_j}^{-1}p_j^{-1}W_{j-1}', q_j) \\
&= (W_{j-1}', 1)(P_j, p_j)(B_{i_j}, h_{i_j})(Q_j, q_j)(w_{j-1}^{-1}W_{j-1}', 1) = (P_j', 1)(W_j, w_j)(Q_j', 1)
\end{aligned}
$$

and so (2.1) holds for $m = j$.

Thus (2.1) holds for every $m \in \{0, \ldots, n\}$. In particular, we have $(W_n', v)R^{\#} = (W_n', w_n)R^{\#} = (U, u)R^{\#}$ and $(W_n', v) = (P_n', 1)(W_n, w_n)(Q_n', 1) = (P_n', 1)(V, v)(Q_n', 1)$. Therefore $W_n' \leqslant V$ and so $(W_n', v) = (W_n', 1)(V, v)$. It follows that $(U, u)R^{\#} = (W_n', 1)R^{\#}(V, v)R^{\#}$ and so $(U, u)R^{\#} \leqslant (V, v)R^{\#}$. Similarly, we obtain $(V, v)R^{\#} \leqslant (U, u)R^{\#}$ and so $(U, u)R^{\#} = (V, v)R^{\#}$. Thus $\varphi$ is normal-convex.

Now, Lemma 1.5 and Theorem 2.1 immediately yield

COROLLARY 2.2. *Every quasi-free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

Since every free inverse semigroup is quasi-free, we also obtain

COROLLARY 2.3. *Every free inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a free group.*

**3. $E$-unitary inverse semigroups.** In this section we prove that every $E$-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.

Let $S$ be an $E$-unitary inverse semigroup. Let $M(S) = \{\varnothing \notin A \subseteq S : E(S) . A \subseteq A \subseteq a\sigma$ for some $a \in S\}$ with the operation described by $AB = \{ab : a \in A$ and $b \in B\}$. The following result is due to O'Carroll.

LEMMA 3.1 [3]. *Let $S$ be an $E$-unitary inverse semigroup. Then $M(S)$ is an $F$-inverse monoid and the map $\varphi : S \to M(S) : s \mapsto \{t \in S : t \leq s\}$ is an embedding. Moreover, if $\sigma_S$ and $\sigma_{M(S)}$ denote respectively the least group congruences of $S$ and $M(S)$, then $\sigma_{M(S)} \cap (S \times S)\varphi = \sigma_S\varphi$.*

We prove that this embedding is in fact normal-convex.

LEMMA 3.2. *Let $S$ be an $E$-unitary inverse semigroup. Then the embedding $\varphi : S \to M(S) : s \mapsto \{t \in S : t \leq s\}$ is normal-convex.*

*Proof.* Let $R$ be a relation on $S$. Without loss of generality, we can assume that $R$ is symmetric. Let $a, b \in S$ be such that $(a\varphi, b\varphi) \in (R\varphi)^{\#}$. We want to prove that $(a, b) \in R^{\#}$.

Since $(a\varphi, b\varphi) \in (R\varphi)^{\#}$, there exist $W_0, \ldots, W_n \in M(S)$ such that

$$W_0 = a\varphi;$$

$$W_n = b\varphi;$$

$$\forall i \in \{1, \ldots, n\} \; \exists P_i, Q_i \in M(S) \; \exists (u_i, v_i) \in R:$$

$$W_{i-1} = P_i(u_i\varphi)Q_i \quad \text{and} \quad W_i = P_i(v_i\varphi)Q_i.$$

We prove the following result. Let $z \in S$ and $C, D \in M(S)$ be such that $C(z\varphi)D \in S\varphi$. Then

$$\exists c, d \in S : c\varphi \subseteq C, \qquad d\varphi \subseteq D \quad \text{and} \quad (czd)\varphi = C(z\varphi)D. \tag{3.1}$$

Since $C(z\varphi)D \in S\varphi$, there exists some $w \in S$ such that $C(z\varphi)D = w\varphi$. Since $w \in w\varphi$, there exist $c \in C$, $z' \in z\varphi$ and $d \in D$ such that $cz'd = w$. Since $c\varphi \subseteq C$, $z'\varphi \subseteq z\varphi$ and $d\varphi \subseteq D$, we obtain $w\varphi = (cz'd)\varphi = (c\varphi)(z'\varphi)(d\varphi) \subseteq (c\varphi)(z\varphi)(d\varphi) \subseteq C(z\varphi)D = w\varphi$. Therefore $(czd)\varphi = C(z\varphi)D$ and (3.1) holds.

Since $S$ is $E$-unitary, it is clear that

$$\forall A \in M(S), \qquad AA^{-1} \subseteq 1\sigma \subseteq E(S). \tag{3.2}$$

Now we show that, for every $j \in \{0, \ldots, n\}$

$$\exists w_j \in S : w_j\varphi \subseteq W_j \quad \text{and} \quad (a, w_j) \in R^{\#}. \tag{3.3}$$

Let $w_0 = a$. It follows that (3.3) holds for $j = 0$.

Now suppose that (3.3) holds for $j = i - 1$, with $i > 0$. Then $w_{i-1}\varphi \subseteq W_{i-1}$ and so, since $S$ is inverse, $w_{i-1}\varphi \subseteq W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi)$. By (3.2), we also have $W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) \subseteq w_{i-1}\varphi$. Hence $w_{i-1}\varphi = W_{i-1}W_{i-1}^{-1}(w_{i-1}\varphi) = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Now we can apply (3.1) with $z = u_i$, $C = P_i$ and $D = Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$. Hence there exist $p_i$, $q_i \in S$ such that $p_i\varphi \subseteq P_i$, $q_i\varphi \subseteq Q_iW_{i-1}^{-1}(w_{i-1}\varphi)$ and $(p_iu_iq_i)\varphi = P_i(u_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = w_{i-1}\varphi$. We define $w_i = p_iv_iq_i$. Now $w_i\varphi = (p_i\varphi)(v_i\varphi)(q_i\varphi) \subseteq P_i(v_i\varphi)Q_iW_{i-1}^{-1}(w_{i-1}\varphi) = W_iW_{i-1}^{-1}(w_{i-1}\varphi) \subseteq W_iW_{i-1}^{-1}W_{i-1}$ and so, by (3.2), we have $w_i\varphi \subseteq W_i . E(S)$. For every $s \in S$ and $e \in E(S)$, we have $ae = aea^{-1}a$, and hence $W_i . E(S) \subseteq E(S) . W_i$. Therefore $w_i\varphi \subseteq W_i . E(S) \subseteq E(S) . W_i \subseteq W_i$. Moreover, $w_iR^{\#} = (p_iv_iq_i)R^{\#} = (p_iu_iq_i)R^{\#} = w_{i-1}R^{\#} = aR^{\#}$ and so (3.3) holds for $j = i$. Thus (3.3) holds for every $j \in \{0, \ldots, n\}$.

In particular, $w_n\varphi \subseteq W_n = b\varphi$ and $(a, w_n) \in R^{\#}$. Hence $w_n \leqslant b$ and $aR^{\#} = w_nR^{\#} \leqslant bR^{\#}$. Similarly, we prove that $bR^{\#} \leqslant aR^{\#}$. Thus $(a, b) \in R^{\#}$ and the lemma is proved.

Now we obtain

THEOREM 3.3. *Every E-unitary inverse semigroup admits a normal-convex embedding into a semidirect product of a semilattice by a group.*

*Proof.* Let $S$ be an $E$-unitary inverse semigroup. By Lemma 3.2, the embedding $\varphi: S \to M(S): s \mapsto \{t \in S : t \leqslant s\}$ is normal-convex. By Lemma 3.1, $M(S)$ is $F$-inverse and so, by Lemma 1.4 and Theorem 2.1, there exists a normal-convex embedding $\psi: M(S) \to P$, where $P$ is a semidirect product of a semilattice by a group. By Lemma 1.2, the composition $\varphi\psi: S \to P$ is a normal-convex embedding and the theorem is proved.

**4. Inverse semigroups.** The results of Section 2 can be used to obtain a general embedding result on inverse semigroups. We shall make use of the following result on quasi-free covers, due to Munn and Reilly.

LEMMA 4.1 [2]. *Let $S$ be an inverse semigroup. Then there exists a quasi-free inverse semigroup $F$ and an idempotent-separating congruence $\tau$ on $F$ such that $S \simeq F/\tau$.*

Now we have

THEOREM 4.2. *Every inverse semigroup admits a normal-convex embedding into an idempotent-separating homomorphic image of a semidirect product of a semilattice by a free group.*

*Proof.* Let $S$ be an inverse semigroup. By Lemma 4.1, we can assume that $S = F/\tau$, with $F$ quasi-free and $\tau$ idempotent-separating. By Lemma 1.5, we can assume that $F = P(G, K, L)$ for some strong McAlister triple $(G, K, L)$, with $G$ free. By Theorem 2.1, the inclusion $\varphi: F \to K \overline{\times} G$ is normal-convex. Therefore, by Lemma 1.1, the induced map $\psi: F/\tau \to (K \overline{\times} G)/(\tau\varphi)^{\#}$ defined by $(a\tau)\psi = a(\tau\varphi)^{\#}$ is injective. We must prove that $\psi$ is normal-convex and $(\tau\varphi)^{\#}$ is idempotent-separating.

First we prove that $\psi$ is normal-convex. Let $T = (K \overline{\times} G)/(\tau\varphi)^{\#}$. Let $R$ be a relation on $S$. We want to show that $(R\psi)^{\#} \cap (S \times S)\psi \subseteq R^{\#}\psi$.

Let $\mu$ be the congruence on $F$ such that $\mu/\tau = R^{\#}$. It follows that, for every $a, b \in F$, $(a, b) \in \mu$ if and only if $(a\tau, b\tau) \in R^{\#}$. We prove that

$$(R\psi)^{\#} \subseteq (\mu\varphi)^{\#}/(\tau\varphi)^{\#}. \tag{4.1}$$

Since $\tau \subseteq \mu$, we have $\tau\varphi \subseteq \mu\varphi$ and so $(\tau\varphi)^{\#} \subseteq (\mu\varphi)^{\#}$. Hence $(\mu\varphi)^{\#}/(\tau\varphi)^{\#}$ is a congruence on $T$ and we only need to show that $R\psi \subseteq (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Let $a, b \in F$ be such

that $(a\tau, b\tau) \in R$. Then $(a\tau, b\tau) \in R^{\#}$ and so, by definition of $\mu$, we have $(a, b) \in \mu$. Hence $(a\varphi, b\varphi) \in \mu\varphi \subseteq (\mu\varphi)^{\#}$. Therefore $(a\varphi(\tau\varphi)^{\#}, b\varphi(\tau\varphi)^{\#}) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$, that is, $((a\tau)\psi, (b\tau)\psi) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Hence (4.1) holds.

Now suppose that $a, b \in F$ and $((a\tau)\psi, (b\tau)\psi) \in (R\psi)^{\#}$. Then, by (4.1), we have $((a\tau)\psi, (b\tau)\psi) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$. Hence $(a\varphi(\tau\varphi)^{\#}, b\varphi(\tau\varphi)^{\#}) \in (\mu\varphi)^{\#}/(\tau\varphi)^{\#}$ and so $(a\varphi, b\varphi) \in (\mu\varphi)^{\#}$. Since $\varphi$ is normal-convex and $\mu$ is a congruence on $F$, we have $(\mu\varphi)^{\#} \cap (F \times F)\varphi \subseteq \mu\varphi$. Hence $(a\varphi, b\varphi) \in \mu\varphi$ and so $(a, b) \in \mu$ and $(a\tau, b\tau) \in R^{\#}$. Therefore $((a\tau)\psi, (b\tau)\psi) \in R^{\#}\psi$ and so $\psi$ is normal-convex.

Now we prove that $(\tau\varphi)^{\#}$ is idempotent-separating. Obviously, $E(K \overline{\times} G) = \{(A, 1): A \in K\}$. Suppose that $A, B \in K$ are such that $(A, 1)(\tau\varphi)^{\#} = (B, 1)(\tau\varphi)^{\#}$. Since $GL = K$, there exists $g \in G$ and $C \in L$ such that $gC = A$. Hence $g^{-1}A = C \in L$ and we have

$$(g^{-1}A, 1)(\tau\varphi)^{\#} = [(g^{-1}A, g^{-1})(A, 1)(A, g)](\tau\varphi)^{\#} = [(g^{-1}A, g^{-1})(B, 1)(A, g)](\tau\varphi)^{\#}$$
$$= ((g^{-1}A)(g^{-1}B), 1)(\tau\varphi)^{\#}.$$

Since $(g^{-1}A)(g^{-1}B) \leq g^{-1}A \in L$ and $L$ is an ideal of $K$, we have $(g^{-1}A)(g^{-1}B) \in L$. Hence $(g^{-1}A, 1), ((g^{-1}A)(g^{-1}B), 1) \in F$. But

$$[(g^{-1}A, 1)\tau]\psi = (g^{-1}A, 1)(\tau\varphi)^{\#} = ((g^{-1}A)(g^{-1}B), 1)(\tau\varphi)^{\#} = [((g^{-1}A)(g^{-1}B), 1)\tau]\psi$$

and so, since $\psi$ is injective, $(g^{-1}A, 1)\tau = ((g^{-1}A)(g^{-1}B), 1)\tau$. Since $\tau$ is idempotent-separating, we obtain $(g^{-1}A, 1) = ((g^{-1}A)(g^{-1}B), 1)$, that is, $g^{-1}A = (g^{-1}A)(g^{-1}B)$. Hence $A = AB$ and $A \leq B$. Similarly, we obtain $B \leq A$ and so $A = B$. Thus $(A, 1) = (B, 1)$ and $(\tau\varphi)^{\#}$ is idempotent-separating.

REFERENCES

**1.** D. B. McAlister, Groups, semilattices and inverse semigroups II, *Trans. Amer. Math. Soc.* **196** (1974), 351–370.

**2.** W. D. Munn and N. R. Reilly, *E*-unitary congruences on inverse semigroups, *Glasgow Math. J.* **17** (1976), 57–75.

**3.** L. O'Carroll, Reduced inverse and partially ordered semigroups, *J. London Math. Soc.* (2)**9** (1974), 293–301.

**4.** C. D. Papakyriakopoulos, A new proof for the invariance of the homology groups of a complex, *Bull. Soc. Math. Grèce* **22** (1943), 1–154.

**5.** M. Petrich, *Inverse semigroups*, (Wiley, 1984).

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