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FINITE 3-GEODESIC TRANSITIVE BUT NOT 3-ARC TRANSITIVE GRAPHS

WEI JIN

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Abstract

In this paper, we first prove that for $g \in \{3, 4\}$, there are infinitely many 3-geodesic transitive but not 3-arc transitive graphs of girth g with arbitrarily large diameter and valency. Then we classify the family of 3-geodesic transitive but not 3-arc transitive graphs of valency 3 and those of valency 4 and girth 4.

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1. Introduction

In this paper, all graphs are finite, simple and undirected. A *geodesic* from a vertex u to a vertex v in a graph Γ is one of the shortest paths from u to v in Γ, and this geodesic is called an *s-geodesic* if the distance between u and v is s. Then Γ is said to be *s-geodesic transitive* if, for each $1 \le i \le s$, the automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the set of i-geodesics of Γ. For a positive integer s, an s-arc of Γ is a sequence of vertices $(v_0, v_1, \ldots, v_s)$ in Γ such that v_i, v_{i+1} are adjacent and $v_{j-1} \ne v_{j+1}$ where $0 \le i \le s-1$ and $1 \le j \le s-1$. In particular, 1-arcs are called *arcs*. Then Γ is said to be *s-arc transitive* if, for each $i \le s$, the group $\mathrm{Aut}(\Gamma)$ is transitive on the set of i-arcs of Γ. Thus if a graph is s-geodesic transitive (s-arc transitive), then it is t-geodesic transitive (t-arc transitive) for each $t \le s$.

Clearly, every 3-geodesic is a 3-arc, but some 3-arcs may not be 3-geodesics. If Γ has girth 4 (the girth of Γ, denoted by $\mathrm{girth}(\Gamma)$, is the length of the shortest cycle in Γ), then the 3-arcs contained in 4-cycles are not 3-geodesics. The graph in Figure 1 is the Hamming graph $\mathrm{H}(3, 2)$, which is 3-geodesic transitive but not 3-arc transitive with valency 3 and girth 4. Thus the family of 3-arc transitive graphs is properly contained in the family of 3-geodesic transitive graphs.

The first remarkable result about 2-arc transitive graphs comes from Tutte [10, 11], and this family of graphs has been studied extensively; see [1, 7, 8, 12]. The local

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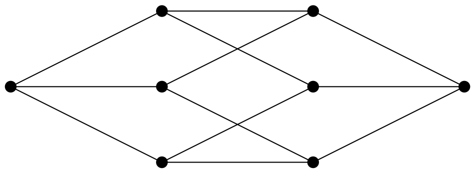

Figure 1. H(3, 2).

structure of the family of 2-geodesic transitive graphs was determined in [3]. In [4], the authors classified 2-geodesic transitive graphs of valency 4. Later, in [5], a reduction theorem for the family of normal 2-geodesic transitive Cayley graphs was produced and those which are complete multipartite graphs were also classified. In this paper, we study the family of 3-geodesic transitive graphs, and the first theorem shows that there exist geodesic transitive but not 3-arc transitive graphs with unbounded large diameter and valency. (The *diameter* $\mathrm{diam}(\Gamma)$ of a connected graph Γ is the maximum distance of u, v over all $u, v \in V(\Gamma)$. If Γ is s-geodesic transitive with $s = \mathrm{diam}(\Gamma)$, then Γ is called *geodesic transitive*.)

Theorem 1.1. *For $g \in \{3, 4\}$, there exist infinitely many geodesic transitive but not 3-arc transitive graphs of girth g with arbitrarily large diameter and valency. In particular, these graphs are 3-geodesic transitive but not 3-arc transitive.*

Remark 1.2. Let Γ be a 3-geodesic transitive but not 3-arc transitive graph. Then $\mathrm{girth}(\Gamma) \le 5$. If $\mathrm{girth}(\Gamma) = 3$, then Γ is not 2-arc transitive, and such graphs have been investigated; see [3–5]. We suppose that Γ is 2-arc transitive, so $\mathrm{girth}(\Gamma) = 4$ or 5. If $\mathrm{girth}(\Gamma) = 5$, then Γ is nonbipartite, and there is a characterisation of such graphs in [9].

Our second theorem is a classification of the family of 3-geodesic transitive graphs which are not 3-arc transitive of valency at most 4. Note that a 3-geodesic transitive graph of valency 2 is a cycle and so is 3-arc transitive.

Theorem 1.3. *Let Γ be a 3-geodesic transitive but not 3-arc transitive graph of valency k. Suppose that* $\mathrm{girth}(\Gamma) \ge 4$.

(1) *If $k = 3$, then Γ is either* H(3, 2) *or the dodecahedron, and both are geodesic transitive.*
(2) *If $k = 4$ and* $\mathrm{girth}(\Gamma) = 4$, *then Γ is either* H(4, 2) *or the complement of the* 2×5 *grid, and both are geodesic transitive.*

We do not have examples of 3-geodesic transitive but not 3-arc transitive graphs of valency 4 with girth 5 at the time of writing, and we conjecture that there is no such graph.

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2. Proof of Theorem 1.1

To facilitate the following discussion, we recall the definition of the Hamming graph. The *Hamming graph* $\Gamma = \mathrm{H}(d, n)$ has vertex set $\Delta^d = \{(x_1, \ldots, x_d) \mid x_i \in \Delta\}$, the cartesian product of d copies of Δ, where $\Delta = \{1, a, \ldots, a^{n-1}\}$, $d \geq 2$ and $n \geq 2$. Then two vertices v and v' are adjacent if and only if they are different in exactly one coordinate. Thus, if we suppose that $|v - v'|$ is the number of different coordinates of v and v', then v and v' are adjacent if and only if $|v - v'| = 1$. Moreover, $v' \in \Gamma_i(v)$ if and only if $|v - v'| = i$, where $1 \leq i \leq \mathrm{diam}(\Gamma)$ and $\Gamma_i(v)$ is the set of vertices of Γ which have distance i from v. The graph Γ has valency $d(n - 1)$.

When $n = 2$ and $d \geq 2$, the Hamming graph $\mathrm{H}(d, 2)$ is often called a *d-cube graph*, see [2, pages 261–262]. If $n = 2$, then $\mathrm{girth}(\mathrm{H}(d, n)) = 4$; if $n \geq 3$, then $\mathrm{girth}(\mathrm{H}(d, n)) = 3$. In the following discussion, we always suppose that Hamming graph $\mathrm{H}(d, n)$ and Δ are as defined above.

A graph Γ is said to be *G-geodesic transitive* if, for each $i \leq \mathrm{diam}(\Gamma)$, the group $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of i-geodesics of Γ.

Lemma 2.1. *Let* $\Gamma = \mathrm{H}(d, n)$ *with vertex set* Δ^d *where* $d \geq 2$ *and* $n \geq 2$. *Let* $G = X \wr S_d \leq \mathrm{Aut}(\Gamma)$ *where* $X \leq S_n$. *If* X *acts* 2*-transitively on* Δ, *then* Γ *is* G*-geodesic transitive. In particular,* Γ *is geodesic transitive.*

Proof. Suppose that X acts 2-transitively on Δ. Then X acts primitively but not regularly on Δ. It follows from [6, Lemma 2.7A] that G acts primitively and hence transitively on $V(\Gamma)$.

First, we prove that Γ is $(G, 1)$-geodesic transitive. Suppose that (v_0, v_1) is a 1-geodesic of Γ. Since G acts transitively on $V(\Gamma)$, we can assume that $v_0 = (1, 1, \ldots, 1)$. Since, for any two vertices u, u' of Γ, $u' \in \Gamma_i(u)$ if and only if $|u - u'| = i$, that is, u and u' have exactly i different entries, it follows that $v_1 = (1, \ldots, b, \ldots, 1)$ for some $b \in \Delta\backslash\{1\}$. Now since X acts 2-transitively on Δ, it follows that the stabiliser X_1 acts transitively on $\Delta\backslash\{1\}$, hence there exists $\sigma \in X_1$ such that $b^\sigma = a$. It follows that there exists $\alpha \in G_{v_0} \cong X_1 \wr S_d$ such that $v_1^\alpha = (1, \ldots, b, \ldots, 1)^\alpha = (a, 1, \ldots, 1)$. Thus Γ is $(G, 1)$-geodesic transitive.

Next, we prove that, for each $j = 2, 3, \ldots, d$, whenever Γ is $(G, j - 1)$-geodesic transitive, then Γ is (G, j)-geodesic transitive.

Let $(v_0, v_1, \ldots, v_{j-1}, v_j)$ be a j-geodesic of Γ where $2 \leq j \leq d$. Suppose that Γ is $(G, j - 1)$-geodesic transitive. Then we can fix a $(j - 1)$-geodesic $(v_0, v_1, \ldots, v_{j-1})$ such that $v_0 = (1, 1, \ldots, 1)$, and for each $i = 1, 2, \ldots, j - 1$, $v_i = (a, a, \ldots, a, 1, \ldots, 1)$ where the first i entries are equal to a and the last $d - i$ entries are equal to 1. Now since $v_j \in \Gamma_j(v_0) \cap \Gamma_{j-1}(v_1) \cap \cdots \cap \Gamma_1(v_{j-1})$ and since $v_j \in \Gamma_k(v)$ if and only if $|v - v_j| = k$, it follows that $v_j = (a, \ldots, a, 1, \ldots, x, \ldots, 1)$ for some $x \in \Delta\backslash\{1\}$, where the first $j - 1$ entries are equal to a. Moreover, since X acts 2-transitively on $\{1, a, \ldots, a^{n-1}\}$, X_1 acts transitively on $\Delta\backslash\{1\}$. Since $X_1 \wr S_{d-(j-1)} \leq G_{v_0,\ldots,v_{j-1}}$, it follows that there exists $\gamma \in G_{v_0,\ldots,v_{j-1}}$ such that $v_j^\gamma = (a, \ldots, a, 1, \ldots, x, \ldots, 1)^\gamma = (a, \ldots, a, a, 1, \ldots, 1)$ where the first j entries are equal to a, and the last $d - j$ entries are equal to 1. Therefore, Γ is

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(G, j)-geodesic transitive. Finally, since $\mathrm{diam}(\Gamma) = d$, it follows that Γ is G-geodesic transitive, and so is also geodesic transitive. □

Proof of Theorem 1.1. Let $\Gamma = \mathrm{H}(d, n)$ be the Hamming graph with $d \geq 2, n \geq 3$. Then Γ has girth 3, diameter d and valency $d(n-1)$. Thus Γ is not 3-arc transitive. Further, by Lemma 2.1, Γ is geodesic transitive.

Let $\Gamma = J(n, k)$ be the Johnson graph with $1 \leq k < [n/2]$ where $[n/2]$ is the integer part of $n/2$. Then Γ has girth 3, diameter k and valency $k(n-k)$. Thus Γ is not 2-arc transitive, and so is not 3-arc transitive. Further, by Devillers *et al.* ('On the transitivities of graphs', Proposition 2.1, submitted for publication), Γ is geodesic transitive.

Let $\Gamma = \mathrm{H}(d, 2)$ with $d \geq 3$. Then, $\mathrm{girth}(\Gamma) = 4$ and Γ has both diameter and valency d. Hence Γ is not 3-arc transitive. It follows from Lemma 2.1 that Γ is geodesic transitive. □

3. Proof of Theorem 1.3

3.1. Valency 3

A graph Γ is said to be *distance transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the ordered pairs of vertices at any given distance. Suppose that Γ is a distance transitive graph of valency k and diameter d. Then the cells of the distance partition with respect to u are orbits of A_u where $A := \mathrm{Aut}(\Gamma)$, and every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i-1}(u)$, say c_i. Similarly, every vertex in $\Gamma_i(u)$ is adjacent to the same number of other vertices in $\Gamma_{i+1}(u)$, say b_i. We denote by $(k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d)$ the *intersection array* of Γ.

The distance from vertex u to vertex v is denoted by $d_\Gamma(u, v)$. We give a useful lemma.

Lemma 3.1. *Let Γ be an i-geodesic transitive graph where $1 \leq i \leq \mathrm{diam}(\Gamma) - 1$. Let u, v be two vertices of Γ such that $d_\Gamma(u, v) = i$. Suppose that $|\Gamma_{i+1}(u) \cap \Gamma(v)| = 1$. Then Γ is geodesic transitive and $b_j = 1$ for each $i \leq j \leq \mathrm{diam}(\Gamma) - 1$.*

Proof. Let $(u_0 = u, u_1, \ldots, u_i = v)$ be an i-geodesic of Γ. Since $|\Gamma_{i+1}(u_0) \cap \Gamma(u_i)| = 1$, it follows that $b_i = 1$ and Γ is $(i+1)$-geodesic transitive. Let j be an integer such that $i \leq j \leq \mathrm{diam}(\Gamma) - 2$. Suppose that $b_k = 1$ for every $i \leq k \leq j$. Then Γ is $(j+1)$-geodesic transitive. Let $(u_0, \ldots, u_{j+2})$ be a $(j+2)$-geodesic. Since Γ is $(j+1)$-geodesic transitive, it follows that $b_j = |\Gamma_{j+1}(u_1) \cap \Gamma(u_{j+1})|$.

Suppose that $x \in \Gamma_{j+2}(u_0) \cap \Gamma(u_{j+1})$. Then $d_\Gamma(x, u_1) \leq j+1$. If $d_\Gamma(x, u_1) < j+1$, then $d_\Gamma(x, u_0) \leq d_\Gamma(x, u_1) + 1 < j + 2$, contradicting the assumption. Thus $d_\Gamma(x, u_1) = j + 1$, that is, $x \in \Gamma_{j+1}(u_1) \cap \Gamma(u_{j+1})$. Thus $\Gamma_{j+2}(u_0) \cap \Gamma(u_{j+1}) \subseteq \Gamma_{j+1}(u_1) \cap \Gamma(u_{j+1})$, and hence $b_{j+1} = |\Gamma_{j+2}(u_0) \cap \Gamma(u_{j+1})| \leq |\Gamma_{j+1}(u_1) \cap \Gamma(u_{j+1})| = b_j = 1$. Thus, Γ is $(j+1)$-geodesic transitive. By induction, Γ is geodesic transitive. □

Lemma 3.2. *Let Γ be a 3-geodesic transitive but not 3-arc transitive graph of valency 3. Suppose that $\mathrm{girth}(\Gamma) \geq 4$. Then Γ is geodesic transitive, and Γ is $\mathrm{H}(3, 2)$ or the dodecahedron.*

Proof. Since Γ is 3-geodesic transitive but not 3-arc transitive, it follows that $\mathrm{girth}(\Gamma) = 4$ or 5.

Let (u, v, w) be a 2-geodesic of Γ. Suppose first that $\mathrm{girth}(\Gamma) = 4$. Then there are six edges between $\Gamma(u)$ and $\Gamma_2(u)$, and $|\Gamma(u) \cap \Gamma(w)| = 2$ or 3. If $|\Gamma(u) \cap \Gamma(w)| = 3$, then Γ is isomorphic to the complete bipartite graph $K_{3,3}$ which is 3-arc transitive, a contradiction. Suppose that $|\Gamma(u) \cap \Gamma(w)| = 2$. Then $|\Gamma_3(u) \cap \Gamma(w)| = 1$, and by Lemma 3.1, Γ is geodesic transitive. Next assume that $\mathrm{girth}(\Gamma) = 5$. Then $|\Gamma(u) \cap \Gamma(w)| = 1$ and $|\Gamma_3(u) \cap \Gamma(w)| = 0$ or 1. If $|\Gamma(u) \cap \Gamma(w)| = 0$, then Γ is geodesic transitive. If $|\Gamma(u) \cap \Gamma(w)| = 1$, then by Lemma 3.1, Γ is also geodesic transitive. Therefore, Γ is distance transitive, and so Γ is one of the graphs listed in [2, pages 221–222, Theorems 7.5.1 and 7.5.2].

Since Γ is 3-geodesic transitive, it follows that Γ has 3-geodesics and so the diameter of Γ is at least 3. By inspecting the candidates in [2, pages 221–222, Theorems 7.5.1 and 7.5.2], Γ is either $H(3, 2)$ or the dodecahedron. □

3.2. Valency 4

Lemma 3.3. *Let Γ be a 3-geodesic transitive but not 3-arc transitive graph of valency 4. Suppose that $\mathrm{girth}(\Gamma) = 4$. Then $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 1$ or 2, for each 2-geodesic (u, v, w).*

Proof. Suppose that $\Gamma(v) = \{u_1, u_2, u_3, u_4\}$. Since $\mathrm{girth}(\Gamma) = 4$, it follows that any pair of vertices in $\Gamma(v)$ are nonadjacent, and $1 \le |\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2)| \le 3$. We will now prove that $|\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2)| \ne 3$.

Suppose to the contrary that $|\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2)| = 3$. Since Γ is 3-geodesic transitive, it follows that for any 2-geodesic (x, y, z) of Γ, $|\Gamma_2(y) \cap \Gamma(x) \cap \Gamma(z)| = 3$. Thus, $|\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_3)| = |\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_4)| = 3$.

Since the valency of Γ is 4, it follows that $|\Gamma_2(v) \cap \Gamma(u_1)| = 3$. Thus $\Gamma_2(v) \cap \Gamma(u_1) = \Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2) = \Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_3) = \Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_4)$. Hence $\Gamma_2(v) = \Gamma_2(v) \cap \Gamma(u_1)$ and $\mathrm{diam}(\Gamma) = 2$, contradicting the hypothesis that Γ contains 3-geodesics. Therefore, $|\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2)| = 1$ or 2. Since Γ is 3-geodesic transitive, $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 1$ or 2 for any 2-geodesic (u, v, w). □

Lemma 3.4. *Let Γ be the complement of the $2 \times (k + 1)$ grid. Then Γ is geodesic transitive with diameter 3 and valency k.*

Proof. By [2, page 222], the intersection array of Γ is $(k, k - 1, 1; 1, k - 1, k)$, so its valency is k and its diameter is 3. Note that Γ is antipodal and each antipodal block has two vertices. The automorphism group of Γ is $S_2 \times S_{k+1}$. We reconstruct Γ in the following way. Let $V(\Gamma) = \{(a_1, 0), (a_2, 0), \ldots, (a_{k+1}, 0), (a_1, 1), (a_2, 1), \ldots, (a_{k+1}, 1)\}$, and make two vertices $(a_i, 0)$, $(a_j, 1)$ adjacent if and only if $i \ne j$. It is clear that Γ is vertex transitive. Let $u = (a_1, 0)$. Then $\Gamma(u) = \{(a_2, 1), \ldots, (a_{k+1}, 1)\}$. As S_{k+1} is $k + 1$ transitive on $\{a_1, \ldots, a_{k+1}\}$, it follows that Γ is arc transitive. Let $v = (a_2, 1)$.

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Then $\Gamma_2(u) \cap \Gamma(v) = \{(a_3, 0), (a_4, 0), \ldots, (a_{k+1}, 0)\}$. As S_{k+1} is $k+1$ transitive on $\{a_1, \ldots, a_{k+1}\}$, it follows that $A_{u,v}$ is transitive on $\Gamma_2(u) \cap \Gamma(v)$, and so Γ is 2-geodesic transitive. Finally, from its intersection array, Γ is geodesic transitive. □

Lemma 3.5. *Let Γ be a 3-geodesic transitive but not 3-arc transitive graph of valency 4. Suppose that $\text{girth}(\Gamma) = 4$. Let (u, v, w) be a 2-geodesic and $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 2$. Then Γ is geodesic transitive and Γ is the complement of the 2×5 grid.*

Proof. Since $\text{girth}(\Gamma) = 4$, any pair of vertices of $\Gamma(v)$ are nonadjacent. Since $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 2$ and $v \in \Gamma(u) \cap \Gamma(w)$, it follows that $|\Gamma(u) \cap \Gamma(w)| = 3$, and so $|\Gamma_3(u) \cap \Gamma(w)| = 1$, and by Lemma 3.1, Γ is geodesic transitive. Hence Γ is distance transitive and Γ is one of the graphs in [2, Theorems 7.5.2 and 7.5.3]. By inspecting these graphs, Γ is the complement of the 2×5 grid. □

Lemma 3.6. *Let Γ be the incidence graph of the 2-$(7, 4, 2)$ design (the complement of the Fano plane). Then Γ is not 3-geodesic transitive.*

Proof. By [2, page 222], Γ is distance transitive and its intersection array is $(4, 3, 2; 1, 2, 4)$. Hence it is arc transitive. Note that its automorphism group is $A \cong \text{PGL}(3, 2))$ of order 168. Let (u, v, w) be a 2-geodesic. Then $|A_u| = 12$, $|A_{u,v}| = 3$ and $|A_{u,v,w}| = 1$. However, $|\Gamma_3(u) \cap \Gamma(w)| = 2$, so $A_{u,v,w}$ is not transitive on $\Gamma_3(u) \cap \Gamma(w)$, that is, Γ is not 3-geodesic transitive. □

Lemma 3.7. *Let Γ be a 3-geodesic transitive but not 3-arc transitive graph of valency 4. Suppose that $\text{girth}(\Gamma) = 4$. Let (u, v, w) be a 2-geodesic and $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 1$. Then $\Gamma \cong \text{H}(4, 2)$.*

Proof. Since Γ is 3-geodesic transitive and $|\Gamma_2(v) \cap \Gamma(u) \cap \Gamma(w)| = 1$, it follows that for every 2-geodesic (x, y, z), $|\Gamma_2(y) \cap \Gamma(x) \cap \Gamma(z)| = 1$. Suppose that $\Gamma(v) = \{u_1 = u, u_2 = w, u_3, u_4\}$. Then $|\Gamma_2(v) \cap \Gamma(u_i) \cap \Gamma(u_j)| = 1$ whenever $i \neq j$.

Suppose that $\Gamma_2(v) \cap \Gamma(u_1) = \{w_1, w_2, w_3\}$ and $\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_2) = \{w_1\}$. Then $\{u_1, u_2\} \subseteq \Gamma(v) \cap \Gamma(w_1)$, and it follows that $|\Gamma(v) \cap \Gamma(w_2)| \geq 2$. Hence w_2 is adjacent to at least one of u_3, u_4. Without loss of generality, assume that w_2 is adjacent to u_3. Since $|\Gamma_2(v) \cap \Gamma(u_1) \cap \Gamma(u_j)| = 1$, where $j = 2, 3, 4$, it follows that w_3 is not adjacent to u_2 or u_3. Since $|\Gamma(v) \cap \Gamma(w_3)| \geq 2$, it follows that w_3 is adjacent to u_4. Moreover, $\Gamma(v) \cap \Gamma(w_1) = \{u_1, u_2\}$. It follows that $|\Gamma(x) \cap \Gamma(z)| = 2$ for every 2-geodesic (x, y, z).

Now suppose that $\Gamma_2(v) \cap \Gamma(u_2) = \{w_1, w_4, w_5\}$. Since $|\Gamma_2(v) \cap \Gamma(u_2) \cap \Gamma(u_3)| = 1$ and u_3, w_1 are not adjacent, it follows that u_3 is adjacent to exactly one of w_4, w_5. Suppose that u_3 is adjacent to w_4. By noting that $|\Gamma_2(v) \cap \Gamma(u_2) \cap \Gamma(u_4)| = 1$, $|\Gamma(v) \cap \Gamma(w_j)| = 2$ where $j = 1, 2, 3, 4, 5$, $\Gamma(v) \cap \Gamma(w_1) = \{u_1, u_2\}$ and $\Gamma(v) \cap \Gamma(w_4) = \{u_2, u_3\}$, we see that u_4 is adjacent to w_5.

Assume that $\Gamma_2(v) \cap \Gamma(u_3) = \{w_2, w_4, w_6\}$. Since $|\Gamma_2(v) \cap \Gamma(u_3) \cap \Gamma(u_4)| = 1$ and u_4 is not adjacent to w_2, w_4, it follows that u_4 is adjacent to w_6. Thus $\Gamma_2(v) = \{w_1, w_2, w_3, w_4, w_5, w_6\}$.

If w_1 is adjacent to one of w_2, w_3, w_4, w_5, then girth$(\Gamma) = 3$, which contradicts the assumption girth$(\Gamma) = 4$. Suppose that w_1 is adjacent to w_6. Then since Γ is 3-geodesic transitive, w_2 is adjacent to one of w_3, w_4, w_5, w_6. If w_2 is adjacent to w_6, then diam$(\Gamma) = 2$ and Γ is distance transitive of 11 vertices. By inspecting candidates from [2, page 222], such a graph does not exist, giving a contradiction. If w_2 is adjacent to w_3 or w_4, then (w_2, w_3, u_1) or (w_2, w_4, u_3) is a triangle, which contradicts the assumption that girth$(\Gamma) = 4$. Thus w_2 is adjacent to w_5. Similarly, w_3 is adjacent to w_4. Thus $\Gamma(u_2) = \{v, w_1, w_4, w_5\} \subseteq \Gamma(u_1) \cup \Gamma_2(u_1)$, and so $\Gamma(u_2) \cap \Gamma_3(u_1) = \emptyset$. Since $u_2 \in \Gamma_2(u_1)$ and Γ is 3-geodesic transitive, it follows that diam$(\Gamma) = 2$ and Γ is distance transitive with 11 vertices. By inspecting the graphs of [2, page 222], such a graph does not exist, giving a contradiction. Thus $\Gamma(w_1) \cap \Gamma_2(v) = \emptyset$, so $|\Gamma(w_1) \cap \Gamma_3(v)| = 2$.

Now suppose that $\Gamma_3(v) \cap \Gamma(w_1) = \{r_1, r_2\}$. Then $\Gamma(w_1) = \{u_1, u_2, r_1, r_2\}$. Since (w_1, u_1, w_2) is a 2-geodesic and $|\Gamma(w_1) \cap \Gamma(w_2)| = 2$, and since w_2 is not adjacent to u_2, it follows that w_2 is adjacent to exactly one of r_1, r_2. Without loss of generality, suppose that w_2 is adjacent to r_1. Similarly, each of w_3, w_4, w_5, is also adjacent to exactly one of r_1, r_2. Thus $\{w_1, w_2, w_3, w_4, w_5\} \subseteq \Gamma_2(v) \cap [\Gamma(r_1) \cap \Gamma(r_2)]$. By the 3-geodesic transitivity, $|\Gamma_2(v) \cap \Gamma(r_1)| \geq 3$. If $|\Gamma_2(v) \cap \Gamma(r_1)| = 4$, then Γ is geodesic transitive with diameter 3 and 14 vertices. Hence Γ is distance transitive. By inspecting the graphs of [2, page 222], only the incidence graph of 2-$(7, 4, 2)$ design has 14 vertices and diameter 3. However, by Lemma 3.6, this graph is not 3-geodesic transitive, giving a contradiction. Hence $|\Gamma_2(v) \cap \Gamma(r_1)| = 3$, and so $|\Gamma_4(v) \cap \Gamma(r_1)| = 1$ or 0. If $|\Gamma_4(v) \cap \Gamma(r_1)| = 0$, then Γ is geodesic transitive of diameter 3, with 15 vertices and intersection array $(4, 3, 2; 1, 2, 3)$. By checking the distance transitive graphs of valency 4 of [2, page 222], such a graph does not exist. If $|\Gamma_4(v) \cap \Gamma(r_1)| = 1$, then by Lemma 3.1, Γ is geodesic transitive, and so is distance transitive. In particular, a part of its intersection array is $(4, 3, 2, 1, \ldots; 1, 2, 3, \ldots)$. Checking the distance transitive graphs of valency 4 of [2, page 222], only H$(4, 2)$ has such a property. Further, it follows from Lemma 2.1 that H$(4, 2)$ is also geodesic transitive. □

The proof of Theorem 1.3 follows from Lemmas 3.2, 3.3, 3.5 and 3.7.

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WEI JIN, School of Statistics, Research Center of Applied Statistics,
Jiangxi University of Finance and Economics,
Nanchang, Jiangxi 330013, PR China
e-mail: jinwei@jxufe.edu.cn