THE SIMPLE PENDULUM WITH UNIFORMLY CHANGING STRING LENGTH

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Abstract

An analysis is made of the motion of a simple undamped pendulum which performs small oscillations in one plane while its string is raised or lowered through the point of support at a constant rate. The behaviour of the system and the variation of its angular and linear amplitudes are expressible in terms of Bessel functions of orders 0, 1 and 2.

1. Introduction

The free length of the string of a simple undamped pendulum is changed uniformly by raising or lowering the string at a constant velocity V through a fixed point O. The motion begins at time $t = 0$ when the free string length is l, so that its free length at $t \geqq 0$ is

$$r = Vt + l. \tag{1}$$

The pendulum is supposed initially to be executing vibrations of small angular amplitude in one plane, and its subsequent behaviour is sought both for the case of a lengthening string (for which $V > 0$ in (1)) and for that of a shortening string ($V < 0$).

The position of the pendulum bob at time t will be specified by its plane polar co-ordinates r, θ with respect to the point O, where θ is the angle between the pendulum string and the vertical through O. While θ is small the equations of motion of the pendulum bob in polar form are, to the first order in θ,

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\theta, \tag{2}$$

$$\ddot{r} - r\dot{\theta}^2 = g - T, \tag{3}$$

where T is the string tension per unit mass of the pendulum bob and g is gravitational acceleration. Conditions under which these first order approximations remain valid will be established in the sequel.

2. Solution of equation (2)

The transformation (1) enables (2) to be written as

$$r\frac{d^2\theta}{dr^2} + 2\frac{d\theta}{dr} + (g/V^2)\theta = 0.$$

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If the independent variable be changed again from r to x, where

$$x = 2(gr)^{\frac{1}{2}}/V, \qquad (V>0), \tag{4a}$$

or

$$x = -2(gr)^{\frac{1}{2}}/V, \quad (V<0), \tag{4b}$$

so that x is always positive and

$$r = V^2x^2/4g, \quad (V \gtrless 0), \tag{5}$$

the above differential equation becomes

$$x\theta''+3\theta'+x\theta = 0,$$

where dashes denote differentiations with respect to x.

Under the substitution

$$y = x\theta$$

the last equation transforms to

$$x^2y''+xy'+(x^2-1)y = 0,$$

which is Bessel's equation of order one. This has the general solution

$$y = c_1J_1(x)+c_2Y_1(x),$$

where $J_1(x)$ and $Y_1(x)$ are Bessel functions of order one of the first and second kinds respectively and c_1, c_2 are arbitrary constants.

The angular displacement of the pendulum string is therefore

$$\theta(x) = x^{-1}[c_1J_1(x)+c_2Y_1(x)], \tag{6a}$$

or, as a function of the string length r by way of (4*a*, *b*),

$$\theta(r) = r^{-\frac{1}{2}}\{C_1J_1[2(gr)^{\frac{1}{2}}/V]+C_2Y_1[2(gr)^{\frac{1}{2}}/V]\}, \tag{6b}$$

where C_1, C_2 are arbitrary constants.

Since the Bessel functions of both kinds execute damped oscillations as their arguments increase continuously, it is clear from (6*a*, *b*) that the angular displacement of a lengthening pendulum undergoes damped oscillations in $t \geqq 0$ and that the condition of small angular amplitude continues to hold for $l \leqq r<\infty$. The range of validity of (6*a*) is seen from (4*a*) to be

$$2(gl)^{\frac{1}{2}}/V \leqq x<\infty, \quad (V>0). \tag{7a}$$

For a shortening pendulum (6*a*, *b*) show that the condition of small angular amplitude will, in general, be violated before the free string length becomes zero, in which case the validity of (6*a*, *b*) will not extend over the whole of the range

$$-2(gl)^{\frac{1}{2}}/V \geqq x \geqq 0, \quad (V<0). \tag{7b}$$

This point will be considered in § 6. Results deduced in §§ 3, 4 and 5 apply only within the range for which (6*a*, *b*) are valid.

The linear displacement s of the pendulum bob from its mean position is

$$s(r) = r\theta(r) \tag{8a}$$

to the first order in θ; or, as a function of x by way of (5),

$$s(x) = V^2x^2\theta(x)/4g. \tag{8b}$$

3. The mean and extreme positions of the pendulum

An initial condition will be prescribed in the form

$$t = 0,\ \theta = 0,\ x = x_1 = \pm 2(gl)^{\frac{1}{2}}/V,\ (V \lessgtr 0), \tag{9}$$

where x_1 is a zero of $J_1(x)$. This condition can always be achieved, for any given value of V, by allotting a suitable value to l; it is chosen because it assigns the value zero to the constants c_2 and C_2 in (6*a*) and (6*b*), thus reducing the solution to the less clumsy forms

$$\theta(x) = c_1 x^{-1} J_1(x) \tag{10}$$

and

$$\theta(r) = C_1 r^{-\frac{1}{2}} J_1[2(gr)^{\frac{1}{2}}/V].$$

The modifications required in the sequel for more general initial conditions will be discussed briefly in § 6.

Since (10) shows that $\theta(x)$ vanishes at the zeros of $J_1(x)$, it follows at once with the aid of (5) that:

Under the initial condition (9), *the lengths of the pendulum at its mean positions* $\theta = 0$ *are*

$$r_k = V^2 x_k^2/4g, \quad (k = 1, 2, 3, \ldots), \tag{11}$$

where x_k *are the successive zeros of* $J_1(x)$ *in the range* $x \geqq x_1$, $(V>0)$, *or* $x \leqq x_1$, $(V<0)$.

It is clear that the condition revealing the extreme positions of swing of the pendulum is $d\theta/dx = 0$. From (10) it is seen that

$$\frac{d\theta}{dx} = c_1[x^{-1}J_1'(x) - x^{-2}J_1(x)],$$

and taking $n = 1$ in the recurrence relation (1)

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

shows that an alternative form is

$$\frac{d\theta}{dx} = -c_1 x^{-1} J_2(x).$$

The extremum condition is therefore $J_2(x) = 0$, so that the extreme positions of the pendulum occur at the zeros of $J_2(x)$. It is therefore clear from (5) that:

Under the initial condition (9), *the lengths of the pendulum at its extreme positions are*

$$R_k = V^2 X_k^2/4g, \quad (k = 1, 2, 3, \ldots),$$

where X_k *are the successive zeros of* $J_2(x)$ *in the range* $x > x_1$, $(V>0)$, *or* $x < x_1$, $(V<0)$.

4. The angular and linear amplitudes

Substitution in (10) of the extremum condition $x = X_k$ reveals the angular amplitudes θ_k of successive swings of the pendulum to be

$$\theta_k = |\,\theta(X_k)| = |\,c_1 X_k^{-1} J_1(X_k)|.$$

The recurrence relation (1)

$$(2n/x)J_n(x) = J_{n-1}(x) + J_{n+1}(x),$$

for the case $n = 1$, yields the alternative form

$$\theta_k = \tfrac{1}{2}\,|\,c_1 J_0(X_k)|, \tag{12}$$

since the X_k are zeros of $J_2(x)$.

The corresponding linear amplitudes s_k are now shown by (8*b*) to be

$$s_k = V^2 X_k^2\,|\,c_1 J_0(X_k)|/8g.$$

The above results may be summarised in ratio form as follows:

Under the initial condition (9), *the ratio of the angular amplitude of the kth extremum to that of the first is*

$$\theta_k/\theta_1 = |\,J_0(X_k)/J_0(X_1)|, \quad (k = 1, 2, 3, \ldots), \tag{13}$$

and the ratio of the corresponding linear amplitudes is

$$s_k/s_1 = (X_k/X_1)^2\,|\,J_0(X_k)/J_0(X_1)|, \quad (k = 1, 2, 3, \ldots), \tag{14}$$

where X_k are the successive zeros of $J_2(x)$ in the range $x > x_1$, $(V > 0)$, or $x < x_1$, $(V < 0)$.

From (12) and (13) it is easy to verify, either numerically or analytically, that the angular amplitude decreases and the linear amplitude increases in the case of a lengthening pendulum, and that the opposite effects occur for a shortening pendulum.

As an illustration of the numerical magnitudes involved in the above results, consider the particular case in which $V = 10$ cm./sec. and $x_1 = 25{\cdot}9037$ (which is the eighth zero of $J_1(x)$ in $x > 0$). It is easily verified that the initial condition (9) is satisfied by choosing an initial string length $l = 17{\cdot}1$ cm. (with g taken as 981 cm./sec.^{2}). With the help of tables (2, 3) it is easily found from (13) and (14) that

$$\theta_{10}/\theta_1 \approx 0{\cdot}39, \quad s_{10}/s_1 \approx 1{\cdot}40,$$

$$\theta_{20}/\theta_1 \approx 0{\cdot}19, \quad s_{20}/s_1 \approx 1{\cdot}76.$$

5. Asymptotic forms of the amplitudes as $r \to \infty$ and $r \to 0$

Case (*i*): $r \to \infty$. From (10) and the asymptotic result (1) that

$$J_n(x) \sim [2/(\pi x)]^{\frac{1}{2}} \cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)$$

as $x \to \infty$, it follows that for large x,

$$\theta(x) \sim (2/\pi)^{\frac{1}{2}} c_1 x^{-\frac{3}{2}} \cos(x - \tfrac{3}{4}\pi).$$

The angular amplitude θ_k therefore satisfies

$$\theta_k \sim (2/\pi)^{\frac{1}{2}}\,|\,c_1\,|\,x^{-\frac{3}{2}}$$

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to a close approximation. In virtue of (4*a*) this result may be written as

$$\theta_k \sim \tfrac{1}{2} \mid c_1 \mid V^{\frac{3}{2}} \pi^{-\frac{1}{2}} (gr)^{-\frac{3}{4}}, \quad (r \to \infty).$$

When used in (8*a*) this shows that the linear amplitude s_k satisfies

$$s_k \sim \tfrac{1}{2} \mid c_1 \mid V^{\frac{3}{2}} \pi^{-\frac{1}{2}} g^{-\frac{3}{4}} r^{\frac{1}{4}}, \quad (r \to \infty).$$

These results show that *as the free string length r becomes large, the angular amplitude of oscillation decreases as* $r^{-\frac{3}{4}}$ *and the linear amplitude increases as* $r^{\frac{1}{4}}$.

Case (*ii*): $r \to 0$. Since $x = 0$ is a zero of $J_2(x)$, and $J_0(x)$ takes its maximum value of unity at $x = 0$, it follows from (12) that

$$\theta_k \leqq \tfrac{1}{2} \mid c_1 \mid \tag{15}$$

throughout the whole of the range (7*b*) if and only if $\frac{1}{2} \mid c_1 \mid$ is a first order small quantity. Assuming this condition to be satisfied, (10) shows that

$$\theta(r) \sim \tfrac{1}{2} c_1, \quad r \to 0,$$

since the series form of $J_1(x)$ shows that $J_1(x) \sim \frac{1}{2}x$ as $x \to 0$. From (8*a*) it follows that

$$s(r) \sim \tfrac{1}{2} c_1 r, \quad r \to 0.$$

6. Concluding remarks

The tension in the string may be found from (3), and it is easily verified that $T = g$ at the extreme positions of each swing and that in the mean positions

$$T = \{[(c_1/x_k^2) J_0(x_k)]^2 + 1\} g,$$

where x_k are the zeros of $J_1(x)$ in the relevant range of x.

The time for a half-swing of the pendulum between two successive mean positions is easily found from (1) and (11) to be

$$t_k = V(x_{k+1}^2 - x_k^2)/4g,$$

where x_k, x_{k+1} are the successive zeros of $J_1(x)$ which correspond to the two mean positions.

The effects of changing the initial condition (9) are easy to see. If x_1 in (9) is chosen as a zero of $Y_1(x)$ instead of $J_1(x)$, the constants c_1, C_1, in (6*a*, *b*) vanish while c_2, C_2 do not. The analysis of §§ 3 and 4 continues to hold with c_2, $Y_n(x)$ replacing c_1, $J_n(x)$, since $Y_n(x)$ satisfies the same recurrence relations as $J_n(x)$. In § 5, the results of case (i) hold with c_2 replacing c_1; those of case (ii) do not because $Y_n(x)$ is unbounded as $x \to 0$, and this causes the condition of small angular displacement to be violated before the string length becomes zero. Initial conditions which are more general than (9) will retain both constants in (6*a*, *b*), and all of the previous results will be replaced by forms involving linear combinations of Bessel functions of both kinds; the dynamical implications will remain unchanged.

It is easy to augment (9) by a second condition which will ensure (theoretically) that the angular displacement remains small as $r \to 0$. For example, the angular amplitude θ_1 at the first extremum may be prescribed; (12) shows that

$$\tfrac{1}{2}\,|\,c_1\,| = \theta_1\,|\,J_0(X_1)|^{-1},$$

and it should therefore always be possible to make $\frac{1}{2}\,|\,c_1\,|$ a first order small quantity by choosing θ_1 small enough, thus ensuring that (15) is satisfied. This can never be achieved in a practical experiment because the condition (9) cannot be realised exactly, and this permits the survival in (6*a*, *b*) of the Bessel function of the second kind which is unbounded as its argument approaches zero.

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