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Projections in the Convex Hull of Surjective Isometries

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Abstract. We characterize those linear projections represented as a convex combination of two surjective isometries on standard Banach spaces of continuous functions with values in a strictly convex Banach space.

1 Introduction

Characterization of various classes of bounded linear projections is a basic problem in Banach Space Theory, (see [2, 3, 7, 11, 12]). In particular, characterizations of bi-contractive as well as bi-circular projections have received considerable attention, (see [3, 6, 9, 10, 13]). Russo and Friedman showed that the bi-contractive projections on $C(\Omega)$, with Ω a compact Hausdorff space, can be written as the average of the identity and an isometric reflection. Continuing this line of research, in [9], Fosner, Ilisevic, and C. K. Li introduced a new class of projections, designated generalized bi-circular projections (GBP). A GBP is a linear projection P so that $P + \lambda(I - P)$ is an isometry, for some modulus 1 complex number $\lambda \neq 1$. It follows that such an isometry must be surjective. It can also be shown that these projections on $C(\Omega)$ (with Ω a compact space) are given as the average of the identity and an isometric reflection, (see [5]). Hence in this case, GBPs are precisely the bi-contractive projections.

Motivated by these results we are interested in projections that are in the convex hull of two distinct surjective isometries. In this paper, we consider projections acting on the Banach space, $\mathcal{C}(\Omega, \mathcal{X})$, of all continuous functions defined on a compact topological space Ω and with values in a strictly convex space $\mathcal{X}$. We denote by $\|\cdot\|_\infty$, the standard norm on $\mathcal{C}(\Omega, \mathcal{X})$. We classify all projections that are in the convex hull of two distinct isometries. More precisely, any such projection must be either a generalized bi-circular projection or a projection of multiplicative type, (see Theorem 2.4). In either case, the resulting projection is bi-contractive. We also state some minimal hypothesis on the isometries involved in order to assure that only generalized bi-circular projections can be expressed in this way.

We begin by recalling the form of generalized bi-circular projections on $C(\Omega, \mathcal{X})$, whenever $\mathcal{X}$ has the strong Banach Stone property. See [1, page 142] for the statement and consequences of the "strong Banach Stone property." For our work, it is sufficient to note that every strictly convex Banach space has this property.

A generalized bi-circular projection P, on $\mathcal{C}(\Omega, \mathcal{X})$, has one of the following representations, (see [4]).

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(i) There exist a bounded operator $u\colon \Omega \to \mathrm{Isom}(\mathcal{X})$ and a nontrivial homeomorphism $\phi\colon \Omega \to \Omega$ so that $\phi^2 = \mathrm{Id}$ and, for every $\omega \in \Omega$, $u_\omega \circ u_{\phi(\omega)} = \mathrm{Id}$, with $u(\omega) = u_\omega$ an invertible isometry on $\mathcal{X}$. The projection P has the following representation

$$P(f)(\omega) = \tfrac{1}{2}\big(f(\omega) + u_\omega(f \circ \phi(\omega))\big),$$

for every $f \in \mathcal{C}(\Omega, \mathcal{X})$ and $\omega \in \Omega$.

(ii) There exists a generalized bi-circular projection on $\mathcal{X}$, P_ω so that $P(f)(\omega) = P_\omega(f(\omega))$, for each $\omega \in \Omega$.

We notice that in case (i), P is a convex combination of two isometries, the average of the identity operator and an isometric reflection. This theorem relies on the characterization of the surjective isometries of $\mathcal{C}(\Omega, \mathcal{X})$ and is a consequence of the vector valued Banach–Stone Theorem, (see [1]). The following theorem is due to Jerison, (see [1, 8]).

Theorem 1.1 *Let $\mathcal{X}$ be a strictly convex Banach space. T is a surjective isometry of $\mathcal{C}(\Omega, \mathcal{X})$ if and only if, for every $\omega \in \Omega$ and $f \in \mathcal{C}(\Omega, \mathcal{X})$, $T(f)(\omega) = u_\omega(f(\phi(\omega)))$, where $\phi\colon \Omega \to \Omega$ is a homeomorphism and $u_\omega\colon \Omega \to \mathrm{Isom}(\mathcal{X})$ is a continuous operator valued mapping into the group of invertible isometries on $\mathcal{X}$.*

In the next section we characterize those projections that can be expressed as the real convex combination of two surjective isometries. We show that every projection given by the average of two isometries must be a generalized bi-circular projection or a multiplicative projection.

2 Projections in the Convex Hull of Two Isometries

We first state a preliminary result concerning projections in the convex combination of two isometries (not necessarily surjective) on $\mathcal{C}(\Omega, \mathcal{X})$, with X an arbitrary Banach space.

Proposition 2.1 *If P_λ is a projection in the linear combination of two distinct isometries, $P_\lambda = \lambda I_1 + (1-\lambda) I_2$, with λ a positive real number less than 1, then $\lambda = \frac{1}{2}$ or $P_\lambda = \mathrm{Id}$.*

Proof If the kernel of P_λ is nontrivial, then there exists $f \in \mathcal{C}(\Omega, \mathcal{X})$ with norm 1, so that $\lambda I_1(f) = -(1-\lambda) I_2(f)$. Therefore $\lambda = \frac{1}{2}$. If, otherwise, P_λ is injective we have that $P_\lambda = \mathrm{Id}$, since $P_\lambda(\mathrm{Id} - P_\lambda) = 0$. ∎

We consider two distinct isometries in $\mathcal{C}(\Omega, \mathcal{X})$, I_1 and I_2, given by

$$I_1(f)(\omega) = u_\omega(f \circ \phi(\omega)) \quad \text{and} \quad I_2(f)(\omega) = v_\omega(f \circ \psi(\omega)),$$

with ϕ and ψ homeomorphisms of Ω and $u, v\colon \Omega \to \mathrm{Isom}(\mathcal{X})$. For every $\omega \in \Omega$, $u(\omega) = u_\omega$ and $v(\omega) = v_\omega$, represent invertible isometries of $\mathcal{X}$. The symbol $\circ$ denotes composition.

We also consider an operator P on $\mathcal{C}(\Omega, \mathcal{X})$, given by

$$P(f)(\omega) = \frac{I_1(f)(\omega) + I_2(f)(\omega)}{2},$$

and we derive conditions under which such an operator is a projection. We first state a lemma to be used in the proof of the forthcoming proposition. This lemma is a straightforward consequence of the strict convexity of $\mathcal{X}$.

Lemma 2.2 *If u and v are isometries of a strictly convex normed space $\mathcal{X}$ and $u(z) + v(z) = 2z \neq 0$ for some $z \in \mathcal{X}$, then $u(z) = v(z) = z$.*

Proposition 2.3 *Let $\mathcal{X}$ be a strictly convex Banach space. The operator P is a projection on $\mathcal{C}(\Omega, \mathcal{X})$ if and only if every $\omega \in \Omega$ satisfies one of the following statements:*

(i) $\omega = \phi(\omega) = \psi(\omega)$ *and* $q_\omega = \dfrac{u_\omega + v_\omega}{2}$ *is a projection on* $\mathcal{X}$.
(ii) $\phi(\omega) = \psi(\omega)$ *and* $u_\omega = -v_\omega$.
(iii) $\omega = \phi(\omega) \neq \psi(\omega)$, $\psi^2(\omega) = \omega$, $\phi \circ \psi(\omega) = \psi(\omega)$, $u_\omega = u_{\psi(\omega)} = \mathrm{Id}$ *and* $v_\omega \circ v_{\psi(\omega)} = \mathrm{Id}$.
(iv) $\omega = \psi(\omega) \neq \phi(\omega)$, $\phi^2(\omega) = \omega$, $\psi \circ \phi(\omega) = \phi(\omega)$, $v_\omega = v_{\phi(\omega)} = \mathrm{Id}$ *and* $u_\omega \circ u_{\phi(\omega)} = \mathrm{Id}$.

Proof P is a projection if and only if

$$\begin{aligned} (2.1)\quad & u_\omega \circ u_{\phi(\omega)} f(\phi^2(\omega)) + u_\omega \circ v_{\phi(\omega)} f(\psi \circ \phi(\omega)) + v_\omega \circ u_{\psi(\omega)} f(\phi \circ \psi(\omega)) \\ & \qquad + v_\omega \circ v_{\psi(\omega)} (f \circ \psi^2(\omega)) = 2u_\omega f(\phi(\omega)) + 2\, v_\omega f(\psi(\omega)). \end{aligned}$$

We first consider $\omega \in \Omega$ so that $\omega \neq \phi(\omega) \neq \psi(\omega) \neq \omega$ and select $z \in \mathcal{X}$ of norm 1. We define α a continuous function such that $\alpha(\phi(\omega)) = 1$ and $\alpha(\psi(\omega)) = \alpha(\phi^2(\omega)) = \alpha(\phi \circ \psi(\omega)) = 0$. Now, we choose $f(\omega) = \alpha(\omega) \cdot z$. Equation (2.1) reduces to

$$\alpha(\psi \circ \phi(\omega))\, u_\omega \circ v_{\phi(\omega)}(z) + \alpha(\psi^2(\omega)) v_\omega \circ v_{\psi(\omega)}(z) = 2u_\omega(z).$$

This equation implies that $\psi \circ \phi(\omega) = \phi(\omega)$ and $\psi^2(\omega) = \phi(\omega)$. Hence $\phi(\omega) = \psi(\omega)$, contradicting our assumption.

Now, we consider $\omega \in \Omega$ so that $\omega = \phi(\omega) \neq \psi(\omega)$. We select a continuous function α satisfying the conditions: $\alpha(\omega) = \alpha(\phi(\omega)) = \alpha(\phi^2(\omega)) = \alpha(\psi^2(\omega)) = 0$ and $\alpha(\psi(\omega)) = \alpha(\psi \circ \phi(\omega)) = 1$. Given $f(\omega) = \alpha(\omega) \cdot z$, equation (2.1) reduces to $u_\omega \circ v_\omega(z) + \alpha((\phi \circ \psi)(\omega)) \cdot v_\omega \circ u_{\psi(\omega)}(z) = 2v_\omega(z)$. Thus $\phi \circ \psi(\omega) = \psi(\omega)$. Furthermore, for $f(\omega) = \alpha(\omega) \cdot z$ and $f(\omega) = (1 - \alpha)(\omega) \cdot z$, with $z \in \mathcal{X}$, equation (2.1) yields the following: $u_\omega \circ v_\omega(z) + v_\omega \circ u_{\psi(\omega)}(z) = 2v_\omega(z)$ and $u_\omega^2(z) + v_\omega \circ v_{\psi(\omega)}(z) = 2u_\omega(z)$ respectively.

Therefore the Id is the average of the isometries u_ω and $u_\omega^{-1} \circ v_\omega \circ v_{\psi(\omega)}$ or u_ω and $v_\omega \circ u_{\psi(\omega)} \circ v_\omega^{-1}$. This implies that $u_\omega = u_{\psi(\omega)} = \mathrm{Id}$ and $v_\omega \circ v_{\psi(\omega)} = \mathrm{Id}$, (see Lemma 2.2). If α is such that $\alpha(\omega) = 1$ and $\alpha(\psi(\omega)) = 0$ we have $u_\omega \circ u_\omega(z) +$

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$\alpha(\psi^2(\omega)) \cdot u_\omega \circ v_{\psi(\omega)}(z) = 2u_\omega(z)$. This implies that $\psi^2(\omega) = \omega$. Similar considerations hold for $\omega = \psi(\omega) \neq \phi(\omega)$. If $\omega \neq \phi(\omega) = \psi(\omega)$, then equation (2.1) is factored as follows:

$$(u_\omega + v_\omega)\big[u_{\phi(\omega)}(f(\phi^2(\omega))) + v_{\phi(\omega)}(f(\psi^2(\omega))) - 2f(\phi(\omega))\big] = 0.$$

This last equation is clearly true whenever $u_\omega + v_\omega = 0$. If there exists $z \in \mathcal{X}$ so that $u_\omega(z) + v_\omega(z) \neq 0$, then we claim that $\omega = \phi(\omega) = \psi(\omega)$. Otherwise, $\phi^2(\omega) \neq \phi(\omega)$ (hence $\phi^2(\omega) \neq \psi(\omega)$). Urysohn's Lemma asserts the existence of α so that $\alpha(\psi^2(\omega)) = \alpha(\phi^2(\omega)) = 0$ and $\alpha(\phi(\omega)) = 1$. Hence, $f(x) = \alpha(x) \cdot z$ does not satisfy equation (2.1), which leads to a contradiction. It remains to analyze the case where $\omega = \phi(\omega) = \psi(\omega)$. Equation (2.1) now reduces to

$$(u_\omega + v_\omega)\big[u_\omega(f(\omega)) + v_\omega(f(\omega)) - 2f(\omega)\big] = 0.$$

Since this is true for every $f \in C(\Omega, \mathcal{X})$, statement (i) follows, and the proof is complete. ∎

Theorem 2.4 *Let $\mathcal{X}$ be a strictly convex Banach space. If a projection P is given by the average of two distinct isometries on $\mathcal{C}(\Omega, \mathcal{X})$, then either*

(i) $P(f)(\omega) = \frac{1}{2}\big(f(\omega) + u_\omega(f \circ \phi(\omega))\big)$, *for every $f \in \mathcal{C}(\Omega, \mathcal{X})$ and $\omega \in \Omega$, u_ω is an invertible isometry on $\mathcal{X}$ such that $u_\omega \circ u_{\phi(\omega)} = \mathrm{Id}$, and $\phi\colon \Omega \to \Omega$ a nontrivial homeomorphism so that $\phi^2 = \mathrm{Id}$, or*

(ii) $P(f)(\omega) = q_\omega(f(\omega))$ *for every $f \in \mathcal{C}(\Omega, \mathcal{X})$ and $\omega \in \Omega$, u_ω and v_ω are invertible isometries on $\mathcal{X}$ such that $q_\omega = (u_\omega + v_\omega)/2$ is a projection.*

Proof We denote by P a projection given as the average of two distinct isometries,

$$P(f)(\omega) = \frac{u_\omega(f \circ \phi(\omega)) + v_\omega(f \circ \psi(\omega))}{2}.$$

Proposition 2.3 allows us to define the following partition of Ω: $S = \{\omega : \phi(\omega) = \psi(\omega)\}$, $A_1 = \{\omega \notin S : \phi(\omega) = \omega\}$, and $A_2 = \{\omega \notin S : \psi(\omega) = \omega\}$. It also follows from Proposition 2.3 that $\phi(A_2) = A_2$ and $\psi(A_1) = A_1$. If A_1 or A_2 is nonempty and ω_0 denotes a boundary point of either then $\omega_0 = \phi(\omega_0) = \psi(\omega_0)$. We define

$$\phi_*(\omega) = \begin{cases} \phi(\omega) & \text{if } \omega \in \overline{A}_2 \\ \psi(\omega) & \text{if } \omega \in \overline{A}_1, \end{cases} \quad \text{and} \quad u_*(\omega) = \begin{cases} v_\omega & \text{if } \omega \in \overline{A}_2 \\ u_\omega & \text{if} \omega \in \overline{A}_1. \end{cases}$$

It is also a direct consequence of Proposition 2.3 that ϕ_* is a homeomorphism and u_* is continuous. Therefore, we have that $P(f)(\omega) = \frac{1}{2}(f(\omega) + u_*(\omega) f(\phi_*(\omega)))$. The statement (i) in Theorem 2.4 now follows from Proposition 2.3. If A_1 and A_2 are both empty, then $S = \Omega$. Thus, we have that $q_\omega = (u_\omega + v_\omega)/2 = 0$, if $\omega \neq \phi(\omega) = \psi(\omega)$.

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On the other hand, if $\omega = \phi(\omega) = \psi(\omega)$, then $(u_\omega + v_\omega)[u_\omega(f(\omega)) + v_\omega(f(\omega)) - 2f(\omega)] = 0$ and $q_\omega = (u_\omega + v_\omega)/2$, as stated in (ii). This concludes the proof. ■

Definition 2.5 We say that two isometries $I_1(f)(\omega) = u_1(\omega)f(\phi_1(\omega))$ and $I_2(f)(\omega) = u_2(\omega)f(\phi_1(\omega))$ are essentially distinct if and only if ϕ_1 and ϕ_2 are distinct homeomorphisms.

Corollary 2.6 *Let $\mathcal{X}$ be a strictly convex Banach space. A projection P is given by the average of two essentially distinct isometries on $\mathcal{C}(\Omega, \mathcal{X})$ if and only if it is a generalized bi-circular projection.*

Remarks In general we do not know if q_ω in Theorem 2.4 above is a generalized bi-circular projection. There are some special cases where this is indeed the case. The operator $q_\omega = \frac{(u_\omega + v_\omega)}{2}$ is a bi-contractive projection. The bi-contractive projections are known for some of the classical Banach spaces, including $L^p(\mu)$ and $C(\Omega)$. In many cases, such projections can be written as the average of the identity and an isometric reflection. If q_ω in the previous theorem has such a representation then it will also be a generalized bi-circular projection.

If $\mathcal{X}$ is a complex Hilbert space, then q_ω is a generalized bi-circular projection if and only if it is self-adjoint, *i.e.*, $q_\omega^* = q_\omega$. More generally, if q_ω is an L^p projection, *i.e.*, for every $x \in \mathcal{X}$, $\|x\|^p = \|q_\omega x\|^p + \|(I - q_\omega)x\|^p$, for $p \in [1, \infty)$ ($\|x\| = \max\{\|q_\omega x\|, \|(I - q_\omega)x\|\}$, for $p = \infty$), it is easy to show that q_ω is a generalized bi-circular projection.

We end this note with a question directly related to our study. Let $\mathcal{X}$ be a Banach space with U and V surjective isometries. Suppose that $P = \frac{U+V}{2}$ is a nontrivial projection. Must P be a generalized bi-circular projection?

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