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A CONVERSE PROBLEM IN MATRIX DIFFERENTIAL EQUATIONS

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Suppose X and Y are $n\times n$ matrix solutions of the $n\times n$ matrix differential equation

$$X''+P(t)X = 0 \qquad \text{on } J \tag{1}$$

such that

$$XX^*+YY^* = I \qquad \text{on } J \tag{2}$$

where J is some interval. Here $P(t)$ is a symmetric $n\times n$ matrix and 0 and I respectively the $n\times n$ zero and identity matrices. Also X^* and tr X, denote, respectively, the transpose and the trace of X.

In the scalar case it is well known (1) and (2) together imply $p(t)\equiv k$, a non-negative constant. See [1] and the references therein. The lower case letters used above indicate scalars which are 1×1 matrices.

Now in the case that $n>1$, we can no longer be sure that $P(t)\equiv K$ a constant matrix, but we can obtain a result which in the scalar case imlies $p(t)\equiv k$. To see that $P(t)\equiv K$ may fail when (1) and (2) are true, we consider the following example. Let

$$U(t) = \begin{pmatrix} \frac{1}{\sqrt{3}}\cos 4t & -\frac{1}{\sqrt{3}}\sin 4t \\ \frac{\sqrt{2}}{5}\cos 4t+\frac{8}{5\sqrt{3}}\sin 4t & \frac{8}{5\sqrt{3}}\cos 4t-\frac{\sqrt{2}}{5}\sin 4t \end{pmatrix}$$

and

$$V(t) = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}}\cos t+\frac{2}{\sqrt{15}}\sin t & \frac{2}{\sqrt{15}}\cos t+\frac{\sqrt{2}}{\sqrt{5}}\sin t \\ \frac{1}{\sqrt{15}}\cos t & -\frac{1}{\sqrt{15}}\sin t \end{pmatrix}$$

Further, let

$$C = \begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 12 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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and, note that

$$e^{Bt} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

and $(e^{Bt})^* = e^{B^*t} = e^{-Bt}$. The following may be easily, but tediously, verified.

$$UU^* + VV^* = I.$$

Next we note that U and V are solutions of $U'' - 2BU' + (C - B^*B)U = 0$, a constant coefficient matrix differential equation. Defining $X = e^{-Bt}U$ and $Y = e^{-Bt}V$ we obtain that X and Y are solutions to $X'' + P(t)X = 0$ where $P(t) = e^{-Bt}Ce^{Bt}$ is a nonconstant matrix. Looking at $XX^* + YY^*$, we see that

$$\begin{aligned} XX^* + YY^* &= e^{Bt}UU^*(e^{Bt})^* + e^{Bt}VV^*(e^{Bt})^* \\ &= e^{Bt}(UU^* + VV^*)e^{-Bt} = e^{Bt}Ie^{-Bt} = I. \end{aligned}$$

Hence (1) and (2) are satisfied but $P(t) \equiv K$ fails in this case where $n=2$. It is easy to see that this system can be incorporated into a higher dimensional system thus preventing the conclusion that $P(t) \equiv K$ for $n > 2$ also.

We are now ready to state and prove a theorem true for all $n \geq 1$.

THEOREM. *Let X and Y be solutions of the matrix differential equation* (1) *where $P(t)$ is a symmetric matrix. Suppose that* (2) *holds. Then $P(t)$ is positive semi-definite and* tr $P(t) \equiv k$ *a constant. Further $P(t) = e^{-\mathrm{B}t}Ce^{\mathrm{B}t}$ where C is symmetric, and B is skew-symmetric.*

COROLLARY. *If x and y are solutions to the scalar differential equation $x'' + p(t)x = 0$ and if $x^2 + y^2 = 1$, then $p(t) \equiv k \geq 0$.*

Proof of Theorem. Differentiating (2) we have

$$(X'X^* + Y'Y^*) + (XX'^* + YY'^*) = 0. \tag{3}$$

Let $B = X'X^* + Y'Y^*$, then (3) becomes

$$B + B^* = 0. \tag{4}$$

Thus B is a skew-symmetric matrix, and, since B is differentiable, B' is a skew-symmetric matrix also. On the other hand, $B' = X'X'^* + Y'Y'^* - P(XX^* + YY^*) = X'X'^* + Y'Y'^* - P$ which is symmetric. Thus $B' = 0$, since the only simultaneously symmetric and skew-symmetric matrix is the zero matrix. So, B is constant. From $B' = 0$ we obtain

$$X'X'^* + Y'Y'^* - P = 0 \tag{5}$$

Thus P is differentiable and positive semi-definite.

The third differentiation gives us

$$P' = -(BP + PB^*) = PB - BP \tag{6}$$

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Taking the trace in (6) we obtain

$$(\operatorname{tr} P)' = \operatorname{tr}(P') = \operatorname{tr}(PB-BP) = \operatorname{tr}(PB)-\operatorname{tr}(BP) = 0.$$

Thus $\operatorname{tr} P \equiv k$.

Since P satisfies (6), we know that $P(t)=e^{-Bt}Ce^{Bt}$ where C is constant. But B is skew-symmetric, and, thus, C is symmetric since $P(t)$ is.

REMARK. An obvious specialization of the proof of the theorem yields for the corollary an easy and straight forward proof different from any found in [1] or its reference [6].

REFERENCE

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