Proceedings of the Edinburgh Mathematical Society (1989) **32**, 147–156 ©

ON EXTENSIONS OF VALUATIONS TO SIMPLE TRANSCENDENTAL EXTENSIONS

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(Received 1st October 1987)

0. Introduction

Let v_0 be a valuation of a field K_0 with residue field k_0 and value group Z, the group of rational integers. Let $K_0(x)$ be a simple transcendental extension of K_0. In 1936, Maclane [3] gave a method to determine all real valuations V of $K_0(x)$ which are extensions of v_0. But his method does not seem to give an explicit construction of these valuations. In the present paper, assuming K_0 to be a complete field with respect to v_0, we explicitly determine all extensions of v_0 to $K_0(x)$ which have Z as the value group and a simple transcendental extension of k_0 as the residue field. If V is any extension of v_0 to $K_0(x)$ having Z as the value group and a transcendental extension of k_0 as the residue field, then using the Ruled Residue theorem [**4**, **2**, **5**], we give a method which explicitly determines V on a subfield of $K_0(x)$ properly containing K_0.

In Section 1, we prove some results needed for the main results. These results, however, turn out to be of independent interest.

1. Certain extensions of any real valuation to a simple transcendental extension

In this section, v_0 is a real valuation of a field K_0 (not necessarily discrete or complete) with residue field k_0 and $K_0(x)$ is a simple transcendental extension of K_0. We shall denote by V_0 the valuation of $K_0(x)$ defined on $K_0[x]$ by

$$V_0\left(\sum_{i=0}^{n} c_i x^i\right)=\min_i v_0(c_i).$$

Let $P(x)$ be a monic polynomial with coefficients in the valuation ring $\mathfrak{o}$ of v_0 such that the corresponding polynomial $\bar{P}(x)$ with coefficients in the residue field k_0 of v_0 is irreducible over k_0. Let θ be any positive real number. By successive division by powers of $P(x)$, any non-zero polynomial $f(x)$ in $\mathfrak{o}[x]$ can be uniquely represented as

$$f(x)=\sum_{i=0}^{m} f_i(x)P(x)^i$$

where the polynomial $f_i(x)$ in $\mathfrak{o}[x]$ is either zero or has degree less than that of $P(x)$. (The above representation of $f(x)$ will be referred to as the canonical representation of $f(x)$). We define $V_{P(x)}$ on $\mathfrak{o}[x]$ by

$$V_{P(x)}(f(x)) = \min_i (V_0(f_i(x)) + i\theta).$$

We shall soon prove that $V_{P(x)}$ is a valuation of $\mathfrak{o}[x]$. Its unique extension to $K_0(x)$ will also be denoted by $V_{P(x)}$.

Lemma 1. *If a non-zero polynomial $F(x)$ in $\mathfrak{o}[x]$ is written, by the division algorithm, as*

$$F(x) = P(x)q(x) + r(x),$$

then

$$V_0(r(x)) \geqq V_0(F(x))$$

and consequently

$$V_0(q(x)) = V_0(P(x)q(x)) \geqq V_0(F(x)).$$

Proof. This follows at once if we write $F(x) = \alpha F_1(x)$ with α in $\mathfrak{o}$ such that $V_0(F_1(x)) = 0$ and then write by the division algorithm $F_1(x)$ as $P(x)q_1(x) + r_1(x)$.

The following remark follows immediately from above.

Remark 1. With notation as in Lemma 1, $V_0(r(x)) > V_0(F(x))$ if and only if $\bar{P}(x)$ divides $\bar{F}_1(x)$ over k_0, where $F_1(x)$ is any constant multiple of $F(x)$ with $V_0(F_1(x)) = 0$.

Lemma 2. *If $a(x)$ and $b(x)$ are two non-zero polynomials in $\mathfrak{o}[x]$, each of degree less than the degree of $P(x)$, then*

$$V_{P(x)}(a(x)b(x)) = V_{P(x)}(a(x)) + V_{P(x)}(b(x)).$$

Proof. Let α, β be elements of $\mathfrak{o}$ such that

$$a(x) = \alpha a_1(x),\ b(x) = \beta b_1(x),\ V_0(a_1(x)) = V_0(b_1(x)) = 0.$$

On dividing $a_1(x)b_1(x)$ by $P(x)$, we can write

$$a_1(x)b_1(x) = P(x)q_1(x) + r_1(x); \tag{1.1}$$

where either $r_1(x) = 0$ or $\deg r_1(x) < \deg P(x)$. Since $\deg(a_1(x)b_1(x)) < \deg P(x)^2$, therefore $\deg q_1(x) < \deg P(x)$. Also $V_0(r_1(x)) = 0$, i.e. the polynomial $\bar{r}_1(x) \neq 0$ in $k_0[x]$; for other-

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wise by (1.1) $\bar{P}(x)$ will divide $\bar{a}_1(x)\bar{b}_1(x)$ and being irreducible must divide at least one of $\bar{a}_1(x)$ or $\bar{b}_1(x)$, which is impossible in view of the degrees of $\bar{a}_1(x)$ and $\bar{b}_1(x)$. On multiplying (1.1) by $\alpha\beta$, we have

$$a(x)b(x) = P(x)\alpha\beta q_1(x) + \alpha\beta r_1(x).$$

Now by definition of $V_{P(x)}$, we have

$$V_{P(x)}(a(x)b(x)) = \min\{V_0(\alpha\beta q_1(x)) + \theta, V_0(\alpha\beta r_1(x))\}$$

$$= \min\{v_0(\alpha\beta) + V_0(q_1(x)) + \theta, v_0(\alpha\beta)\}$$

$$= v_0(\alpha\beta) = V_{P(x)}(a(x)) + V_{P(x)}(b(x)),$$

and the lemma is proved.

Theorem 1. *$V_{P(x)}$ is a valuation on* $\mathfrak{o}[x]$.

Proof. Let $f(x)$ and $g(x)$ be non-zero polynomials over $\mathfrak{o}$ with canonical representations

$$f(x) = \sum_{i=0}^{m} f_i(x)P(x)^i,\ f_m(x) \neq 0 \tag{1.2}$$

$$g(x) = \sum_{j=0}^{n} g_j(x)P(x)^j,\ g_n(x) \neq 0. \tag{1.3}$$

On adding (1.2) and (1.3) we obtain the canonical representation for $f(x)+g(x)$ and the triangle law, i.e.,

$$V_{P(x)}(f+g) \geqq \min\{V_{P(x)}(f), V_{P(x)}(g)\}$$

follows immediately. Also it is easy to prove using Lemma 2 and the triangle law that

$$V_{P(x)}(fg) \geqq V_{P(x)}(f) + V_{P(x)}(g).$$

Now, it remains to prove that

$$V_{P(x)}(fg) \leqq V_{P(x)}(f) + V_{P(x)}(g). \tag{1.4}$$

Let t and u be the smallest indices such that

$$V_{P(x)}(f(x)) = V_0(f_t(x)) + t\theta,.\ V_{P(x)}(g(x)) = V_0(g_u(x)) + u\theta.$$

Let $r_i(x)$ and $q_i(x)$ be the polynomials over $\mathfrak{o}$ determined by the division algorithm from the following equations.

$$f_0(x)g_0(x)=q_o(x)P(x)+r_0(x)$$

$$f_0(x)g_1(x)+f_1(x)g_0(x)+q_0(x)=q_1(x)P(x)+r_1(x)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$f_m(x)g_n(x)+q_{m+n-1}(x)=q_{m+n}(x)P(x)+r_{m+n}(x).$$

Observe that the degree of each $q_i(x)$ and $r_i(x)$ is less than the degree of $P(x)$. Thus the representation of $f(x)g(x)$ as

$$f(x)g(x)=\sum_{i=0}^{m+n} r_i(x)P(x)^i+q_{m+n}(x)P(x)^{m+n+1}$$

is the canonical representation.

The inequality (1.4) follows at once if we prove that

$$V_0(r_{t+u}(x))=V_0(f_t(x))+V_0(g_u(x)). \tag{1.5}$$

We first show that

$$V_0(f_i(x)g_j(x))>V_0(f_t(x)g_u(x)) \qquad \text{if } i+j<t+u \tag{1.6}$$

and

$$V_0(f_i(x)g_j(x))>V_0(f_t(x)g_u(x)) \qquad \text{if } i+j=t+u, \quad i\neq t. \tag{1.7}$$

Both (1.6) and (1.7) follow immediately from the following observation.

For $0\leqq i\leqq m$, $V_0(f_i)+i\theta\geqq V_0(f_t)+t\theta$ with strict inequality if $i<t$; and for $0\leqq j\leqq n$, $V_0(g_j)+j\theta\geqq V_0(g_u)+u\theta$ with strict inequality if $j<u$.

Define a polynomial $F(x)$ over $\mathfrak{o}$ by

$$F(x)=\sum_{i+j=t+u} f_i(x)g_j(x)+q_{t+u-1}(x).$$

Recall that $q_{t+u}(x)$ and $r_{t+u}(x)$ are respectively the quotient and remainder when $F(x)$ is divided by $P(x)$ i.e.

$$F(x)=q_{t+u}(x)P(x)+r_{t+u}(x).$$

In view of (1.6) and Lemma 1, it is clear that

$$V_0(q_{t+u-1})>V_0(f_tg_u). \tag{1.8}$$

In view of (1.7), we have

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$$V_0\left(\sum_{i+j=t+u} f_i g_j\right) = V_0(f_t g_u);$$

consequently

$$V_0(f_t g_u) = V_0(F(x)). \tag{1.9}$$

Let α and β be elements of $\mathfrak{o}$ such that $f_t(x) = \alpha F_t(x)$, $g_u(x) = \beta G_u(x)$ with $V_0(f_t(x)) = v_0(\alpha)$ and $V_0(g_u(x)) = v_0(\beta)$.

Let $F_1(x)$ be the polynomial over $\mathfrak{o}$ defined by $F(x) = \alpha\beta F_1(x)$. In view of (1.7) and (1.8) it is clear that

$$\bar{F}_1(x) = \bar{F}_t(x)\bar{G}_u(x).$$

Since both $\bar{F}_t(x)$ and $\bar{G}_u(x)$ are of degree less than that of $\bar{P}(x)$, therefore $\bar{P}(x)$ does not divide $\bar{F}_1(x)$. It now follows from equation (1.9), Lemma 1 and the remark following the lemma that

$$V_0(f_t g_u) = V_0(F) = V_0(r_{t+u}),$$

which proves (1.5) and hence completes the proof of the fact that $V_{P(x)}$ is a valuation of $\mathfrak{o}[x]$.

Notation. If α is an element of the valuation ring of a valuation V of a field K, then $\bar{\alpha}$ will denote its image in the residue field of V.

The following theorem determines the residue fields of the valuations V_0 and $V_{P(x)}$. The residue field of V_0 is wellknown [**1**, § 10.2, Prop. 2]. For the sake of completeness we determine it here also.

Theorem 2. *With v_0, k_0, V_0, θ, $V_{P(x)}$ as before and with G_0 as the value group of v_0, we have:*

(i) *The residue field of the valuation V_0 is $k_0(\bar{x})$ with $\bar{x}$ (the image of x in the residue field of V_0) transcendental over k_0.*

(ii) *If θ is free modulo G_0, then the residue field of $V_{P(x)}$ is $k_0(\bar{x})$ with $\bar{x}$ (the image of x in the residue field of $V_{P(x)}$) algebraic over k_0.*

(iii) *If θ is torsion modulo G_0, with s as the smallest positive integer such that $s\theta(=v_0(\alpha))$ is in G_0, then the residue field of $V_{P(x)}$ is $k_0[\bar{x}](t)$ where $t =$ the residue class of $P(x)^s/\alpha$ (in the residue field of $V_{P(x)}$), is transcendental over $k_0[\bar{x}]$ and $\bar{x}$ is algebraic over k_0.*

Proof. In all the three cases, we denote by Δ the residue field of the valuation under consideration and by $\xi=(f(x)/g(x))^{-}$ an arbitrary non-zero element of Δ, with $f(x)$ and $g(x)$ in $\mathfrak{o}[x]$.

(i) It is easy to verify that the image $\bar{x}$ of x in the residue field of V_0 is transcendental over k_0 in this case. Let β be an element of $\mathfrak{o}$ such that

$$V_0(f(x))=V_0(g(x))=v_0(\beta).$$

Then

$$V_0(f(x)/\beta)=0,\ V_0(g(x)/\beta)=0.$$

So $\xi=\xi_1\xi_2^{-1}$ where $\xi_1=(f(x)/\beta)^{-}$ and $\xi_2=(g(x)/\beta)^{-}$ are in $k_0[\bar{x}]$. This proves that $\Delta=k_0(\bar{x})$.

(ii) Let

$$f(x)=\sum_i f_i(x)P(x)^i$$

$$g(x)=\sum_j g_j(x)P(x)^j$$

be the canonical expression for $f(x)$ and $g(x)$ respectively. For non-zero polynomials $f_i(x)$ and $g_j(x)$, define polynomials $F_i(x)$ and $G_j(x)$ with coefficients in $\mathfrak{o}$ by

$$f_i(x)=\beta_iF_i(x),\ g_j(x)=\gamma_jG_j(x)$$

where $v_0(\beta_i)=V_0(f_i(x))$ and $v_0(\gamma_j)=V_o(g_j(x))$. Thus

$$f(x)=\sum_i \beta_iF_i(x)P(x)^i, \tag{1.10}$$

$$g(x)=\sum_j \gamma_jG_j(x)P(x)^j. \tag{1.11}$$

Since $V_{P(x)}(f(x))=V_{P(x)}(g(x))$, i.e., $\min_i(v_0(\beta_i)+i\theta)=\min_j(v_0(\gamma_j)+j\theta)$, therefore there exist subscripts h and k such that

$$v_0(\beta_h)+h\theta=v_0(\gamma_k)+k\theta.$$

In this case, θ being free modulo G_0, the above equality is possible only if $h=k$ and $v_0(\beta_h)=v_0(\gamma_h)$, also $v_0(\beta_i)+i\theta>v_0(\beta_h)+h\theta$ if $i\neq h$ and $v_0(\gamma_j)+j\theta>v_0(\beta_h)+h\theta$ if $j\neq h$. So if we write $\xi_1=f(x)/\beta_hP(x)^h$ and $\xi_2=g(x)/\beta_hP(x)^h$, we have $\xi=\xi_1/\xi_2=\bar{F}_h(\bar{x})((\gamma_h/\beta_h)^{-}\bar{G}_h(\bar{x}))^{-1}$ is in $k_0(\bar{x})$; here $\bar{G}_h(\bar{x})\neq 0$, because the degree of the polynomial $\bar{G}_h(y)$ is less than the degree of $\bar{P}(y)$ which is the minimal polynomial of $\bar{x}$ over k_0. This proves that $\Delta=k_0(\bar{x})$ which is an algebraic extension of k_0.

(iii) Let s be the smallest positive integer such that $s\theta\in G_0$, (say) $s\theta=v_0(\alpha)$ with α in $\mathfrak{o}$.

We first prove that the residue class $(P(x)^s/\alpha)^- = t$ (say) is transcendental over k_0. Suppose t is algebraic over k_0. Let $y^m + \bar{a}_1 y^{m-1} + \cdots + \bar{a}_m$ be a polynomial over k_0 satisfied by t. Therefore

$$V_{P(x)}((P(x)^s/\alpha)^m + a_1(P(x)^s/\alpha)^{m-1} + \cdots + a_m) > 0$$

i.e., if we write

$$F(x) = P(x)^{sm} + a_1 \alpha P(x)^{s(m-1)} + \cdots + a_m \alpha^m \tag{1.12}$$

then

$$V_{P(x)}(F(x)) > v_0(\alpha^m) = ms\theta,$$

which is impossible because (1.12) is a canonical expression for $F(x)$ and by definition of $V_{P(x)}$, we must have $V_{P(x)}(F(x)) \leqq sm\theta$. This contradiction proves that t is transcendental over k_0; in fact t is transcendental over $k_0(\bar{x})$ because $\bar{x}$, satisfying the polynomial $\bar{P}(y)$, is algebraic over k_0.

Let expressions for $f(x)$ and $g(x)$ be as in (1.10) and (1.11). Let β be an element of $\mathfrak{o}$ and h an integer such that

$$v_0(\beta) + h\theta = \min_i (v_0(\beta_i) + i\theta) = \min_j (v_0(\gamma_j) + j\theta).$$

Write $\xi_1 = f(x)/\beta P(x)^h$, $\xi_2 = g(x)/\beta P(x)^h$. Then

$$\bar{\xi}_1 = \sum_i{}' \bar{F}_i(\bar{x})(\beta_i P(x)^i/\beta P(x)^h)^-$$

where the sum $\sum'$ is carried over all those i for which $v_0(\beta_i) + i\theta = v_0(\beta) + h\theta$ (the rest of the terms are zero in the residue field). For each i in $\sum'$, $v_0(\beta_i) + i\theta = v_0(\beta) + h\theta$, i.e., $(i-h)\theta = v_0(\beta) - v_0(\beta_i)$. So $(i-h)$ is an integral multiple of s, say $(i-h) = m_i s$. Therefore the residue class of $\beta_i P(x)^i/\beta P(x)^h = (P(x)^s/\alpha)^{m_i} \cdot (\beta_i \alpha^{m_i}/\beta)$ is an integral power of $t = (P(x)^s/\alpha)^-$ multiplied by an element of k_0. Thus $\bar{\xi}_1$ is in the field $k_0[\bar{x}](t)$. Similarly $\bar{\xi}_2$ and hence $\bar{\xi}$ are in the same field. This proves (iii).

Remark 2. As in [1, § 10.2, Prop. 2], it is easy to prove that if V is a real valuation of $K_0(x)$ extending the valuation v_0 of K_0 with $V(x) = 0$ and if $\bar{x}$ is transcendental over k_0 then $V = V_0$.

2. Construction of extensions of v_0 with residue field $k_0(t)$

In what follows, K_0 is a complete valuation field with respect to a valuation v_0 having the value group Z, the valuation ring $\mathfrak{o}$ and the residue field k_0. As before x is an indeterminate, We shall consider only those extensions V of v_0 to $K_0(x)$ for which $V(x) \geqq 0$.

Theorem 3. *Let V be an extension of v_0 to $K_0(x)$ with value group Z and residue field Δ such that $\Delta \cap$ (the algebraic closure of $k_0) = k_0(\bar{x})$, then $V = V_{P(x)}$ for some monic polynomial $P(x)$ over $\mathfrak{o}$ where $\bar{P}(y)$ is the minimal polynomial of $\bar{x}$ over k_0.*

Proof. Let π be a uniformizer of v_0 in K_0 and $\phi(y)$ be the minimal polynomial of $\bar{x}$ over k_0 of degree n. We claim that there exists a monic polynomial $P(x)$ over $\mathfrak{o}$ with $\bar{P}(x) = \phi(x)$ such that the residue class $(P(x)/\pi^n)^-$ (in the residue field of V) is transcendental over k_0, r being given by $V(P(x)) = r$. Let $P_1(x)$ be any monic polynomial with coefficients in $\mathfrak{o}$ such that $\bar{P}_1(x) = \phi(x)$. Since $\bar{P}_1(\bar{x}) = \phi(\bar{x}) = \bar{0}$, therefore $s_1 = V(P_1(x)) > 0$. If $(P_1(x)/\pi^{s_1})^-$ is transcendental over k_0 then our claim is proved. If it is algebraic over k_0 then by hypothesis there exists a polynomial $f_1(x)$ in $\mathfrak{o}[x]$ of degree $\leqq n-1$ such that

$$(P_1(x)/\pi^{s_1})^- = \bar{f}_1(\bar{x}).$$

So

$$V(P_1(x) - \pi^{s_1} f_1(x)) > s_1.$$

Write $P_2(x) = P_1(x) - \pi^{s_1} f_1(x)$ and define an integer $s_2 > s_1$ by $V(P_2(x)) = s_2$. If $(P_2(x)/\pi^{s_2})^-$ is transcendental over k_0, we stop here otherwise we continue the process. We show that the process cannot continue indefinitely. Suppose it does. So we obtain a sequence of polynomial $f_i(x)$ in $\mathfrak{o}[x]$ each of degree $\leqq n-1$ and a strictly increasing sequence of positive integers $s_1 < s_2 < \cdots$ such that

$$V\left(P_1(x) - \sum_{j=1}^{t} f_j(x)\pi^{s_j}\right) = s_{t+1}.$$

Since K_0 is complete and since each $f_j(x)$ is of degree $\leqq n-1$, therefore $\sum_{j=1}^{\infty} f_j(x)\pi^{s_j}$ is a polynomial over $\mathfrak{o}$ of degree $\leqq n-1$, which we shall denote by $F(x)$. By choice $V(P_1(x) - F(x)) > s_t$ for all t, so $P_1(x) - F(x)$ must be the zero polynomial. Which is impossible because $P_1(x) - F(x)$ is a monic polynomial of degree n over $\mathfrak{o}$. This contradiction proves the claim.

Let $P(x)$ be a monic polynomial over $\mathfrak{o}$ such that $\bar{P}(x) = \phi(x)$, $V(P(x)) = r$ and $(P(x)/\pi^r)^-$ is transcendental over k_0 and hence over $k_0(\bar{x})$. We now prove that the valuation V is nothing but $V_{P(x)}$ with $\theta = V_{P(x)}(P(x)) = r$. Since the minimal polynomial satisfied by $\bar{x}$ over k_0 has degree n, therefore for any polynomial $a(x)$ over $\mathfrak{o}$ of degree $\leqq n-1$, $V(a(x)) = V_0(a(x))$ holds. Let $f(x)$ be any non-zero element of $\mathfrak{o}[x]$ and let

$$f(x) = \sum_{i=1}^{m} f_i(x) P(x)^i \tag{2.1}$$

be the canonical representation of $f(x)$. If $f_i(x) \neq 0$ then $\deg f_i(x) \leqq n-1$. So

$$V(f_i(x)) = V_0(f_i(x)) = a_i, \qquad \text{(say)}.$$

By definition of $V_{P(x)}$, we have

$$V_{P(x)}(f(x)) = \min_i (a_i + ir)$$

where the minimum is carried over those i for which $f_i(x) \neq 0$. We shall prove that

$$V(f(x)) = \min_i (a_i + ir) = V_{P(x)}(f(x)). \tag{2.2}$$

Since $V(f_i(x)P(x)^i) = a_i + ir$, it follows from (2.1) that

$$V(f(x)) \geqq \min_i (a_i + ir). \tag{2.3}$$

We now prove (2.2). Let h (be the smallest subscript such that $\min_i(a_i + ir) = a_h + hr$. For a non-zero polynomial $f_i(x)$ define $F_i(x)$ in $\mathfrak{o}[x]$ by $f_i(x) = \pi^{a_i} F_i(x)$, so that $V_0(F_i(x)) = V(F_i(x)) = 0$. Suppose strict inequality holds in (2.3). Then there exist positive integers $h = h_0 < h_1 < \cdots < h_l \leqq m$, such that

$$V\left(\sum_{i=0}^{l} \pi^{a_{h_i}} F_{h_i}(x) P(x)^{h_i}\right) > a_h + hr \tag{2.4}$$

and

$$V(\pi^{a_{h_i}} F_{h_i}(x) P(x)^{h_i}) = a_h + hr \tag{2.5}$$

for $0 \leqq i \leqq l$. It follows from (2.5) that

$$(a_h - a_{h_i}) = (h_i - h_0)r = n_i r, \qquad \text{(say)} \tag{2.6}$$

for $1 \leqq i \leqq l$. Observe that $n_1 < n_2 < \cdots < n_l$. It follows at once from (2.4) and (2.6) that

$$V\left(F_{h_0}(x) + \sum_{i=1}^{l} (P(x)/\pi^r)^{n_i} F_{h_i}(x)\right) > 0$$

which shows that $(P(x)/\pi^r)^-$ is algebraic over $k_0[\bar{x}]$. This contradiction proves that equality holds in (2.3) and hence $V = V_{P(x)}$.

Remark 3. If V is as in the above theorem then we have shown in Theorem 2, part (iii) of Section 1 that the residue field of V is a simple transcendental extension of $k_0(\bar{x})$.

Theorem 4. *Let V be an extension of v_0 to $K_0(\bar{x})$ with value group Z and residue field a simple transcendental extension of k_0 then either $V = V_0$ or $V = V_{P(x)}$ for some monic linear polynomial $P(x)$ over $\mathfrak{o}$.*

Proof. If $\bar{x}$ is transcendental over k_0 then by the remark in the end of Section 1, $V=V_0$. Suppose now that $\bar{x}$ is algebraic over k_0, therefore $\bar{x}$ is in k_0. So the minimal polynomial of $\bar{x}$ over k_0 is a linear polynomial. The desired assertion now follows immediately from Theorem 3.

3. Method of construction of valuations with residue field transcendental over k_0

Notation and assumptions are as in the previous section. Now we assume that k_0 is a perfect field. Let V be an extension of v_0 to $K_0(x)$ with value group Z and residue field Δ transcendental over k_0. By the Ruled Residue theorem [5], there exists a finite extension k_1 of k_0 such that $\Delta=k_1(t)$ with t transcendental over k_0. If $k_1=k_0$, then by Theorem 4, either $V=V_0$ or $V=V_{P(x)}$ where $P(x)$ is a linear polynomial over $\mathfrak{o}$. Suppose now that k_1 is a proper extension of k_0. Since k_0 is perfect, therefore there exists y in $K_0(x)$ such that $V(y)=0$ and $k_1=k_0(\bar{y})$. Then y does not belong to K_0, so $K_0(x)$ is a finite extension of $K_0(y)$. Let V_1 denote the restriction of V to $K_0(y)$. The hypotheses of Theorem 3 are clearly satisfied for the valuation V_1 of $K_0(y)$, so by this theorem $V_1=V_{P(y)}$ for some monic polynomial $P(y)$ with coefficients in $\mathfrak{o}$. Thus the valuation V is completely determined on $K_0(y)$.

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