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COMBINATORIAL GELFAND MODELS FOR SOME SEMIGROUPS AND q-ROOK MONOID ALGEBRAS

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Abstract Inspired by the results of Adin, Postnikov and Roichman, we propose combinatorial Gelfand models for semigroup algebras of some finite semigroups, which include the symmetric inverse semigroup, the dual symmetric inverse semigroup, the maximal factorizable subsemigroup in the dual symmetric inverse semigroup and the factor power of the symmetric group. Furthermore, we extend the Gelfand model for the semigroup algebras of the symmetric inverse semigroup to a Gelfand model for the q-rook monoid algebra.

Keywords: Gelfand model; rook monoid; inverse semigroup; involution

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1. Introduction

Let A be a finite-dimensional unital associative algebra over $\mathbb{C}$ and let M be an A-module. The module M is said to be a *Gelfand model* for A if it is isomorphic to a multiplicity-free direct sum of all simple A-modules. This paper is inspired by the results of [**1**], where beautiful combinatorial Gelfand models for the group algebra $\mathbb{C}S_n$ of the symmetric group, and for the corresponding Hecke algebra $\boldsymbol{H}_n(q)$, are constructed. We refer the reader to the list of references in [**1**] for the history of the problem and an account of known Gelfand models.

The aim of this paper is to extend the results of [**1**] to some classes of finite semigroups, which include several inverse generalizations of the symmetric group (in particular, the full symmetric inverse semigroup $\mathcal{IS}_n$) and to the Hecke algebra analogue for $\mathcal{IS}_n$, known as the q-rook monoid algebra. The latter has recently been defined by Solomon [**17**]; however, a special case has already appeared in [**16**]. The q-rook monoid algebra has been studied by several authors (see [**3**, **9**, **10**, **14**, **17**] and references therein). Our motivation

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comes from an attempt to better understand the connection between the combinatorial and representation theoretical properties of these objects.

The paper is organized as follows. In § 2 we recall the combinatorial Gelfand model for $\mathbb{C}S_n$, constructed in [**1**]. In § 3 we show how the latter model can be used to construct combinatorial Gelfand models for semigroup algebras of all finite semigroups, for which each trace of a regular $\mathcal{D}$-class is an inverse semigroup in which maximal subgroups are direct sums of symmetric groups. Examples of such semigroups include the symmetric inverse semigroup [**7**, 2.5], the dual symmetric inverse semigroup [**4**] and the maximal factorizable subsemigroup in the dual symmetric inverse semigroup [**4**]. Another, rather surprising, natural example is the factor power of the symmetric group [**5**], which, in particular, is not even regular. In § 4 we recall (an appropriate modification of) the combinatorial Gelfand model for the Hecke algebra $\boldsymbol{H}_n(q)$, constructed in [**1**]. Finally, in § 5 we extend the latter model to a combinatorial Gelfand model for the q-rook monoid algebra $\boldsymbol{I}_n(q)$ from [**17**]. For $\boldsymbol{I}_n(q)$ we use the presentation from [**9**], which is different from that used in [**17**].

2. Combinatorial Gelfand model for $\mathbb{C}S_n$

Let S_n be the symmetric group on $\{1, 2, \dots, n\}$ and let $\mathcal{I}_n$ be the set of all involutions in S_n (recall that $\pi \in S_n$ is an involution provided that $\pi^2 = \mathrm{id}$, in particular, the identity element id itself is an involution). For $\pi \in S_n$ we define the *inversion set* of π as follows:

$$\mathrm{Inv}(\pi) = \{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}.$$

For $w \in \mathcal{I}_n$ set

$$\mathrm{Pair}(w) = \{(i, j) : i < j \text{ and } w(i) = j\}.$$

Set $\mathrm{Inv}_w(\pi) = \mathrm{Inv}(\pi) \cap \mathrm{Pair}(w)$ and $\mathrm{inv}_w(\pi) = |\mathrm{Inv}_w(\pi)|$. Finally, let V_n be the vector space with the basis $\{I_w : w \in \mathcal{I}_n\}$.

Theorem 2.1 (Adin *et al.* [1]). *The assignment*

$$\pi \cdot I_w = (-1)^{\mathrm{inv}_w(\pi)} I_{\pi w \pi^{-1}}, \quad \pi \in S_n,\ w \in \mathcal{I}_n,$$

defines on V_n the structure of a $\mathbb{C}S_n$-module. Moreover, this module is a Gelfand model for $\mathbb{C}S_n$.

Remark 2.2. Theorem 2.1 can be extended to direct sums of symmetric groups in a straightforward way.

Remark 2.3. For $k = 0, 1, \dots, \lfloor \frac{1}{2}n \rfloor$ let $\mathcal{I}_n^k$ denote the subset of $\mathcal{I}_n$ consisting of all involutions, which can be written as a product of exactly k pairwise different and commuting transpositions. We obviously have that $\mathcal{I}_n$ is a disjoint union of the $\mathcal{I}_n^k$s. Moreover, the linear span V_n^k of $\{I_w : w \in \mathcal{I}_n^k\}$ is invariant under the $\mathbb{C}S_n$-action for every k. The Robinson–Schensted correspondence [**15**, 3.1] assigns to each $\pi \in S_n$ a pair $(a(\pi), b(\pi))$ of standard Young tableaux of the same shape. Moreover, $\pi \in S_n$ is an involution if and only if $a(\pi) = b(\pi)$ [**15**, Theorem 3.6.6]. Using the properties of Viennot's shadow diagrams (see [**15**, 3.6]) one may show that two elements $w, w' \in \mathcal{I}_n$ belong to the same $\mathcal{I}_n^k$ provided that $a(w)$ and $a(w')$ have the same shape. Using the main result of [**12**] and tensoring with the sign representation, one may further show that V_n^k is isomorphic to the direct sum of Specht modules S^λ, where λ runs through the set of all shapes of $a(w)$ for $w \in \mathcal{I}_n^k$.

3. Combinatorial Gelfand models for semigroup algebras of some finite semigroups

We use [**7, 11**] as general references for standard notions from semigroup theory. Let S be a finite semigroup and let $E(S)$ be its set of idempotents. For $e \in E(S)$ consider the $\mathcal{D}$-class D_e containing e. Then $\mathrm{D}_e \cup \{0\}$ with multiplication given by

$$a \star b = \begin{cases} ab, & ab \in \mathrm{D}_e, \\ 0, & \text{otherwise}, \end{cases}$$

is called the *trace* of D_e. From now on we assume that, for every $e \in E(S)$,

- the trace $\mathrm{D}_e \cup \{0\}$ is an inverse semigroup,
- the maximal subgroup G_e of S, corresponding to e, is a direct sum of symmetric groups.

Let $e_1, \dots, e_k \in E(S)$ be a fixed collection of idempotents, one for each $\mathcal{D}$-class. Furthermore, let m_i, $i = 1, \dots, k$, denote the number of $\mathcal{L}$-classes inside D_{e_i}. For each $i = 1, \dots, k$ we fix an isomorphism of the group G_{e_i} with

$$S_{n_1^{(i)}} \oplus \cdots \oplus S_{n_{l_i}^{(i)}}$$

and an isomorphism φ_i of $\mathrm{D}_{e_i} \cup \{0\}$ with the Brandt semigroup associated with the group

$$S_{n_1^{(i)}} \oplus \cdots \oplus S_{n_{l_i}^{(i)}}$$

and the cardinality m_i (for details see [**2**, § 3.3]). This means that we have $\varphi_i(0) = 0$ and for any $x \in \mathrm{D}_{e_i}$ we have $\varphi_i(x) = (a, y, b)$, where

$$y \in S_{n_1^{(i)}} \oplus \cdots \oplus S_{n_{l_i}^{(i)}},$$

and $a, b \in \{1, \dots, m_i\}$. The multiplication in the Brandt semigroup is given by

$$(a, y, b) \star (a', y', b') = \begin{cases} (a, yy', b'), & b = a', \\ 0, & \text{otherwise}. \end{cases}$$

We set $\bar{\varphi}_i(x) = y$.

An element $w \in S$ will be called an *involution* provided that $w \in G_e$ for some $e \in E(S)$ and $w^2 = e$. Let $\mathcal{I}_S$ denote the set of all involutions in S, and let V_S denote the vector space with the basis $\{l_w : w \in \mathcal{I}_S\}$. Our first result is the following.

Theorem 3.1. *Let $x \in S$ and $w \in \mathcal{I}_S$ be such that $w \in G_e$ for some $e \in E(S)$ and $e \in \mathrm{D}_{e_i}$ for some $i \in \{1, 2, \dots, k\}$. Then the assignment*

$$x \cdot I_w = \begin{cases} (-1)^{\mathrm{inv}_{\bar{\varphi}_i(w)}(\bar{\varphi}_i(xe))} I_{(xe)w(xe)^{-1}}, & xe \in \mathrm{D}_e, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

defines on V_S the structure of a $\mathbb{C}S$-module. Moreover, this module is a Gelfand model for $\mathbb{C}S$.

Proof. Let $x, y \in S$ be such that $x \cdot (y \cdot I_w) = 0$. We show that in this case we also have $xy \cdot I_w = 0$. Suppose first that $y \cdot I_w = 0$. Let $\leqslant_{\mathcal{J}}$ denote the natural partial order on S associated with Green's $\mathcal{J}$-relation. Then $ye <_{\mathcal{J}} e$ and thus $xye \leqslant_{\mathcal{J}} ye <_{\mathcal{J}} e$, which means that $xy \cdot I_w = 0$. Suppose now that $y \cdot I_w \neq 0$, but $x \cdot (y \cdot I_w) = 0$. Set $v = (ye)w(ye)^{-1}$. Let f be an idempotent such that $v \in G_f$. Then $x \cdot I_v = 0$, which implies that $xf <_{\mathcal{J}} f$. The definition of f yields that $fye\mathcal{R}ye$, which in turn gives us that $fye = ye$. Thus, we have

$$xye = xfye \leqslant_{\mathcal{J}} xf <_{\mathcal{J}} f.$$

It follows that $xy \cdot I_w = 0$ as well.

Now let $x, y \in S$ be such that $x \cdot (y \cdot I_w) \neq 0$. Define $G = G_{e_i}$ and set $\bar{x} = \bar{\varphi}_i(x)$ for every $x \in \mathrm{D}_{e_i}$. Without loss of generality we will assume that $\varphi_i(G_e) = (1, G, 1)$ and that $\bar{g} = g$ whenever $g \in G_e$. Since $y \cdot I_w \neq 0$, it follows that $ye\mathcal{J}e$ and thus $ye\mathcal{L}e$. In particular, $\varphi_i(ye) = (l, \overline{ye}, 1)$ for some l. Then we have $\varphi_i((ye)^{-1}) = (1, \overline{ye}^{-1}, l)$, where $\overline{ye}^{-1}$ is the inverse of $\overline{ye}$ in G. Let $f = (ye)(ye)^{-1}$ and $v = (ye)w(ye)^{-1}$. Then $\varphi_i(f) = (l, e_i, l)$ and $\varphi_i(v) = (l, \overline{ye}w\overline{ye}^{-1}, l)$. Since $x \cdot I_v \neq 0$, it follows that $xf\mathcal{J}f$ and thus $xf\mathcal{L}f$. Set $u = (xf)(ye)w(ye)^{-1}(xf)^{-1}$. Applying the equality $xfye = xye$, we show that $\varphi_i(u) = (k, \overline{xye}w\overline{xye}^{-1}, k)$ for some k.

Now the first part of the proof amounts to checking the equality

$$(-1)^{\mathrm{inv}_w(\overline{ye})} \cdot (-1)^{\mathrm{inv}_{\overline{ye}w\overline{ye}^{-1}}(\overline{xf})} = (-1)^{\mathrm{inv}_w(\overline{xye})}. \tag{3.2}$$

Let $(i, j) \in \mathrm{Pair}(w)$. Suppose that $(i, j) \in \mathrm{Inv}_w(\overline{ye})$, that is $i < j$ and $\overline{ye}(i) > \overline{ye}(j)$. If $\overline{xf}(\overline{ye}(i)) < \overline{xf}(\overline{ye}(j))$, then we have that $(\overline{ye}(j), \overline{ye}(i))$ belongs to $\mathrm{Inv}_{\overline{ye}w\overline{ye}^{-1}}(\overline{xf})$, and at the same time $(i, j) \notin \mathrm{Inv}_w(\overline{xye})$. If $\overline{xf}(\overline{ye}(i)) > \overline{xf}(\overline{ye}(j))$, then $(\overline{ye}(j), \overline{ye}(i))$ does not belong to $\mathrm{Inv}_{\overline{ye}w\overline{ye}^{-1}}(\overline{xf})$, and at the same time $(i, j) \in \mathrm{Inv}_w(\overline{xye})$. Analogously, we consider the case $(i, j) \notin \mathrm{Inv}_w(\overline{ye})$, and (3.2) follows. Therefore, V_S is indeed a $\mathbb{C}S$-module.

We are left to show that V_S is a Gelfand model for S. We will use the fact that simple modules over the complex semigroup algebra of a finite semigroup S are in bijective correspondence with simple modules of G_{e_i}, $1 \leqslant i \leqslant k$ (see [**2**, Chapter 5] or [**8**, Theorem 7] for a more modern approach). In view of this, it is sufficient to show that, for a maximal subgroup G of S and a simple $\mathbb{C}G$-module V, the corresponding $\mathbb{C}S$-module V' is isomorphic to a submodule of V_S, and then to make sure that the sum of the dimensions of all V's equals the dimension of V_S.

Let $1 \leqslant i \leqslant k$ and $\mathcal{I}_{n,i}$ be the set of involutions contained in maximal subgroups of the $\mathcal{D}$-class D_{e_i}. Then the linear span V_S^i of all I_w, $w \in \mathcal{I}_{n,i}$, is a direct summand of V_S. The dimension of this direct summand equals $m_i \cdot |\{I_w : w \in \mathcal{I}_{G_{e_i}}\}|$. The action of G_{e_i} on the linear span $V(G_{e_i})$ of $\{I_w : w \in \mathcal{I}_{G_{e_i}}\}$, coincides with the action from [**1**], and thus from [**1**, Theorem 1.1.2] and Remark 2.2 it follows that $V(G_{e_i})$ is a multiplicity-free direct sum of all simple G_{e_i}-modules. Let V be a simple direct summand of $V(G_{e_i})$ (as a G_{e_i}-module). Suppose that the image of G_{e_i} under φ_i is $(1, G_{e_i}, 1)$, thus identifying V with $(1, V, 1)$ (the latter is a subalgebra of the semigroup algebra of the Brandt semigroup we work with). Then the vector space

$$\hat{V} = \bigoplus_{k=1}^{m_i} (k, V, 1)$$

is a simple S-module, corresponding to V, and by construction is a direct summand of V_S^i. We have

$$\dim(\hat{V}) = m_i \cdot \dim(V)$$

and hence V_S^i is isomorphic to the multiplicity-free direct sum of all $\hat{V}$, where V runs through the set of all simple G_{e_i}-modules. This completes the proof. □

Theorem 3.1 applies to many semigroups; some examples follow.

- The symmetric inverse semigroup $\mathcal{IS}_n$ of all partial injections on $\{1, 2, \dots, n\}$ (also called the rook monoid) [**7**, 2.5]: the conditions are satisfied because of [**7**, 2.6 and 5.1].
- The dual symmetric inverse semigroup $\mathcal{I}_n^*$ (or the monoid of block bijections) from [**4**]: the conditions are satisfied because of [**4**, Theorem 2.2].
- The maximal factorizable submonoid of $\mathcal{I}_n^*$ (or the monoid of uniform block bijections) from [**4**]: the conditions are satisfied because of [**4**, § 3].
- The factor power $\mathcal{FP}^+(S_n)$ from [**5**,**6**]: unlike the previous examples, this semigroup is *not* inverse; moreover, it is not even regular. However, all the required conditions are satisfied because of [**6**, Theorem 1] and [**13**].

4. Combinatorial Gelfand model for the Hecke algebra

For a permutation $\pi \in S_n$, define the *support* of π as follows:

$$\mathrm{supp}(\pi) = \{x \in \{1, 2, \dots, n\} : \pi(x) \neq x\}.$$

For $1 \leqslant i < n$ let s_i denote the transposition $(i, i+1)$.

For $q \in \mathbb{C}^*$ consider the *Hecke algebra* $\boldsymbol{H}_n(q)$, which is a $\mathbb{C}$-algebra with generators $\{T_i : 1 \leqslant i < n\}$ and defining relations

$$(T_i - q)(T_i + 1) = 0, \qquad 1 \leqslant i < n, \tag{4.1}$$

$$T_i T_j = T_j T_i, \qquad 1 \leqslant i < j - 1 < n - 1, \tag{4.2}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leqslant i < n - 1. \tag{4.3}$$

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We have $\boldsymbol{H}_n(1) \cong \mathbb{C}S_n$ canonically. Let $V_{n,q}$ denote the formal linear span of $\{I_w : w \in \mathcal{I}_n\}$. The following theorem is a slightly modified version of [**1**, Theorem 1.2.2], which is better suited to our purposes.

Theorem 4.1. *Let $1 \leqslant i < n$ and $w \in S_n$ be an involution. The assignment*

$$T_i \cdot I_w = \begin{cases} qI_w, & i, i+1 \notin \operatorname{supp}(w), \\ -I_w, & i, i+1 \in \operatorname{supp}(w), \\ I_{s_i w s_i}, & i \in \operatorname{supp}(w),\ i+1 \notin \operatorname{supp}(w), \\ qI_{s_i w s_i} + (q-1)I_w, & i \notin \operatorname{supp}(w),\ i+1 \in \operatorname{supp}(w), \end{cases} \tag{4.4}$$

defines on $V_{n,q}$ the structure of an $\boldsymbol{H}_n(q)$-module. If we additionally assume that q is not a root of unity, then $V_{n,q}$ is a Gelfand model for $\boldsymbol{H}_n(q)$.

5. Combinatorial Gelfand model for the q-rook monoid algebra

For $q \in \mathbb{C}^*$ the *q-rook monoid algebra* $\boldsymbol{I}_n(q)$ is defined [**9**] as a $\mathbb{C}$-algebra with generators $\{T_i : 1 \leqslant i < n\} \cup \{P_i : 1 \leqslant i \leqslant n\}$, defining relations (4.1)–(4.3) and

$$T_i P_j = P_j T_i = qP_j, \qquad 1 \leqslant i < j \leqslant n, \tag{5.1}$$

$$T_i P_j = P_j T_i, \qquad 1 \leqslant j < i \leqslant n-1, \tag{5.2}$$

$$P_i^2 = P_i, \qquad 1 \leqslant i \leqslant n, \tag{5.3}$$

$$P_{i+1} = P_i T_i P_i - (q-1)P_i, \quad 1 \leqslant i \leqslant n-1. \tag{5.4}$$

This is not a semigroup algebra of some 'q-rook monoid' but rather a one-parameter (q-)deformation of the semigroup algebra of the rook monoid; in particular, $\boldsymbol{I}_n(1) \cong \mathbb{C}\mathcal{IS}_n$ canonically. For generic q there is a (non-canonical) isomorphism $\boldsymbol{I}_n(q) \cong \mathbb{C}\mathcal{IS}_n$ [**14**, **16**, **17**].

To proceed, we need to fix some notation. Let $X = \{1, 2, \ldots, n\}$. $\mathcal{IS}_n$ acts on X in the standard way by partial permutations. For a partial transformation α we denote by $\operatorname{dom}(\alpha)$ the *domain* of α. For a subset A of X denote by e_A the identity transformation of A, and by $G(A)$ the $\mathcal{H}$-class of e_A, which consists of all $\pi \in \mathcal{IS}_n$ whose domains and images are equal to A.

Set $V = V_q^{\mathcal{IS}_n}$ to be the vector space with the basis I_w, where w is an involution of $\mathcal{IS}_n$.

Let $A \subset X$. For every $\pi \in G(A)$ define $\psi_A(\pi) \in S_n$ as the element whose action on A coincides with that of π, and which acts identically on the set $X \setminus A$. The map ψ_A gives rise to a monomorphism $\bar{\psi}_A$ from the linear span of $\{I_w : w \in \mathcal{I}_{G(A)}\}$ to $V_{n,q}$ defined on the basis via $\bar{\psi}_A(I_w) = I_{\psi(w)}$. If $i, i+1 \in A$ and $w \in G(A)$ is an involution, we define

$$T_i \circ I_w = \bar{\psi}_A^{-1}(T_i \cdot \bar{\psi}_A(I_w)),$$

where the action $T_i \cdot I_{\psi_A(w)}$ is given by (4.4).

Now, for every generator T_i, $1 \leqslant i \leqslant n-1$, and P_i, $1 \leqslant i \leqslant n$, of $\boldsymbol{I}_n(q)$ we define a linear transformation of V as follows:

$$T_i \cdot \mathsf{l}_w = \begin{cases} T_i \circ \mathsf{l}_w, & i, i+1 \in \operatorname{dom}(w), \\ q\mathsf{l}_w, & i, i+1 \notin \operatorname{dom}(w), \\ \mathsf{l}_{s_i w s_i}, & i \in \operatorname{dom}(w),\ i+1 \notin \operatorname{dom}(w), \\ q\mathsf{l}_{s_i w s_i} + (q-1)\mathsf{l}_w, & i \notin \operatorname{dom}(w),\ i+1 \in \operatorname{dom}(w), \end{cases} \tag{5.5}$$

$$P_i \cdot \mathsf{l}_w = \begin{cases} \mathsf{l}_w, & \operatorname{dom}(w) \subset \{i+1, \dots, n\}, \\ 0, & \operatorname{dom}(w) \not\subset \{i+1, \dots, n\}. \end{cases} \tag{5.6}$$

Theorem 5.1. *The assignments (5.5) and (5.6) define on V the structure of an $\boldsymbol{I}_n(q)$-module. If we additionally assume that q is not a root of unity, then V is a Gelfand model for $\boldsymbol{I}_n(q)$.*

To prove the theorem we need some preparation. The group S_n acts on $\mathcal{I}_{\mathcal{IS}_n}$ by conjugation. This action gives rise to an action of S_n on V defined as follows: $\pi \mathsf{l}_w \pi^{-1} = \mathsf{l}_{\pi w \pi^{-1}}$, $w \in \mathcal{I}_{\mathcal{IS}_n}$. We will need the following technical lemma.

Lemma 5.2. *Let $w \in \mathcal{I}_{\mathcal{IS}_n}$.*

(i) *If $i, i+1 \in \operatorname{dom}(w)$, then $\pi(T_i \circ \mathsf{l}_w)\pi^{-1} = T_{i+1} \circ \mathsf{l}_{\pi w \pi^{-1}}$ for any $\pi \in S_n$ such that $\pi(k) = k+1$ for $k = i, i+1$.*

(ii) *If $i, i+1 \in \operatorname{dom}(w)$ and $|j-i| > 1$, then $s_j(T_i \circ \mathsf{l}_w)s_j = T_i \circ \mathsf{l}_{s_j w s_j}$.*

Proof. Let $\psi = \psi_{\operatorname{dom}(w)}$ and $\tau = \psi_{\operatorname{dom}(\pi w \pi^{-1})}$. Applying the definition of $\circ$, one reduces the first equality to

$$\pi(\bar{\psi}^{-1}(T_i \cdot \bar{\psi}(\mathsf{l}_w)))\pi^{-1} = \bar{\tau}^{-1}(T_{i+1} \cdot \bar{\tau}(\mathsf{l}_{\pi w \pi^{-1}})).$$

Observe that for $k = i, i+1$ we have $k \in \operatorname{supp}(\psi(w))$ if and only if $k+1 \in \operatorname{supp}(\tau(\pi w \pi^{-1}))$. It follows that if $T_i \cdot (\bar{\psi}(\mathsf{l}_w))$ is a linear combination of some $\mathsf{l}_{\psi(u)}$s, then $T_{i+1} \cdot (\bar{\tau}(\mathsf{l}_{\pi w \pi^{-1}}))$ is the same linear combination of the corresponding $\mathsf{l}_{\tau(\pi u \pi^{-1})}$s. Hence, we are left to check the equality

$$\pi(\bar{\psi}^{-1}(\mathsf{l}_{\psi(u)}))\pi^{-1} = \bar{\tau}^{-1}(\mathsf{l}_{\tau(\pi u \pi^{-1})}).$$

The latter equality reduces to $\pi \mathsf{l}_u \pi^{-1} = \mathsf{l}_{\pi u \pi^{-1}}$, which follows from the definitions. This proves (i).

To prove (ii) we set $\psi = \psi_{\operatorname{dom}(w)}$ and $\tau = \psi_{\operatorname{dom}(s_j w s_j)}$. The required equality reduces to

$$s_j(\bar{\psi}^{-1}(T_i \cdot \bar{\psi}(\mathsf{l}_w)))s_j = \bar{\tau}^{-1}(T_i \cdot \bar{\tau}(\mathsf{l}_{s_j w s_j})).$$

Observe that for $k = i, i+1$ we have $k \in \operatorname{supp}(\psi(w))$ if and only if $k \in \operatorname{supp}(\tau(s_j w s_j))$. It follows that if $T_i \cdot (\bar{\psi}(\mathsf{l}_w))$ is a linear combination of some $\mathsf{l}_{\psi(u)}$s, then $T_i \cdot (\bar{\tau}(\mathsf{l}_{s_j w s_j}))$ is the same linear combination of the corresponding $\mathsf{l}_{\tau(s_j u s_j)}$s. Hence, we are left to check the equality

$$s_j(\bar{\psi}^{-1}(\mathsf{l}_{\psi(u)}))s_j = \bar{\tau}^{-1}(\mathsf{l}_{\tau(s_j u s_j)}).$$

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This reduces to $s_j I_u s_j = I_{s_j u s_j}$, which follows from the definitions. This completes the proof. □

Proof of Theorem 5.1. First we show that V is indeed an $\boldsymbol{I}_n(q)$-module. For this we have to check the defining relations.

Relation (4.1)

Observe that with respect to the fixed basis of V the matrix corresponding to the action of T_i is a direct sum of blocks of three possible types: (q), (-1) and

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}.$$

Each of these blocks satisfies (4.1).

Relation (4.2)

There are 16 possible cases depending on whether or not each of the elements $i, i+1, j, j+1$ belongs to $\mathrm{dom}(w)$. The cases where $i, i+1 \notin \mathrm{dom}(w)$ or $j, j+1 \notin \mathrm{dom}(w)$ are trivial since the action of T_i or T_j just multiplies the vectors I_w, $I_{s_i w s_i}$ or $I_{s_j w s_j}$, respectively, by q.

As i and j appear in (4.2) symmetrically, we are left to consider six cases. They all follow by a routine calculation using Theorem 4.1 and Lemma 5.2 (ii), so we present only the most complicated case and leave the rest to the reader: let $i+1, j+1 \in \mathrm{dom}(w)$, $i, j \notin \mathrm{dom}(w)$. We have

$$I_w \xmapsto{T_i} qI_{s_i w s_i} + (q-1)I_w \xmapsto{T_j} q^2 I_{s_j s_i w s_i s_j} + q(q-1)I_{s_i w s_i} + q(q-1)I_{s_j w s_j} + (q-1)^2 I_w,$$
$$I_w \xmapsto{T_j} qI_{s_j w s_j} + (q-1)I_w \xmapsto{T_i} q^2 I_{s_i s_j w s_j s_i} + q(q-1)I_{s_j w s_j} + q(q-1)I_{s_i w s_i} + (q-1)^2 I_w,$$

and the claim follows as $s_j s_i = s_i s_j$.

Relation (4.3)

We consider eight possible cases depending on whether or not the elements $i, i+1, i+2$ belong to $\mathrm{dom}(w)$.

Case 1 ($\boldsymbol{i, i+1, i+2 \in \mathrm{dom}(w)}$). This follows immediately from Theorem 4.1.

Case 2 ($\boldsymbol{i, i+1 \in \mathrm{dom}(w),\ i+2 \notin \mathrm{dom}(w)}$). We have

$$I_w \xmapsto{T_i} T_i \circ I_w \xmapsto{T_{i+1}} s_{i+1}(T_i \circ I_w)s_{i+1} \xmapsto{T_i} s_i s_{i+1}(T_i \circ I_w)s_{i+1}s_i,$$
$$I_w \xmapsto{T_{i+1}} I_{s_{i+1} w s_{i+1}} \xmapsto{T_i} I_{s_i s_{i+1} w s_{i+1} s_i} \xmapsto{T_{i+1}} T_{i+1} \circ I_{s_i s_{i+1} w s_{i+1} s_i},$$

and the claim follows applying Lemma 5.2 (i) for $\pi = s_i s_{i+1}$.

Case 3 ($i, i+2 \in \mathrm{dom}(w)$, $i+1 \notin \mathrm{dom}(w)$). We have

$$I_w \xmapsto{T_i} I_{s_i w s_i} \xmapsto{T_{i+1}} T_{i+1} \circ I_{s_i w s_i} \xmapsto{T_i} q s_i (T_{i+1} \circ I_{s_i w s_i}) s_i + (q-1) T_{i+1} \circ I_{s_i w s_i}, \tag{5.7}$$

$$\begin{aligned} I_w &\xmapsto{T_{i+1}} q I_{s_{i+1} w s_{i+1}} + (q-1) I_w \xmapsto{T_i} q T_i \circ I_{s_{i+1} w s_{i+1}} + (q-1) I_{s_i w s_i} \\ &\xmapsto{T_{i+1}} q s_{i+1} (T_i \circ I_{s_{i+1} w s_{i+1}}) s_{i+1} + (q-1) T_{i+1} \circ I_{s_i w s_i}. \end{aligned} \tag{5.8}$$

The claim now follows from the equality

$$s_i s_{i+1} (T_i \circ I_{s_{i+1} w s_{i+1}}) s_{i+1} s_i = T_{i+1} \circ I_{s_i w s_i},$$

which in turn follows from Lemma 5.2 (i).

Case 4 ($i+1, i+2 \in \mathrm{dom}(w)$, $i \notin \mathrm{dom}(w)$). We have

$$\begin{aligned} I_w &\xmapsto{T_i} q I_{s_i w s_i} + (q-1) I_w \xmapsto{T_{i+1}} q^2 I_{s_{i+1} s_i w s_i s_{i+1}} + q(q-1) I_{s_i w s_i} + (q-1) T_{i+1} \circ I_w \\ &\xmapsto{T_i} q^2 T_i \circ I_{s_{i+1} s_i w s_i s_{i+1}} + q(q-1) I_w + (q-1) q s_i (T_{i+1} \circ I_w) s_i + (q-1)^2 T_{i+1} \circ I_w \\ &=: A + B + C + D, \end{aligned} \tag{5.9}$$

$$\begin{aligned} I_w &\xmapsto{T_{i+1}} T_{i+1} \circ I_w \xmapsto{T_i} q s_i (T_{i+1} \circ I_w) s_i + (q-1) T_{i+1} \circ I_w \\ &\xmapsto{T_{i+1}} q^2 s_{i+1} s_i (T_{i+1} \circ I_w) s_i s_{i+1} + q(q-1) s_i (T_{i+1} \circ I_w) s_i + (q-1) T_{i+1} \circ (T_{i+1} \circ I_w) \\ &=: E + F + G. \end{aligned} \tag{5.10}$$

We have $C = F$. Furthermore, $B + D = G$ follows from (4.1). Finally, $A = E$ follows from Lemma 5.2 (i), applying arguments similar to those used in the previous case.

Case 5 ($i+2 \in \mathrm{dom}(w)$, $i, i+1 \notin \mathrm{dom}(w)$). We have

$$\begin{aligned} I_w &\xmapsto{T_i} q I_w \xmapsto{T_{i+1}} q^2 I_{s_{i+1} w s_{i+1}} + q(q-1) I_w \\ &\xmapsto{T_i} q^3 I_{s_i s_{i+1} w s_{i+1} s_i} + q^2 (q-1) I_{s_{i+1} w s_{i+1}} + q^2 (q-1) I_w, \end{aligned} \tag{5.11}$$

$$\begin{aligned} I_w &\xmapsto{T_{i+1}} q I_{s_{i+1} w s_{i+1}} + (q-1) I_w \\ &\xmapsto{T_i} q^2 I_{s_i s_{i+1} w s_{i+1} s_i} + q(q-1) I_{s_{i+1} w s_{i+1}} + q(q-1) I_w \\ &\xmapsto{T_{i+1}} q^3 I_{s_i s_{i+1} w s_{i+1} s_i} + q(q-1) I_w + (q-1) q^2 I_{s_{i+1} w s_{i+1}} + (q-1)^2 q I_w. \end{aligned} \tag{5.12}$$

The right-hand sides of (5.11) and (5.12) are equal.

Case 6 ($i+1 \in \mathrm{dom}(w)$, $i, i+2 \notin \mathrm{dom}(w)$). We have

$$\begin{aligned} I_w &\xmapsto{T_i} q I_{s_i w s_i} + (q-1) I_w \xmapsto{T_{i+1}} q^2 I_{s_i w s_i} + (q-1) I_{s_{i+1} w s_{i+1}} \\ &\xmapsto{T_i} q^2 I_w + q(q-1) I_{s_{i+1} w s_{i+1}}, \end{aligned} \tag{5.13}$$

$$I_w \xmapsto{T_{i+1}} I_{s_{i+1} w s_{i+1}} \xmapsto{T_i} q I_{s_{i+1} w s_{i+1}} \xmapsto{T_{i+1}} q^2 I_w + q(q-1) I_{s_{i+1} w s_{i+1}}. \tag{5.14}$$

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Case 7 ($i \in \mathrm{dom}(w)$, $i+1, i+2 \notin \mathrm{dom}(w)$). In this case we have

$$I_w \stackrel{T_i}{\longmapsto} I_{s_i w s_i} \stackrel{T_{i+1}}{\longmapsto} I_{s_{i+1}s_i w s_i s_{i+1}} \stackrel{T_i}{\longmapsto} qI_{s_{i+1}s_i w s_i s_{i+1}},$$

$$I_w \stackrel{T_{i+1}}{\longmapsto} qI_w \stackrel{T_i}{\longmapsto} qI_{s_i w s_i} \stackrel{T_{i+1}}{\longmapsto} qI_{s_{i+1}s_i w s_i s_{i+1}}.$$

Case 8 ($i, i+1, i+2 \notin \mathrm{dom}(w)$). In this case the actions of both $T_iT_{i+1}T_i$ and $T_{i+1}T_iT_{i+1}$ map I_w to $q^3 I_w$.

Relation (5.1)

We consider two possible cases.

Case 1 ($\mathrm{dom}(w) \subset \{j+1, \dots, n\}$). Then we also have $i, i+1 \notin \mathrm{dom}(w)$. Therefore, $T_i \cdot I_w = qI_w$ and thus $T_iP_j \cdot I_w = P_jT_i \cdot I_w = qP_j \cdot I_w = qI_w$.

Case 2 ($\mathrm{dom}(w) \not\subset \{j+1, \dots, n\}$). In this case it follows from the definitions that $T_iP_j \cdot I_w = P_jT_i \cdot I_w = qP_j \cdot I_w = 0$.

Relation (5.2)

We consider two possible cases.

Case 1 ($\mathrm{dom}(w) \subset \{j+1, \dots, n\}$). Then $T_iP_j \cdot I_w = P_jT_i \cdot I_w = T_i \cdot I_w$ as $T_i \cdot I_w$ is a linear combination of I_w and $I_{s_i w s_i}$ and both $\mathrm{dom}(w)$ and $\mathrm{dom}(s_i w s_i)$ are contained in $\{j+1, \dots, n\}$.

Case 2 ($\mathrm{dom}(w) \not\subset \{j+1, \dots, n\}$). In this case $T_iP_j \cdot I_w = P_jT_i \cdot I_w = 0$ follows from the definitions.

Relation (5.3)

This follows immediately from the definitions.

Relation (5.4)

Consider three possible cases.

Case 1 ($\mathrm{dom}(w) \subset \{i+2, \dots, n\}$). In this case we also have $i, i+1 \notin \mathrm{dom}(w)$. Therefore, $T_i \cdot I_w = qI_w$ and hence

$$P_{i+1} \cdot I_w = (P_iT_iP_i - (q-1)P_i) \cdot I_w = I_w.$$

Case 2 ($\mathrm{dom}(w) \subset \{i+1, \dots, n\}$ and $i+1 \in \mathrm{dom}(w)$). In this case $P_{i+1} \cdot I_w = 0$ and $P_i \cdot I_w = I_w$. We are left to check that $P_iT_i \cdot I_w = (q-1)I_w$. Observe that $P_i \cdot I_{s_i w s_i} = 0$ as $\mathrm{dom}(s_i w s_i) \not\subset \{i+1, \dots, n\}$. Taking this into account we obtain

$$P_iT_i \cdot I_w = P_i \cdot (qI_{s_i w s_i} + (q-1)I_w) = (q-1)I_w.$$

Case 3 ($\mathrm{dom}(w) \not\subset \{i+1, \dots, n\}$). This case follows directly from the definitions. This completes the proof of the fact that V is an $\boldsymbol{I}_n(q)$-module.

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To prove that V is a Gelfand model for $\boldsymbol{I}_n(q)$ we use the results of [**14**] and arguments analogous to those used in the second part of the proof of Theorem 3.1. Set $J_0 = \boldsymbol{I}_n(q)$, $J_k = \boldsymbol{I}_n(q)P_k\boldsymbol{I}_n(q)$, $k = 1, \dots, n$, and $J_{n+1} = 0$. In [**14**, § 4] it is shown that there is an algebra isomorphism

$$\boldsymbol{I}_n(q) \cong \bigoplus_{k=0}^{n} J_k/J_{k+1}$$

and that J_k/J_{k+1} is isomorphic to $\boldsymbol{M}_{l_k}(\mathbb{C}) \otimes \boldsymbol{H}_{n-k}(q)$ (where $\boldsymbol{M}_{l_k}(\mathbb{C})$ is the matrix algebra and $l_k = \binom{n}{k}$), with a multiplication, which is appropriately twisted to take into account powers of q, which may appear when multiplying in J_k/J_{k+1}. This twist can be neglected by using [**14**, Corollary 15].

For $k = 0, \dots, n$ let $V^{(k)}$ denote the subspace of V, spanned by all $\boldsymbol{l}_w$, where $|\mathrm{dom}(w)| = n - k$. Denote also by $\tilde{V}^{(k)}$ the subspace of $V^{(k)}$, spanned by all $\boldsymbol{l}_w$, where $\mathrm{dom}(w) = \{1, 2, \dots, n-k\}$. Finally, we identify $\boldsymbol{H}_{n-k}(q)$ with the subalgebra of $\boldsymbol{I}_n(q)$, generated by T_i, $i < n - k$. From the definitions we immediately obtain

(i) $V = \bigoplus_{k=0}^{n} V^{(k)}$ as $\boldsymbol{I}_n(q)$-modules,

(ii) $J_{k+1} \cdot V^{(k)} = 0$.

Furthermore, from the definitions we have that $\tilde{V}^{(k)}$ is an $\boldsymbol{H}_{n-k}(q)$-module and, moreover, from Theorem 4.1 we have that $\tilde{V}^{(k)}$ is a Gelfand model of $\boldsymbol{H}_{n-k}(q)$.

As $\dim V^{(k)} = l_k \cdot \dim \tilde{V}^{(k)}$, we obtain that $V^{(k)}$ is a Gelfand model for $\mathrm{M}_{l_k}(\mathbb{C}) \otimes \boldsymbol{H}_{n-k}(q)$. Summarizing, we thus obtain that V is a Gelfand model for $\boldsymbol{I}_n(q)$. This completes the proof. □

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