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SUM THEOREMS FOR COUNTABLY PARACOMPACT SPACES

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In this paper, we extend the class of spaces to which the Σ and β theorems of Hodel apply, as well as the sum and subset theorems of [**2**]. Instead of the open cover definition of countable paracompactness, we utilize an equivalent formulation of countable paracompactness, due to Ishikawa [**3**]. Using the same technique, it is then possible to extend these results to spaces having property $\mathscr{B}$, introduced by Zenor in [**7**].

Finally, we exhibit a class of completely regular, T_2 spaces which have property $\mathscr{B}$.

Definition. A space X is said to be countably paracompact provided every countable open cover has an open locally-finite refinement.

Definition. A space X is said to have property $\mathscr{B}$ if for any well-ordered monotone decreasing family $\{F(\alpha)|\alpha \in A\}$ of closed sets with empty intersection, there is a monotone decreasing family of domains $\{G(\alpha)|\alpha \in A\}$ such that:

(1) For all $\alpha \in A$, $F(\alpha) \subset G(\alpha)$,

(2) $\bigcap_{\alpha\in A} G(\alpha)^- = \emptyset$.

LEMMA 1. *A space X is countably paracompact provided that for any countable descending family $\{F(i)|i \in \mathbf{Z}^+\}$, of closed sets with empty intersection, there is a countable descending family $\{G(i)|i \in \mathbf{Z}^+\}$, of open sets such that*

(1) *For all $i \in \mathbf{Z}^+$, $G(i) \supset F(i)$,*

(2) $\bigcap\{G(i)^-|i \in \mathbf{Z}^+\} = \emptyset$.

Proof. See [**3**].

THEOREM 1. *Let $X = \bigcup K(\alpha)$, where $\mathscr{K} = \{K(\alpha)|\alpha \in A\}$ is a locally finite family of closed, countably paracompact subsets of X. Then X is countably paracompact.*

Proof. Let $\{F(i)|i \in \mathbf{Z}^+\}$ be a descending family of closed subsets of X such that $\bigcap F(i) = \emptyset$. For all $\alpha \in A$, $\{F(i) \cap K(\alpha)|i \in \mathbf{Z}^+\}$ is a descending family of closed subsets of $K(\alpha)$, with void intersection, whence there is a family $\{T(\alpha, i)|i \in \mathbf{Z}^+\}$, of open subsets of $K(\alpha)$ such that $T(\alpha, i+1) \subset T(\alpha, i)$, and $\bigcap \{\mathrm{Cl}_{K(\alpha)}T(\alpha, i)|i \in \mathbf{Z}^+\} = \emptyset$. Since each $K(\alpha)$ is a closed subset of X, we can say that $\mathrm{Cl}_{K(\alpha)}T(\alpha, i) = T(\alpha, i)^-$, so that for $\alpha \in A$, $\bigcap \{T(\alpha, i)^-|i \in \mathbf{Z}^+\} = \emptyset$. (The closure bar means closure in X.) Now for

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$x \in X$, let $A(x) = \{\alpha \in A | x \in K(\alpha)\}$. Then $A(x)$ is finite, and since $\mathscr{K}$ is locally finite, we can say that for $x \in X$, there is a set $V(x)$, open in X such that $x \in V(x) \subset \bigcup \{K(\alpha) | \alpha \in A(x)\}$. Now for all $i \in \mathbf{Z}^+$, and $x \in F(i)$, let

$$S(x, i) = V(x) \cap (\bigcap\{X - (K(\alpha) - T(\alpha, i)) | \alpha \in A(x)\}).$$

Then it is easy to see that $S(x, i)$ is an open subset of X, and contain the point x. Further, $S(x, i) \subset \bigcup \{T(\alpha, i) | \alpha \in A(x)\}$, for if $p \in S(x, i)$, then $p \in V(x)$, whence there exists $\tilde{\alpha} \in A(x)$ such that $p \in K(\tilde{\alpha})$. Also, $p \in \bigcap \{X - (K(\alpha) - T(\alpha, i)) | \alpha \in A(x)\}$, so that in particular, $p \in X - (K(\tilde{\alpha}) - T(\tilde{\alpha}, i))$, and $p \notin K(\tilde{\alpha}) - T(\tilde{\alpha}, i)$. Since p is a member of $K(\tilde{\alpha})$, then p must be a member of $T(\tilde{\alpha}, i)$, which is a subset of $\bigcup \{T(\alpha, i) | \alpha \in A(x)\}$.

Now for $i \in \mathbf{Z}^+$, let $W(i) = \bigcup \{S(x, i) | x \in F(i)\}$. Then the family $\{W(i)\}$ "works", that is, $\{W(i)\}$ is a family of open subsets, $W(i) \supset F(i)$, $\bigcap W(i)^- = \emptyset$, and $\{W(i)\}$ is a descending family. That each $W(i)$ is open and contains $F(i)$ is clear from the definition of $W(i)$.

To see that $\bigcap W(i)^- = \emptyset$, note that

$$W(i)^- \subset \mathrm{Cl}_X(\bigcup \{T(\alpha, i) | x \in F(i), \alpha \in A(x)\}).$$

Since $\{K(\alpha) | \alpha \in A\}$ is locally finite, then for all $i \in \mathbf{Z}^+$, $\{T(\alpha, i) | \alpha \in A(x)\}$ is also locally finite, hence closure-preserving. In particular, then, the family $\{T(\alpha, i) | x \in F(i), \alpha \in A(x)\}$ is closure-preserving. Thus

$$W(i)^- \subset \bigcup \{T(\alpha, i)^- | x \in F(i), \alpha \in A(x)\};$$

hence to show that $\bigcap W(i)^- = \emptyset$, it suffices to show that

$$\bigcap_{i=1}^{\infty} \{\bigcup\{T(\alpha, i)^- | \, x \in F(i), \alpha \in A(x)\}\} = \phi.$$

Now let $p \in X$. Let $\alpha_1, \alpha_2, \ldots \alpha_k$ be those members of A for which $p \in K(\alpha_1)$, $K(\alpha_2), \ldots K(\alpha_k)$. Since $\bigcap_{i=1}^{\infty} T(\alpha_j, i)^- = \emptyset$, for $\alpha_j = \alpha_1, \alpha_2, \ldots \alpha_k$, there are integers $i_1, i_2, \ldots i_k$ such that $p \notin T(\alpha_j, i_j)^-$, for $\alpha_j = \alpha_1, \alpha_2, \ldots \alpha_k$. Let $i^* = \max(i_1, i_2, \ldots i_k)$. Then since each family $\{T(\alpha, i)^-\}$ is descending, we have that $p \notin T(\alpha_j, i^*)^-$, for $\alpha_j = \alpha_1, \alpha_2, \ldots \alpha_k$. This shows that $p \notin \bigcup \{T(\alpha, i^*)^- | x \in F(i^*), \alpha \in A(x)\}$, for if $\alpha \in A$, with $p \notin K(\alpha)$, then $p \notin T(\alpha, i^*)$, since $T(\alpha, i^*) \subset K(\alpha)$, and p meets only those sets $K(\alpha)$, for which α is one of the indices $\alpha_1, \alpha_2, \ldots \alpha_k$. If α is one of the indices $\alpha_1, \alpha_2, \ldots \alpha_k$, then by the choice of i^*, $p \notin T(\alpha, i^*)^-$. Thus

$$p \notin \bigcup \{T(\alpha, i^*)^- | x \in F(i^*), \alpha \in A(x)\},$$

and we have that $\bigcap W(i)^- = \emptyset$.

We show one further fact. If $j > i$, then $W(j) \subset W(i)$. To do this, we show that if $x \in F(j)$, then $S(x, i) \supset S(x, j)$.

For all $\alpha \in A(x)$, $T(\alpha, j) \subset T(\alpha, i)$, so $K(\alpha) - T(\alpha, j)$ contains $K(\alpha) - T(\alpha, i)$, whence $X - (K(\alpha) - T(\alpha, j))$ is a subset of $X - (K(\alpha) - T(\alpha, i))$, and

$$V(x) \cap \left(\bigcap_{\alpha \in A(x)} (X - (K(\alpha) - T(\alpha, j))) \right)$$

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is a subset of

$$V(x) \cap \left(\bigcap_{\alpha \in A(x)} (X - (K(\alpha) - T(\alpha, i))) \right),$$

that is, $S(x, j) \subset S(x, i)$.

Finally, then, $W(j) = \bigcup \{S(x, j) | x \in F(j)\} \subset \bigcup \{S(x, i) | x \in F(i)\} = W(i)$, and the theorem is proved.

THEOREM 2. *Let X be a space which is the union of a locally finite family of closed sets, each hvaing property $\mathscr{B}$. Then X has property $\mathscr{B}$.*

Proof. The proof is analogous to that of Theorem 1. Instead of a countable descending family, $\{F(i) | i \in \mathbf{Z}^+\}$, of closed sets with void intersection, we have to deal with a well-ordered monotone decreasing family, $\{F(i) | i \in I\}$, of closed sets, with empty intersection. It is easy to see that all remarks in Theorem 1 concerning the index set $\mathbf{Z}^+$ depended only on the fact that the positive integers are well-ordered. Hence the proof applies to the more general case. The proof in Theorem 1 was given for countable paracompactness, instead of the more general $\mathscr{B}$ property, only to simplify the notation as much as possible.

THEOREM 3. *Let X be a topological space such that every open subset of X ix countably paracompact (has property $\mathscr{B}$). Then every subset is countably paracompact (has property $\mathscr{B}$).*

Proof. The proof is clear for countable paracompactness. Let F be a subset of X, where X has property $\mathscr{B}$. Let $\{F(\alpha) | \alpha \in A\}$ be a well-ordered monotone collection of closed subsets of F, with empty intersection. Then $\{\mathrm{Cl}_X F(\alpha) | \alpha \in A\}$ is a well-ordered, monotone collection of closed subsets of X.

Let $Y = \bigcap \{\mathrm{Cl}_X F(\alpha) | \alpha \in A\}$. Then $\{\mathrm{Cl}_X F(\alpha) \cap (X - Y) | \alpha \in A\}$ is a well-ordered monotone collection of closed subsets of $X - Y$, with empty intersection. Since Y is closed, $X - Y$ is open, hence has property $\mathscr{B}$. Thus there is a monotone collection, $\{G(\alpha) | \alpha \in A\}$, of open subsets of $X - Y$, such that $G(\alpha) \supset \mathrm{Cl}_X F(\alpha) \cap (X - Y)$ and $\bigcap \{\mathrm{Cl}_{X-Y} G(\alpha) | \alpha \in A\} = \emptyset$.

For $\alpha \in A$, let $G'(\alpha) = G(\alpha) \cap F$. Then $\{G'(\alpha) | \alpha \in A\}$ is a monotone collection of open subsets of F. We show that this collection has two further properties:

(1) For all $\alpha \in A$, $G'(\alpha) \supset F(\alpha)$, and

(2) $\bigcap \{\mathrm{Cl}_F G'(\alpha) | \alpha \in A\} = \emptyset$.

Note first that $F \subset X - Y$. Suppose $x \in F$. Then there exists $\alpha \in A$ such that $x \notin F(\alpha)$. Since $F(\alpha)$ is closed in F, x is not a limit point of $F(\alpha)$. In particular, $x \notin \mathrm{Cl}_X F(\alpha)$. But then $x \notin Y$, and $F \subset X - Y$.

1. Let $x \in F(\alpha)$. Then $x \in F \subset X - Y$. Further, $x \in \mathrm{Cl}_X F(\alpha)$, so that $x \in \mathrm{Cl}_X F(\alpha) \cap (X - Y) \subset G(\alpha)$. Then $x \in G(\alpha) \cap F = G'(\alpha)$, so that $F(\alpha) \subset G'(\alpha)$.

2. It is clear that $\mathrm{Cl}_F G'(\alpha) \subset \mathrm{Cl}_{X-Y} G'(\alpha)$, which is in turn a subset of $\mathrm{Cl}_{X-Y} G(\alpha)$. Since $\bigcap \{\mathrm{Cl}_{X-Y} G(\alpha) | \alpha \in A\} = \emptyset$, then the intersection of the smaller sets, $\{\mathrm{Cl}_F G'(\alpha) | \alpha \in A\} = \emptyset$. This shows that F has property $\mathscr{B}$.

THEOREM 4. *Let X be countably paracompact (have property $\mathscr{B}$). Let F be a closed subset of X. Then F is countably paracompact (has property $\mathscr{B}$).*

Proof. The proof is clear. Theorems 1, 2, 3, and 4 suffice to show that countable paracompactness and the $\mathscr{B}$ property satisfy the sum and subset theorems of [**2**], this is: (1) if X is a space which admits a σ-locally finite open cover, the closure of each member being countably paracompact or $\mathscr{B}$, then X is countably paracompact or $\mathscr{B}$; (2) if X admits a σ-locally finite elementary cover, with each member of the cover countably paracompact or $\mathscr{B}$, then X is countably paracompact or $\mathscr{B}$; (3) if X is regular, and admits a σ-locally finite open cover, each member with compact boundary and each member countably paracompact or $\mathscr{B}$, then X is countably paracompact or $\mathscr{B}$; (4) if X is totally normal and is countably paracompact or $\mathscr{B}$, then X is hereditarily countably paracompact or hereditarily $\mathscr{B}$.

We now exhibit a class of completely regular, T_2 spaces with the $\mathscr{B}$ property. This class is described in terms of the Stone-Čech compactification, in a fashion similar to that introduced by Tamano in [**5**] and [**6**], in order to characterize various classes of spaces.

THEOREM 5. *Let X be completely regular and Hausdorff. Suppose that for each compact subset K, of $\beta X - X$, that $X \times K$, and the diagonal, $\Delta(X)$, have disjoint neighborhoods in the space $X \times \beta X$. Then X has property $\mathscr{B}$.*

Proof. Let $\{F(\alpha) | \alpha \in \Gamma\}$ be a well-ordered, monotone collection of closed subsets of X, with empty intersection. Let $K = \bigcap \{\mathrm{Cl}_{\beta X} F(\alpha) | \alpha \in \Gamma\}$. Let G, H be disjoint open subsets of $X \times \beta X$ such that $G \supset \Delta(X)$, and $H \supset X \times K$. For each $\alpha \in \Gamma$, let

$$G(\alpha) = \{x | \{x\} \times \mathrm{Cl}_{\beta X} F(\alpha)) \cap G \neq \emptyset\}.$$

We exhibit four properties of the collection $\{G(\alpha) | \alpha \in \Gamma\}$.

(1) For each $\alpha \in \Gamma$, $F(\alpha) \subset G(\alpha)$, for if $p \in F(\alpha)$, then $(p, p) \in \{p\} \times \mathrm{Cl}_{\beta X} F(\alpha)$ and $(p, p) \in \Delta(X) \subset G$, so that $(\{p\} \times \mathrm{Cl}_{\beta X} F(\alpha)) \cap G \neq \emptyset$, whence $p \in G(\alpha)$.

(2) For each $\alpha \in \Gamma$, $G(\alpha)$ is open. Let $x \in G(\alpha)$. Then

$$(\{x\} \times \mathrm{Cl}_{\beta X} F(\alpha)) \cap G \neq \emptyset.$$

Let (x, p) be a point in the intersection. G is open, and $(x, p) \in G$, so that there exist sets V, W, open in X, βX, respectively, such that $(x, p) \in V \times W \subset G$. But $V \subset G(\alpha)$, for if $a \in V$, then $(a, p) \in V \times W \subset G$, and $(a, p) \in \{a\} \times \mathrm{Cl}_{\beta X} F(\alpha)$ as well, so that $a \in G(\alpha)$. Thus $x \in V \subset G(\alpha)$, so that $G(\alpha)$ is open.

(3) $\{G(\alpha)|\alpha \in \Gamma\}$ is easily seen to be monotone.

(4) $\bigcap \{\text{Cl}_X G(\alpha)|\alpha \in \Gamma\} = \emptyset$. Let $x \in X$. Then

$$\bigcap \{\{x\} \times \text{Cl}_{\beta X} F(\alpha)|\alpha \in \Gamma\} = \{x\} \times (\bigcap \{\text{Cl}_{\beta X} F(\alpha)|\alpha \in \Gamma\}) \subset X \times K \subset H.$$

But $\bigcap\{\{x\} \times \text{Cl}_{\beta X} F(\alpha)|\alpha \in \Gamma\}$ is the intersection of compact sets and lies inside the open set H. Therefore the intersection of some finite number of these sets lies inside H, say

$$\bigcap_{i=1}^{n} \{\{x\} \times \text{Cl}_{\beta X} F(\alpha_i)\}.$$

Since the collection $\{F(\alpha)|\alpha \in \Gamma\}$ is monotone, one of the finite number of sets is smaller than the others, say $\{x\} \times \text{Cl}_{\beta X} F(\alpha)$, for some $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$. Then

$$\{x\} \times \text{Cl}_{\beta X} F(\alpha) \subset H.$$

But then there is an open subset of X, say $N(x)$, such that $x \in N(x)$, and $N(x) \times \text{Cl}_{\beta X} F(\alpha) \subset H$. But then $N(x) \cap G(\alpha) = \emptyset$, for if $p \in N(x)$, then $\{p\} \times \text{Cl}_{\beta X} F(\alpha) \subset H$, and $H \cap G = \emptyset$. Thus $N(x)$ is an open set about x, which does not meet $G(\alpha)$, whence $x \notin \text{Cl}_X G(\alpha)$. Therefore

$$\bigcap \{\text{Cl}_X G(\alpha)|\alpha \in \Gamma\} = \emptyset.$$

The existence of the four properties described above shows that X has property $\mathscr{B}$.

There is a partial converse to the previous theorem.

THEOREM 6. *Let X be a completely regular Hausdorff space with property $\mathscr{B}$. Let K be a compact subset of $\beta X - X$. Let $\{W(\alpha)|\alpha < \sigma\}$ be a well-ordered monotone decreasing family of open subsets of βX such that $K = \bigcap W(\alpha) = \bigcap \text{Cl}_{\beta X} W(\alpha)$. Then $X \times K$ and ΔX have disjoint neighborhoods in $X \times \beta X$.*

Proof. For $\alpha < \sigma$, let $F(\alpha) = P_X[X \times \text{Cl}_{\beta X} W(\alpha)) \cap \Delta X]$. Then the family $\{F(\alpha)|\alpha < \sigma\}$ is a well-ordered, monotone decreasing family of closed subsets of X with empty intersection.

For $\alpha < \sigma$, let $K(\alpha) = F(\alpha)$ if α is not a limit ordinal and let $K(\alpha) = \bigcap \{F(\beta)|\beta < \alpha\}$ if α is a limit ordinal. Then $\{K(\alpha)|\alpha < \sigma\}$ is a well-ordered monotone decreasing family of closed subsets of X, with empty intersection. Since X has the $\mathscr{B}$ property, there is a monotone decreasing family $\{V(\alpha)|\alpha < \sigma\}$ of open subsets of X such that $V(\alpha) \supset K(\alpha)$, and $\bigcap V(\alpha)^- = \emptyset$.

Now let

$$A = \bigcup_{\alpha<\sigma} ((X - V(\alpha)^-) \times W(\alpha + 1)).$$

Then A is an open set and is easily seen to contain $X \times K$. Moreover $(\text{Cl}_{X\times\beta X} A) \cap \Delta X = \emptyset$. To see this, let x be an arbitrary point of X and let α^* be the least member of $\{\alpha < \sigma|x \notin \text{Cl}_{\beta X} W(\alpha)\}$. Then (x, x) is a member of $X \times (\beta X - \text{Cl}_{\beta X} W(\alpha^*))$, which is open, and does not meet

$$\bigcup_{\alpha\geqq\alpha^*} ((X - V(\alpha)^-) \times W(\alpha + 1)).$$

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Thus, to show that (x, x) is not a limit point of A, it suffices to show that (x, x) is not a limit point of $\bigcup_{\alpha<\alpha^*}((X - V(\alpha)^-) \times W(\alpha + 1))$.

Suppose first that α^* is a limit ordinal. Since $x \in \mathrm{Cl}_{\beta X} W(\alpha)$, for all $\alpha < \alpha^*$, then $x \in F(\alpha)$, for all $\alpha < \alpha^*$, and $x \in \bigcap \{F(\alpha) | \alpha < \alpha^*\} = K(\alpha^*) \subset V(\alpha^*)$ and $V(\alpha^*) \times \beta X$ is an open set about (x, x) which does not meet $\bigcup_{\alpha<\alpha^*}((X - V(\alpha)^-) \times W(\alpha + 1)$, so in the case that α^* is a limit ordinal, (x, x) is not a limit point of A.

Suppose now that α^* has an immediate predecessor, say α_0. Then

$$\bigcup_{\alpha<\alpha^*} ((X - V(\alpha)^-) \times W(\alpha + 1)) = \bigcup_{\alpha\leqq\alpha_0} ((X - V(\alpha)^-) \times W(\alpha + 1)).$$

Since $\alpha_0 < \alpha^*$, then $x \in \mathrm{Cl}_{\beta X} W(\alpha_0)$, whence $x \in F(\alpha_0)$, which is a subset of $V(\alpha_0)$. Then $V(\alpha_0) \times \beta X$ is an open set about (x, x) which does not meet $\bigcup_{\alpha\leqq\alpha_0}((X - V(\alpha)^-) \times W(\alpha + 1))$.

Thus (x, x) is not a limit point of A, so that $(\mathrm{Cl}_{X\times\beta X} A) \cap \Delta X = \emptyset$, whence the sets A and $(X \times \beta X) - \mathrm{Cl}_{X\times\beta X} A$ are disjoint open subsets of $X \times \beta X$ which contain $X \times K$ and ΔX, respectively.

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