Astin Bulletin (2023), **53**, pp. 418–442
doi:10.1017/asb.2023.8

International Actuarial Association
Association Actuarielle Internationale

RESEARCH ARTICLE

The 3-step hedge-based valuation: fair valuation in the presence of systematic risks

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Received: 4 September 2022; **Revised:** 2 February 2023; **Accepted:** 2 February 2023; **First published online:** 14 March 2023

Keywords: Fair valuation; fair hedger; actuarial valuation; systematic risk

Abstract
In this paper, we introduce the 3-step hedge-based valuation for the valuation of hybrid claims. We consider an insurance portfolio which is exposed to traded risks, diversifiable risks and non-traded systematic risks. The class of 3-step hedge-based valuations is equivalent with the class of fair valuations. Closed-form solutions are derived for a portfolio of unit-linked contracts under the assumption of independence between financial and non-financial risks. We also consider the additive 3-step valuation and show that this additive valuation is a member of the more general class of 3-step hedge-based valuations.

1. Introduction

The value of a random future liability should correspond with the amount of money required today to set up an appropriate risk management strategy and to ensure this strategy leads to a sufficiently large likelihood to meet the future liability. Insurance liabilities are complex combinations of different types of risks. There is a pletora of papers proposing valuation frameworks when faced with only a single type of risk. For example, the field of mathematical finance developed the risk neutral valuation framework for hedgeable claims, whereas premium principles are introduced in the actuarial literature to deal with claims depending on diversifiable risks. In this paper, we introduce the 3-step hedge-based valuation that decomposes a claim into a hedgeable, diversifiable and residual part and determines the value of the claim by combining appropriate valuation principles.

Traditional insurance liabilities in life and non-life insurance are often assumed to be diversifiable. For example, the underlying assumptions when determining pure premiums for life insurance and annuity benefits is that the insurer will aggregate a large amount of independent policies. Non-life insurers employ statistical methods to determine rating variables which are then used to classify policies in homogeneous groups. Actuarial valuation principles (or premium principles) such as the standard deviation principle can be used for valuating such diversifiable claims.

A hedgeable claim is a liability that can be replicated by investing in an appropriate linear combination of traded assets. For example, stocks, bonds and other traded derivatives are examples of traded assets that can be used to replicate the payoff of a future liability. If such a replicating portfolio exists, buying the replicating portfolio eliminates the risks of the liability. The claim is perfectly 'hedged', and the value of such a hedgeable claim should correspond with the market value of the replicating portfolio to avoid arbitrage opportunities. The value of the replicating portfolio can be observed in the market and therefore valuation of a hedgeable claim is model free.

In this paper, we propose a 3-step hedge-based valuation principle for insurance claims. We consider claims that can be expressed as a product between a financial part and a conditional diversifiable part.

Indeed, given the realization of the traded assets and the non-traded systematic risks, the second part of the claim is a sum of identical and independent random variables. In a first step, we determine an appropriate hedging portfolio. The hedging portfolio will not exactly replicate the hybrid claim. The residual part of the claim corresponds with the hybrid claim after subtracting the random income generated by the hedging portfolio. In the second step, we use a conditional actuarial valuation, which delivers a random variable that represents the actuarial value of the residual claim for each realization of the traded and non-traded systematic risks. The third valuation step then consists of valuating this residual claim using a systematic valuation.

The 3-step hedge-based valuation is a combination of the hedge-based valuation introduced in Dhaene *et al*. (2017) and the 2-step valuation introduced in Pelsser and Stadje (2014). Both valuations assume that a market with actuarial and traded risks and decompose a hybrid claim in a financial and an actuarial part. These valuations are therefore a combination of financial and actuarial pricing principles. We will show that our 3-step hedge-based valuation is equivalent with the hedge-based valuation and the 2-step valuation.

The 3-step hedge-based valuation is a *fair* valuation, as defined in Dhaene *et al*. (2017). A fair valuation is a valuation which is market, but also model consistent. Market consistency ensures that the hedgeable part of a claim is valuated at its hedging cost. Model consistency implies that claims which are independent of the financial market and the systematic risk factors are valuated using an actuarial valuation. In a one-period model, independence of the financial markets implies that observing the asset prices does not give information about the realization of the claim. Therefore, hedging using a model-consistent hedger such as the mean-variance hedger will not be able to reduce the risk of the claims. Independence of the systematic risk factors implies the claim can be managed by diversification and therefore an actuarial valuation may be most appropriate. In Neuberger and Hodges (1989), the authors consider market-consistent valuation using a utility indifference approach; see also Malamud *et al*. (2008). Market-consistent valuation requires a combination of actuarial and financial valuation methods, as was first pointed out by Brennan and Schwartz (1976a), who considered the valuation of guarantees in unit-linked insurance contracts; see also Embrechts (2000). Recent approaches to define fair valuations are Pelsser and Stadje (2014), Pelsser and Schweizer (2016), Wuthrich (2016), Dhaene *et al*. (2017), Delong *et al*. (2019a), Barigou and Dhaene (2019), Barigou *et al*. (2019), Barigou *et al*. (2021), Deelstra *et al*. (2020), Bacinello *et al*. (2021), Chen *et al*. (2021).

The idea of decomposing a hybrid claim in a hedgeable, diversifiable and residual part was also proposed in Dhaene (2022) and Deelstra *et al*. (2020). In both situations, the diversifiable and financial risks are assumed to be independent. We generalize these approaches in that the 3-step hedge-based valuation allows for dependencies between the financial and diversifiable risks. In Deelstra *et al*. (2020), the authors use a different 3-step valuation principle. Indeed, after decomposing the hybrid claim into three parts, the valuation of the hybrid claim is assumed to be the sum of the valuations of the different parts. We show in this paper that such an additive 3-step valuation is a subset of the larger class of 3-step hedge-based valuations.

The contributions of this paper are as follows. First, we introduce the 3-step hedge-based valuation, which takes into account non-traded systematic risks. Second, we show the equivalence between the class of 3-step hedge-based valuations and the class of fair valuations. This then implies that the class of 3-step hedge-based valuations is the same as the class of valuations described in Dhaene *et al*. (2017) and Pelsser and Stadje (2014). Third, we derive closed-form expressions for a portfolio of unit-linked contracts under the assumption the financial and non-financial risks are independent. We assume an incomplete financial market and derive closed-form expressions for the mean-variance hedger. Fourth, we consider an additive 3-step valuation and show that this valuation is a special case of the 3-step hedge-based valuations.

This paper is organized as follows. In Section 2, we introduce the valuation of hedgeable and diversifiable claims. The class of fair valuations is discussed in Section 3 as well as the hedge-based valuations.

We propose our 3-step hedge-based valuation in Section 4, together with important properties and an illustration. The case where financial and non-financial risks are independent is considered in Section 5. In Section 6, we introduce the additive 3-step valuation and show that this valuation is a special case of the more general class of 3-step hedge-based valuations.

2. Valuation of hybrid claims

A claim is a future cashflow which has to be paid by the insurance company to the policyholders. In this paper we consider a one period valuation where claims are modeled by random variables. We assume today is time $t = 0$ and the claim has a deterministic maturity $T > 0$. No actions (e.g., rebalancing of investment portfolios) are allowed between the time of valuation and the maturity of the claim. The aim is to determine the time-0 value of such a future liability. This value should correspond with the amount of money required to set up an appropriate risk management strategy for the claim. In the remainder of the paper, we assume there is a risk-free bank account paying a deterministic and constant continuously compounded interest rate r. We assume that all random variables encountered in this paper are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and have finite first two moments. The set of all random variables is denoted by $\mathcal{C}$. Similar as in Dhaene *et al*. (2017), we define the class of valuations for claims in $\mathcal{C}$ as functions $\rho : \mathcal{C} \to \mathbb{R}$ attaching the real number $\rho[S]$ to any claim $S \in \mathcal{C}$ such that the function ρ is normalized and translation invariant. The real number $\rho[S]$ can be interpreted as the time-0 value of a claim S.

2.1. Hedgeable claims

We assume that there is an arbitrage-free financial market with n^f traded payoffs with maturity T. The traded assets can be bought and sold by all market participants at any quantity. The bank account at time T is denoted by Y_0. Since we assume a deterministic rate r, we have that, starting from an initial investment of 1 at time $t = 0$, the bank account is given by $Y_0 = \mathrm{e}^{rT}$. The payoff of asset i at time T is denoted by Y_i. We use the random vector $\boldsymbol{Y} = (Y_0, Y_1, Y_2, \ldots, Y_{n^f})$ to denote the bank account and the n^f traded assets. The unique time-0 price y_i of the payoff Y_i can be observed in the market. We assume that $y_0 = 1$. A *financial derivative* S is a function of the traded assets, that is, $S = h(\boldsymbol{Y})$, for some Borel-measurable function h. We denote the set of all financial derivatives by $\mathcal{C}^{\boldsymbol{Y}}$. Throughout the paper we assume that all functions we encounter are Borel measurable.

A claim $S^h \in \mathcal{C}$ is a *hedgeable* claim if we have a real vector $\boldsymbol{\nu} \in \mathbb{R}^{n^f+1}$ such that $S^h = \boldsymbol{\nu} \cdot \boldsymbol{Y}$, almost surely. The set of hedgeable claims is denoted by $\mathcal{C}^h$. The vector $\boldsymbol{\nu}$ is called the *hedge* of the claim S^h. We assume that the tradeable assets are non-redundant, that is, we cannot use a subset of the traded assets to replicate the payoff of another traded asset. Assume $\boldsymbol{\theta} \in \mathbb{R}^{n^f+1}$. Then $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$ implies $\boldsymbol{\theta} = (0, 0, \ldots, 0)$. Since we assume the traded assets to be non-redundant, the hedge $\boldsymbol{\nu}$ is unique. Of course, a hedgeable claim is a financial derivative, that is, we have $\mathcal{C}^h \subseteq \mathcal{C}^{\boldsymbol{Y}}$. In case both sets coincide, the market is complete.

A hedgeable claim can be managed by buying at time $t = 0$ the hedge $\boldsymbol{\nu}$ at the market price $\boldsymbol{\nu} \cdot \boldsymbol{y}$. Since the hedge will replicate the payoff of the liability, the residual liability, that is, the liability after taking into account the payout of the hedge, is zero. In order to avoid arbitrage opportunities, the price of a hedgeable claim should be equal to the unique price of the hedge. We assume there is a risk neutral probability measure. Note that in our setting, the existence of a risk neutral probability measure is equivalent to the absence of arbitrage. $\mathbb{Q}$ which is equivalent to the real-world probability measure $\mathbb{P}$. The time-0 prices can be expressed as $y_i = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}[Y_i]$, for $i = 0, 1, 2, \ldots, n^f$. The price of a hedgeable claim should be determined using a discounted risk neutral expectation. The financial market, however, is assumed to be incomplete, which implies there are different risk neutral measures and there may be financial derivatives which cannot be perfectly replicated.

2.2. Diversifiable claims

The claim $S^d \in \mathcal{C}$ is said to be a diversifiable claim if there exist i.i.d. risks $V_1, V_2, \ldots, V_n \in \mathcal{C}$ such that

$$S^d = \frac{1}{n}\sum_{i=1}^{n} V_i, \tag{2.1}$$

where $n > 1$. A diversifiable claim is a combination of independent claims V_i in (2.1) which are attributed to individual policies. An insurer holding a diversifiable claim insures a finite number of individual and independent policies. However, we assume the insurer can always decide to increase the number of policies in the portfolio if necessary, that is, we assume there exist more copies of V_i than the insurer currently holds in its portfolio. Indeed, in order for a claim to be diversifiable, the insurer holding the claim should be able to diversify, that is, increase the number of policies (n) in its portfolio. The claim S^d is not perfectly diversified since only a finite number of policies are included. Since we assume that $\mathbb{E}[V_1]$ is finite which implies that the weak law of large numbers can be applied if we allow n to grow. Longevity risk, for example, impacts the future lifetime of a large group of policyholders simultaneously in the same direction. Although the longevity risk of different groups may be independent (e.g., longevity in the United States and China), one cannot find a large number of independent copies which means an insurer may not find enough independent copies to appropriately diversify the risk.

We can determine a value and the corresponding risk management strategy for a diversifiable claim. A diversifiable claim could be a linear combination of n independent hedgeable claims V_i. In this case, the diversifiable claim is also hedgeable and one can construct a hedging strategy. The value of the diversifiable claim then corresponds with the price to set up the hedging strategy, that is, pricing is based on risk neutral valuation. Assume now the diversifiable claim is not hedgeable. The law of large numbers states that if n tends to infinity, the claim S^d converges to $\mathbb{E}[V_1]$, where this expectation is taken under the real-world distribution $\mathbb{P}$. In reality, an insurer can never hold a portfolio with infinitely many contracts. Therefore, the realization of S^d fluctuates around the expectation $\mathbb{E}[V_1]$. For a sufficiently large n, the central limit theorem shows that the distribution of S^d is approximately normal distributed, that is

$$S^d \approx \mathbb{E}\left[S^d\right] + \sqrt{\mathrm{Var}\left[S^d\right]}\,\Phi^{-1}(U), \tag{2.2}$$

where Φ is the cdf of a standard normal distribution and U is a uniform distribution.

The value of a non-hedgeable diversifiable claim S^d should correspond with the amount of money required to be able to pay the claim with a sufficiently large likelihood. A possible value for the diversifiable claim can be determined using the following valuation:

$$\rho^a\left[S^d\right] = \mathrm{e}^{-rT}\left(\mathbb{E}\left[S^d\right] + \lambda\left[S^d\right]\right), \tag{2.3}$$

where $\lambda\left[S^d\right] \in \mathbb{R}$ is a risk margin capturing deviations of S^d from the average claim amount. The capital $\rho^a\left[S^d\right]$ is invested in the risk-free bank account until the maturity T of the claim. Taking into account (2.2), the probability of insolvency is expressed as follows:

$$\mathbb{P}\left[S^d > \mathrm{e}^{rT}\rho^a\left[S^d\right]\right] \approx 1 - \Phi\left(\frac{\lambda\left[S^d\right]\sqrt{n}}{\sqrt{\mathrm{Var}[V_1]}}\right). \tag{2.4}$$

We conclude that if we have a non-hedgeable diversifiable claim and a valuation with a given risk margin determined by λ, we can reduce the insolvency probability by increasing the size of the portfolio. For a given insolvency probability, the risk margin is a decreasing function of the portfolio size.

A valuation ρ^a which has the form of (2.3) and where $\lambda\left[S^d\right]$ is $\mathbb{P}$-law invariant is called an *actuarial valuation*. Note that a valuation ρ^a is $\mathbb{P}-$law invariant if $S_1 \overset{d}{=} S_2$ implies $\rho^a[S_1] = \rho^a[S_2]$, where '$\overset{d}{=}$' stands for equality in distribution under the probability measure $\mathbb{P}$. The set with all actuarial valuations is denoted by $\mathcal{A}$. The risk margin λ can be determined using the standard deviation of the claim, which results in the standard deviation principle. The value of a claim $S \in \mathcal{C}$ valuated using the standard

https://doi.org/10.1017/asb.2023.8 Published online by Cambridge University Press

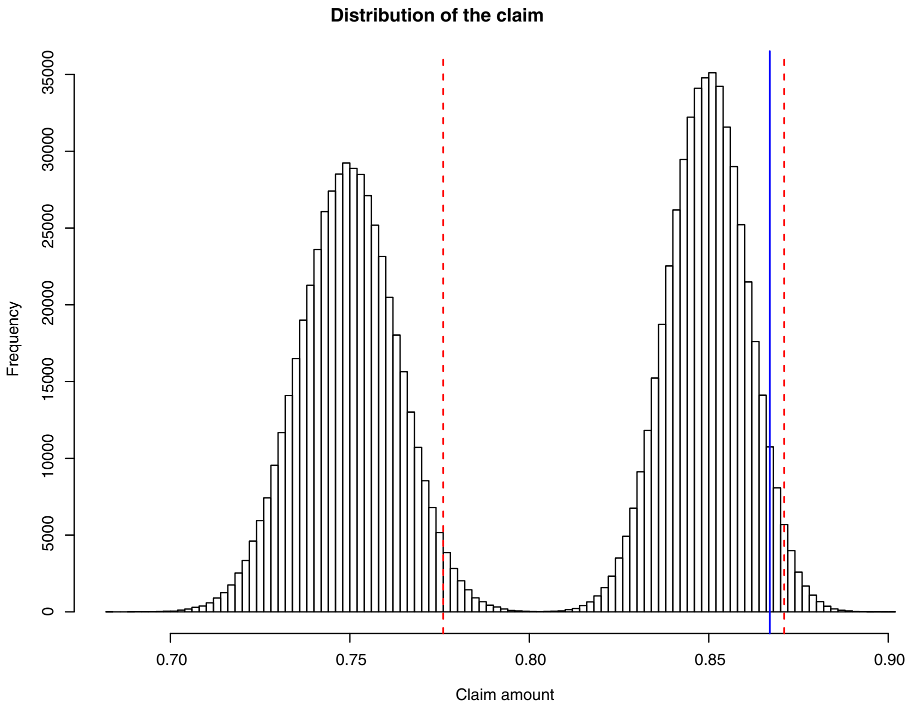

Figure 1. *The distribution of the claim S given by $S = \frac{1}{n^a}\sum_{i=1}^{n^a} X_i$, where $p_0 = 0.75, p_1 = 0.85$ and $n^a = 1000$. The solid vertical line corresponds with the value of the claim when a standard deviation principle with $\beta = 1$ is used. The dashed lines represent the value of the conditional claims S_0 and S_1.*

deviation principle is then given by

$$\rho^a[S] = \mathbb{E}[S] + \beta \times \sqrt{\mathrm{Var}[S]}, \tag{2.5}$$

where $\beta > 0$ is called the safety loading. A $\mathbb{P}-$law invariant valuation ρ^a can always be written as $\rho^a\,[S] = \mathrm{e}^{-rT}\left(\mathbb{E}\,[S] + \left(\mathrm{e}^{rT}\rho^a\,[S] - \mathbb{E}\,[S]\right)\right)$, which shows we can write the valuation $\rho^a\,[S]$ as in (2.3). Therefore, the class of actuarial valuations can equivalently be defined as the class of all $\mathbb{P}-$law invariant valuations.

Example 1. Consider an insurance company holding a portfolio of n^a pure endowment policies. The claim S denotes the per-policy loss and is given by $S = \frac{1}{n^a}\sum_{i=1}^{n^a} X_i$, where the random variable X_i takes value 1 if policyholder i survives and 0 otherwise. We assume there is one systematic risk factor denoted by Z and $\mathbb{P}[X_i = 1 \,|\, Z = 0] = p_0$ and $\mathbb{P}[X_i = 1 \,|\, Z = 1] = p_1$. The random variable Z can be interpreted as a longevity index for a population and we assume here $\mathbb{P}[Z = 0] = \mathbb{P}[Z = 1] = 0.5$. We assume $p_0 < p_1$ and given Z, the random variables $X_1, X_2, \ldots, X_{n^a}$ are independent. The survival probabilities depend on the realization of the longevity index Z. The distribution of S is shown in Figure 1, where we use $p_0 = 0.75, p_1 = 0.85$ and $n^a = 1000$. The x-axis shows the possible realizations of the random variable S, that is, the possible claim amounts. The corresponding frequencies are shown on the y-axis. This distribution is not approximately normal, as was the case for a diversifiable claim; see (2.2). Increasing the size of the portfolio, that is, increasing n^a will not result in a normal distribution for the claim. Moreover, if we use the standard deviation principle given by (2.5), the risk margin will not decrease to zero if we increase the portfolio size. The vertical solid line corresponds with the value of the claim S when a standard deviation principle is applied with $\beta = 1$. A similar example was considered in Milevsky *et al*. (2006) to illustrate that a portfolio of life insurance policies cannot be priced using a standard

https://doi.org/10.1017/asb.2023.8 Published online by Cambridge University Press

deviation principle in case there is a longevity risk factor affecting the survival probabilities of the policyholders.

If we condition on the longevity index Z, the claim is diversifiable. Indeed, consider the random variable $S_k = S \,|Z = k$, where $k = 0, 1$. Then S_k is a diversifiable claim. Moreover, if we use a standard deviation principle ρ^a to valuate the claim, we find that $\rho^a\,[S_k] \to \mathbb{E}[S_k]$ if $n^a \to +\infty$. The two dashed lines in Figure 1 correspond with the value of the claims S_k, when a standard deviation principle is used. In this paper, we propose a new valuation that takes into account that valuating the claim S using solely a valuation based on diversification is not appropriate, but will exploit the fact that the conditional claims S_0 and S_1 are diversifiable. ▼

2.3. Hybrid claims

In this paper, we consider an insurance company holding a portfolio with liabilities. The portfolio consists of a total of n^a policyholders. We assume the random variable X_i is modeling policyholder specific risks. For example, X_i can be a random variable taking the value 1 if the policyholder survives until time T and 0 otherwise. We assume that claims depending on the insurer's policyholder specific risks are not traded in the financial market and are therefore not perfectly replicable by the traded assets $\boldsymbol{Y}$. However, the random variables X_i, $i = 1, 2 \ldots, n^a$ can be dependent on the financial market, meaning that they may be partially hedgeable.

We assume the portfolio of the insurer is also exposed to a number of non-traded systematic risks. We denote these non-traded systematic risks by the random vector $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{n^s})$. We assume that the random variables X_i denoting the policyholder specific risks are conditionally independent:

$$\mathbb{P}\left[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_{n^a} \leq x_{n^a} |\, \boldsymbol{Y} = \boldsymbol{y}, \boldsymbol{Z} = \boldsymbol{z}\right] = \prod_{i=1}^{n^a} \mathbb{P}\left[X_i \leq x_i |\, \boldsymbol{Y} = \boldsymbol{y}, \boldsymbol{Z} = \boldsymbol{z}\right]. \tag{2.6}$$

We thank the anonymous referee to point out that this condition is a simplifying assumption. The complexity of some practical situations may distort the conditional independence.

Consider a claim S which can be expressed as $S = h\,(\boldsymbol{Z})$ for some function h. This claim is called a systematic claim, and we denote the set with all such claims by $\mathcal{C}^{\boldsymbol{Z}}$. The set with all claims containing financial and systematic information is denoted by $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$, that is, if $S \in \mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$, then $S = h\,(\boldsymbol{Y}, \boldsymbol{Z})$ for some function h.

Definition 2.1. *A claim S is said to be a pure actuarial claim if S is independent of the financial and non-traded systematic risks. We denote by $\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$ the set of all pure actuarial claims*

$$\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}} = \{S \in \mathcal{C} \text{ and } S \text{ is independent of } \boldsymbol{Y} \text{ and } \boldsymbol{Z}\}\,.$$

We will investigate a portfolio of policies, where the payoff for policyholder i at maturity T is given by $S^f \times g\,(X_i, \boldsymbol{Z})$. The claim S^f is a financial derivative and g is a known function. Note that the financial derivative S^f is not necessarily hedgeable, that is, it may be that $S^f \notin \mathcal{C}^h$.

The per-policy liability to the insurance company is then given by

$$S = S^f \times \frac{\sum_{i=1}^{n^a} g\,(X_i, \boldsymbol{Z})}{n^a}. \tag{2.7}$$

The structure (2.7) together with Condition (2.6) allows to decompose the hybrid claim in three distinctive parts which individually require a different type of valuation. The aim of this paper is to propose a valuation that shows how to combine these different type of valuations.

In this paper, we assume that hedgeable claims are valuated using risk neutral valuation and pure actuarial claims are valuated with a given actuarial valuation; see (2.3). The idea of the 3-step hedge-based valuation is that we have an actuarial valuation at hand that we want to use for valuating the diversifiable part of the claim. Consider a claim S of the form (2.7) and define the conditional random

https://doi.org/10.1017/asb.2023.8 Published online by Cambridge University Press

variable $S_{y,z}$ which is given by $S_{y,z} = (S|\, \boldsymbol{Y} = \boldsymbol{y},\ \boldsymbol{Z} = \boldsymbol{z})$. Then we find from Condition (2.6) that $S_{y,z}$ is a diversifiable claim (see (2.1)) and therefore can be valuated using an actuarial valuation. Moreover, the random vector $\boldsymbol{Z}$ can be interpreted as the non-traded systematic effects. The valuation proposed in this paper then assumes that the systematic effects should be valuated with a different valuation than the diversifiable part. We assume that we know how to deal with diversifiable claims through a particular choice of an actuarial valuation ρ^a. Since we only have a claim that is diversifiable given the outcomes of the traded and non-traded systematic effects, we can only determine the value of the claim in each systematic scenario. Otherwise stated, once we know $\boldsymbol{y}$ and $\boldsymbol{z}$, we can valuate this claim $S_{y,z}$ using the actuarial valuation ρ^a. Since the outcomes of the systematic scenario are random, we will determine a value $\rho^a\left[S_{y,z}\right]$ for each possible realization of $\boldsymbol{Y}$ and $\boldsymbol{Z}$. In the last step of our valuation, we apply a separate valuation that takes into account the uncertainty about the systematic scenario, that is, the values for $\boldsymbol{y}$ and $\boldsymbol{z}$, that will eventually unfold.

Note that not all claims in $\mathcal{C}$ are product claims of the form (2.7). For such a general hybrid claim, however, one may not be able to define a diversifiable part which should be valuated with an actuarial valuation. Indeed, diversifiability of a claim implies the insurer pools many policyholders with the same policy and is able to increase the size of the portfolio. If the hybrid claim does not contain a diversifiable part, that is, in case we do not have the part $\frac{\sum_{i=1}^{n^a} g(X_i,\boldsymbol{Z})}{n^a}$ together with the Condition (2.6), it may not be intuitively clear why one has to combine different valuation principles for the residual part of the claim. An example of a claim that does not have the structure as in (2.7) are option-type payoffs, for example, $S = \left(S^f \times \frac{\sum_{i=1}^{n^a} g(X_i,\boldsymbol{Z})}{n^a} - K\right)_+$, where $(x)_+ = \max(x, 0)$. Another example is the claim $S = S^f \times \prod_{i=1}^{n} X_i$. If the X_i takes value 1 in case policyholder i survives and 0 otherwise, the claim S only pays the amount S^f if everyone survives. However, these examples are rather theoretical and by restricting the set of claims to product claims of the form (2.7), we cover most realistic situations.

By considering claims S which can be expressed as in (2.7), we assume that each of the n^a policyholders receives the same financial benefit S^f. If an insurer sells different types of contracts, we have that the financial benefit depends on the policyholder i, that is, we have to write S_i^f. In this case, we first subdivide the contracts in portfolios where each policy receives the same financial benefit and secondly, we valuate each portfolio separately. In Bacinello *et al*. (2011), the authors also consider product claims and describe different choices for S^f, encompassing the most common minimum death and living guarantees. Their valuation then assumes a perfectly diversified portfolio, that is, there is no risk margin in the actuarial valuation and the financial market is independent of the mortality risk, that is, $\boldsymbol{Y}$ is independent of $\boldsymbol{Z}$ and $\boldsymbol{X}$; see also Brennan and Schwartz (1976b), Deelstra *et al*. (2020) for the one-period case and Ballotta *et al*. (2021), Delong *et al*. (2019b) for the multiperiod case. Product claims with dependent financial and non-financial risks are considered in Deelstra *et al*. (2016), Salahnejhad and Pelsser (2020), Barigou *et al*. (2022).

We write g instead of $g(X_i, \boldsymbol{Z})$ if no confusion is possible. The claim S is called a hybrid claim, since it is a combination of different type of risks. The aim of this paper is to determine a valuation framework to valuate the hybrid claim S.

3. Fair valuations

3.1. Definition

A hybrid claim depends on the information about the traded assets $\boldsymbol{Y}$. It is then reasonable to impose that a valuation of a hybrid claim takes into account the market prices $\boldsymbol{y}$ of the traded assets. The valuation of the hybrid claim should be 'consistent' with the available market information. Such valuations are called *market-consistent* valuations. However, employing the market information may not be appropriate for a pure actuarial claim. Therefore, we also require that valuations are *model consistent*. Combining market consistency with model consistency leads to the class of fair valuations; see also Dhaene *et al*. (2017).

Definition 3.1 (Fair valuation). *A valuation ρ is*

- market consistent *if for any claim $S \in \mathcal{C}$ and trading strategy $\boldsymbol{\nu}$, we have:*

$$\rho\left[S + \boldsymbol{\nu} \cdot \boldsymbol{Y}\right] = \rho[S] + \boldsymbol{\nu} \cdot \boldsymbol{y}, \tag{3.1}$$

- model consistent *if there exists an actuarial valuation $\pi \in \mathcal{A}$ such that for any pure actuarial claim $S^{\perp} \in \mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$, we have:*

$$\rho\left[S^{\perp}\right] = \pi\left[S^{\perp}\right], \tag{3.2}$$

- fair *if it is market consistent and model consistent.*

If we decompose a hybrid claim in a hedgeable part $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ and a residual part S, then any market-consistent value is the sum of the value of the hedgeable and residual part. One can buy the hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ at the price $\boldsymbol{\nu} \cdot \boldsymbol{y}$ and therefore the value of the hedgeable part is unambiguously determined by its hedging cost. The residual part S, on the contrary, has to be valuated using the valuation ρ. If the value of a hybrid claim is determined with a market-consistent valuation, this value is consistent with the prices of the available traded financial assets in that it takes into account the market price to buy the hedge.

The set of all model-consistent valuations is denoted by $\mathcal{M}$. Expression (3.2) states that a model-consistent valuation employs a given actuarial valuation π, which is usually different from ρ, for pricing a pure actuarial claim. Note that the set of actuarial valuations is a subset of the model-consistent valuations, that is, $\mathcal{A} \subseteq \mathcal{M}$. For general hybrid claims, the valuation ρ will be a combination of the hedging cost $\boldsymbol{\nu} \cdot \boldsymbol{y}$ and a model-consistent valuation.

In Dhaene *et al*. (2017), the authors consider a slightly different definition of a model-consistent valuation. Indeed, in their setting, a valuation is model consistent if an actuarial valuation is used for any claim which is independent of the financial assets. Hence, claims which depend on non-traded systematic information, but are independent of the financial market are also valuated using an actuarial valuation. Note, however, that using a less restrictive definition for a model-consistent, and therefore also a fair valuation does not change the results of Dhaene *et al*. (2017) which we will need in this paper.

We have to define the notion of a hedger. These concepts were first defined in Dhaene *et al*. (2017) and we refer to this paper for a detailed discussion and more properties on hedgers and hedge-based valuations. A *hedger* is a function $\boldsymbol{\theta} : \mathcal{C} \to \mathbb{R}^{n^f+1}$ which is normalized, that is, $\boldsymbol{\theta}_0 = (0, 0, \ldots, 0)$ and translation invariant, that is, $\boldsymbol{\theta}_{S+a} = \boldsymbol{\theta}_S + \left(e^{-rT}a, 0, 0, \ldots, 0\right)$, for $S \in \mathcal{C}$ and $a \in \mathbb{R}$. The hedging strategy $\boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n^f)}\right)$ is also called the hedge for the claim S. The component $\theta_S^{(i)}$ of the hedge determines the amount of units to be bought of the traded asset Y_i. It is reasonable to assume that the hedge of a hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ corresponds with $\boldsymbol{\nu}$. Consider the set $\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$ containing the claims that are independent of the traded assets $\boldsymbol{Y}$ and the non-traded systematic risks $\boldsymbol{Z}$. It is also reasonable to assume that the hedge for a claim in $\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$ only consists of a position in the risk-free bank account. Therefore, we assume that all hedgers we encounter in this paper are fair hedgers, as defined in Dhaene *et al*. (2017).

The idea of market-consistent, model-consistent and fair hedgers was first introduced in Dhaene *et al*. (2017). The following theorem was proven in Theorem 3 of Dhaene *et al*. (2017) and characterizes the class of valuations that are both market and model consistent.

Theorem 3.1. *Consider a valuation $\rho : \mathcal{C} \to \mathbb{R}$. The following two statements are equivalent:*

1. *The valuation ρ is a fair valuation, that is, it is market consistent and model consistent.*
2. *The valuation ρ is a hedge-based valuation: there exist a model-consistent valuation $\pi \in \mathcal{M}$ and a fair hedger $\boldsymbol{\theta}$ such that*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right]. \tag{3.3}$$

The valuation defined in (3.3) is called a *hedge-based valuation* and was introduced in Dhaene *et al*. (2017) in a one-period setting. In Barigou and Dhaene (2019), Barigou *et al*. (2019) and Chen *et al*.

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(2021), the authors consider the hedge-based valuations in a multi-period setting, whereas continuous-time versions of the hedge-based valuations are considered in Delong *et al*. (2019a,b).

A hedged-based valuation employs a fair hedger $\boldsymbol{\theta}$ to determine the hedgeable part of a hybrid claim. This hedgeable part can be priced using the available market information. The remaining, residual, part of the claim is then priced using an appropriate model-consistent valuation. We say that $S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$ is the residual part of the hybrid claim S.

4. Fair valuations with non-traded systematic risks

4.1. Definition of the 3-step hedge-based valuation

One may argue that, when using the hedge-based valuation, one can use an actuarial valuation to valuate the residual part. The actuarial valuation should then be determined such that it corresponds with setting up a sufficiently large capital buffer as the risk management strategy to cope with the residual part. For example, one may use a standard deviation principle with a sufficiently large safety loading, or the VaR/TVaR with a confidence level close to 1. An example of such a valuation is explored in Example 2. Theorem 3.1, however, states that there exist fair valuations which employ a model-consistent, but not an actuarial, valuation to valuate the residual part. In this paper, we will consider these fair valuations. A first example of such a valuation is introduced in Example 3.

Example 2 (Capital for the residual part). By decomposing a claim S in a hedgeable part and a residual part using a hedger $\boldsymbol{\theta}$, we mainly have to focus on the valuation of the residual part. Indeed, the value of the hedgeable part is uniquely determined by its hedging cost. One possibility to valuate the residual part is to determine a sufficiently large capital buffer. For example, one could use the following valuation:

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right],$$

where π is an actuarial valuation. The amount $\pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right]$ is a capital buffer which is invested in the risk-free asset until the maturity T of the claim. For example, one can use the VaR to determine the capital buffer. ▼

Example 3 (The quantile hedging valuation of Barigou *et al*. (2021)). In Barigou *et al*. (2021), the authors start from the hedge-based valuation, but propose to use a quantile hedging approach for the residual part of the claim. Assume that $i \in (0, 1)$ is the cost-of-capital rate. The quantile hedging valuation is defined as follows:

$$\rho\left[S\right] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + i \times \boldsymbol{\eta}^{(l)}_{S-\boldsymbol{\theta}_S\cdot Y} \cdot \boldsymbol{y}, \tag{4.1}$$

where $\boldsymbol{\theta}$ is a fair hedger and $\boldsymbol{\eta}^{(l)}$ is a hedger defined as follows:

$$\boldsymbol{\eta}^{(l)}_S = \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^{n^f+1}} \mathbb{E}\left[l\left(S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} - \boldsymbol{\beta} \cdot \boldsymbol{Y}\right)\right], \tag{4.2}$$

and $l : \mathbb{R} \to [0, +\infty)$ is a convex function satisfying $l(x) = 0 \leftrightarrow x = 0$. The residual part is valuated using a model-consistent valuation which is not necessarily actuarial. Instead of using a capital buffer for coping with the residual part of the hybrid claim, the underlying risk management strategy is buying an appropriate investment portfolio. Note that the hedger $\boldsymbol{\eta}^l$ is a fair hedger. ▼

The difference between Examples 2 and 3 lies in the valuation of the residual part. Example 2 uses a 'passive' risk management strategy where the risks of the residual part are covered by a capital buffer. Example 3, on the other hand, uses an 'active' risk management strategy which invests in an appropriate investment portfolio. In this section, we will build a valuation framework that incorporates both approaches.

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We can now define the class of $(\boldsymbol{Y},\boldsymbol{Z})$ −conditional valuations, which are valuations that map a hybrid claim into the set $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$.

Definition 4.1 ($(\boldsymbol{Y},\boldsymbol{Z})$–conditional valuation). *A* $(\boldsymbol{Y},\boldsymbol{Z})$ −conditional valuation *is a function* $\pi\left[\cdot|\ \boldsymbol{Y},\boldsymbol{Z}\right]:\mathcal{C}\to\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ *attaching the claim* $\pi\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]\in\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ *to a hybrid claim* $S\in\mathcal{C}$ *such that*

1. $\pi\left[\cdot|\ \boldsymbol{Y},\boldsymbol{Z}\right]$ *is normalized:* $\pi\left[0|\ \boldsymbol{Y},\boldsymbol{Z}\right]=0$,
2. $\pi\left[\cdot|\ \boldsymbol{Y},\boldsymbol{Z}\right]$ *is conditionally translation invariant:* $\pi\left[S+S^h|\ \boldsymbol{Y},\boldsymbol{Z}\right]=\pi\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]+S^h$, *for* $S^h\in\mathcal{C}^h$.

Note that we only require conditionally translation invariance for hedgeable claims. In the remainder of the paper, we will write 'conditional valuation' instead of '$(\boldsymbol{Y},\boldsymbol{Z})-$ conditional valuation' if no confusion is possible.

Assume that we have an actuarial valuation $\rho^a\in\mathcal{A}$. In case a claim is independent of the financial and the non-traded systematic risks, we require that the claim is valuated using ρ^a. Indeed, since we consider claims of the form (2.7), independence of the financial market and the systematic risks implies that the claim is diversifiable, which justifies the use of an actuarial valuation. If a conditional valuation is consistent with this requirement, we say that the valuation is *conditional model-consistent*.

Definition 4.2. *The conditional valuation* $\pi\left[\cdot|\ \boldsymbol{Y},\boldsymbol{Z}\right]:\mathcal{C}\to\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ *is a conditional model-consistent valuation if pure actuarial claims are valuated with an actuarial valuation:*

$$e^{-rT}\pi\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]=\rho^a\left[S\right],\ \text{for}\ S\in\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}},\tag{4.3}$$

where $\rho^a\in\mathcal{A}$ *is a model-consistent valuation.*

An example of a conditional model-consistent valuation is the conditional standard deviation principle, which is defined as follows:

$$\rho^a\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]=\mathbb{E}\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]+\beta\sqrt{\mathrm{Var}\left[S|\ \boldsymbol{Y},\boldsymbol{Z}\right]}.\tag{4.4}$$

We can now define the *3-step hedge-based valuation*. In a first step, an appropriate hedging strategy $\boldsymbol{\theta}_S$ is used to offset a hedgeable part of a hybrid claim S. In the second valuation step, the residual claim $S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}$ is transformed by using a conditional model-consistent valuation. Indeed, since we consider claims of the form (2.7), conditioning on $\boldsymbol{Y}$ and $\boldsymbol{Z}$ will transform the residual claim in a diversifiable claim for which the actuarial valuation ρ^a can be used. The last, and third step valuates the remaining claim using a model-consistent valuation ρ^s.

Definition 4.3. (3-step hedge-based valuation). *A valuation* ρ *is said to be a 3-step hedge-based valuation if for any claim* $S\in\mathcal{C}$*, it can be expressed as follows:*

$$\rho[S]=\boldsymbol{\theta}_S\cdot\boldsymbol{y}+\rho^s\left[\rho^a\left[S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}|\ \boldsymbol{Y},\boldsymbol{Z}\right]\right],\tag{4.5}$$

where $\boldsymbol{\theta}$ *is a fair hedger,* ρ^s *is a model-consistent valuation principle and* ρ^a *is a conditional model-consistent valuation principle.*

The first term in (4.5) corresponds with using risk neutral valuation for the hedgeable part of a claim. The conditional valuation corresponds with using a valuation which is based on diversification. There are different choices for the valuation ρ^s, depending on the risk management strategy one wants to implement. In the following example, we discuss a possible choice for the systematic valuation ρ^s.

Example 4 (Linear systematic valuation $\boldsymbol{\rho^s}$ consistent with the market prices). In this example, we consider the systematic valuation which was proposed in Deelstra *et al*. (2020). Assume that ρ^s can be expressed as follows:

$$\rho^s\left[S\right]=\mathrm{e}^{-rT}\mathbb{E}\left[\varphi\times S\right],\tag{4.6}$$

for some non-negative random variable φ, where we assume φ to be in $\mathcal{C}^{\boldsymbol{Y},\boldsymbol{Z}}$ with $\mathbb{E}\left[\varphi\right]=1$. The valuation ρ^s uses the expectation of the claim, but only after transforming the claim S with an appropriate

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'distortion' φ. The distortion allows to use a prudent valuation approach by considering the claim in a less preferable, or stressed, scenario. The value $\rho^s[S]$ can then be interpreted as the expected payout of the claim S in the stressed scenario. If we define the probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$ using its Radon–Nikodyn derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = \varphi$, then we can write $\rho^s[S] = e^{-rT}\mathbb{E}_{\mathbb{Q}}[S]$. The liability S is priced using an expectation under a stressed probability measure. This valuation corresponds with a risk management strategy which provides a sufficient large capital buffer to absorb extreme losses which arise in less preferable systematic scenarios. In order to determine the distortion φ, one can follow the approach of Deelstra *et al*. (2020), where the authors require that the valuation ρ^s should be consistent with the market prices, that is, $\rho^s[Y_i] = y_i$, for $i = 1, 2, \ldots, n^f$. One can employ the Esscher transform, which was introduced in Esscher (1932) to find the φ in this way; see also Gerber and Shiu (1995). ▼

Example 5 (Modified standard deviation principle). Assume we aim for a valuation ρ which is consistent with risk-neutral valuation for hedgeable claims and with the standard deviation principle for pure actuarial claims. We can then use the following 3-step hedge-based valuation:

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^s\left[\mathbb{E}\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \mid \boldsymbol{Y}, \boldsymbol{Z}\right] + \beta\sqrt{\mathrm{Var}\left[S \mid \boldsymbol{Y}, \boldsymbol{Z}\right]}\right]. \tag{4.7}$$

Indeed, it is easy to verify that $\rho[S^h] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S^h\right]$ for $S^h \in \mathcal{C}^h$ and $\rho[S] = e^{-rT}\left(\mathbb{E}[S] + \beta\sqrt{\mathrm{Var}[S]}\right)$ for a claim $S \in \mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$. ▼

It is straightforward to see that the valuation introduced in Example 2 is a 3-step hedge-based valuation. Below, we show that the quantile hedging valuation of Barigou *et al*. (2021) is also an example of a 3-step hedge-based valuation.

Example 6 (The quantile hedging valuation (continued)). Consider the valuation introduced in Example 3. The hedger $\boldsymbol{\eta}^{(l)}$ defined in (4.2) is a fair hedger, see Theorem 4 in Dhaene *et al*. (2017). We can define the conditional valuation as follows $\rho^a[S \mid \boldsymbol{Y}, \boldsymbol{Z}] = \boldsymbol{\eta}_S^{(l)} \cdot \boldsymbol{Y}$. Since the hedger $\boldsymbol{\eta}^{(l)}$ is model consistent, the valuation $\rho^a[\cdot \mid \boldsymbol{Y}, \boldsymbol{Z}]$ is conditional model consistent. The valuation ρ^s is defined as follows $\rho^s[S] = i \times \boldsymbol{\mu}_S \cdot \boldsymbol{y}$, for some fair hedger $\boldsymbol{\mu}$. Indeed, we then find that $\rho^s[\rho^a[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \mid \boldsymbol{Y}, \boldsymbol{Z}]] = i \times \boldsymbol{\eta}^{(l)}_{S-\boldsymbol{\theta}_S \cdot \boldsymbol{Y}} \cdot \boldsymbol{y}$. Moreover, since $\boldsymbol{\mu}$ is a fair hedger, the valuation ρ^s is a model-consistent valuation. We can then conclude that the quantile hedging valuation is a 3-step hedge-based valuation. ▼

Example 7 (The two-step valuation of Pelsser and Stadje (2014)). We assume that the financial market is complete and there exists a pricing measure $\mathbb{Q}$. The 2-step valuation is given by $\rho^{2-step}[S] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\pi\left[S \mid \boldsymbol{Y}\right]\right]$, where $\pi\left[\cdot \mid \boldsymbol{Y}\right] : \mathcal{C} \to \mathcal{C}^{\boldsymbol{Y}}$ is a conditional valuation. We can rewrite this expression as follows

$$\rho^{2-step}[S] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\pi\left[S - H_S^h \mid \boldsymbol{Y}\right]\right], \tag{4.8}$$

where $H_S^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$ and $\boldsymbol{\theta}$ is a fair hedger. This expression shows that the 2-step valuation is also a 3-step hedge-based valuation provided the conditional valuation π is conditional model consistent. The first term of (4.8) corresponds to the price to buy the hedge for the claim S when a hedger $\boldsymbol{\theta}$ is used. The conditional valuation is used to separate the financial risks from the non-financial risks in the residual part of the claim. Since the market is assumed to be complete, the claim $\pi\left[S - H_S^h \mid \boldsymbol{Y}\right]$ is hedgeable and therefore the second term in (4.8) corresponds with the price to buy the investment strategy to replicate $\pi\left[S - H_S^h \mid \boldsymbol{Y}\right]$. Different choices for the conditional valuation π are considered in Example 3.4 of Pelsser and Stadje (2014).

4.2. Characterisation

The following theorem shows that the 3-step hedge-based valuation is a fair valuation.

Theorem 4.1. *A 3-step hedge-based valuation, is a fair valuation.*

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Proof. Consider the 3-step hedge-based valuation ρ given by (4.5). Since $\boldsymbol{\theta}$ is a fair hedger, it is also a market-consistent hedger and we directly find that ρ is market consistent. The hedger is also model consistent, from which we find that

$$\rho\left[S^{\perp}\right]=\rho^{s}\left[\rho^{a}\left[S^{\perp}\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right],\ \text{for } S^{\perp}\in\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}.$$

Since $\rho^{a}\left[\cdot\mid \boldsymbol{Y},\boldsymbol{Z}\right]$ is a conditional model-consistent valuation, $\rho^{a}\left[S^{\perp}\mid \boldsymbol{Y},\boldsymbol{Z}\right]=\rho^{a}\left[S^{\perp}\right]$. Model consistency of the valuation ρ then follows from the model consistency of the valuation ρ^{a}. □

Theorem 4.2. *Consider a fair valuation ρ. Then, there exists a model-consistent valuation ρ^{s} and a conditional model-consistent valuation $\rho^{a}\left[\cdot\mid \boldsymbol{Y},\boldsymbol{Z}\right]$ such that*

$$\rho[S]=\boldsymbol{\theta}_S\cdot\boldsymbol{y}+\rho^{s}\left[\rho^{a}\left[S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}\mid \boldsymbol{Y},\boldsymbol{Z}\right]\right].$$

Proof. Consider a fair hedger $\boldsymbol{\theta}$. Since ρ is market consistent, we can write

$$\rho[S]=\boldsymbol{\theta}_S\cdot\boldsymbol{y}+\rho\left[S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}\right].$$

We use Theorem 1 of Dhaene *et al*. (2017) which states that the fair valuation ρ can be expressed as follows:

$$\rho[S]=\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\boldsymbol{\mu}_S\cdot\boldsymbol{Y}\right], \tag{4.9}$$

where $\mathbb{Q}$ is a risk neutral measure and $\boldsymbol{\mu}$ is a model-consistent hedger. Define the conditional valuation as follows:

$$\rho^{a}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]=\boldsymbol{\mu}_S\cdot\boldsymbol{Y}, \tag{4.10}$$

and the valuation ρ^{s} as follows

$$\rho^{s}\left[S\right]=\begin{cases}\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{P}}[S] & \text{if } S\in\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}},\\ \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}[S] & \text{if } S\neq\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}.\end{cases} \tag{4.11}$$

Note that ρ^{s} is model consistent. Since $\boldsymbol{\mu}$ is model consistent, there exists an actuarial valuation $\rho^{a}\in\mathcal{A}$ such that $\rho^{a}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]=\rho^{a}\left[S\right]$ for $S\in\mathcal{C}^{\perp,\boldsymbol{Y},\boldsymbol{Z}}$ and therefore it is a conditional model-consistent valuation. Combining (4.10) and (4.11) together with (4.9) proves the result. □

We thank the anonymous referee to point out that the notation of the conditional valuation might be misleading, since with our choice $\rho^{a}\left[S\mid \boldsymbol{Y},\boldsymbol{Z}\right]$ does not depend on $\boldsymbol{Z}$.

Theorem 4.2 shows that a fair valuation has to be a 3-step hedge-based valuation. Combining Theorems 4.1 and 4.2, we find that the class of fair valuations is equivalent with the class of the 3-step hedge-based valuations:

$$\rho \text{ is a fair valuation } \Leftrightarrow \rho \text{ is a 3-step hedge-based valuation.}$$

It was shown in Dhaene *et al*. (2017) that the class of hedge-based valuations is equivalent with the class of fair valuations. Moreover, the 2-step valuation introduced in Pelsser and Stadje (2014) also characterizes the class of fair valuations. We can conclude that the 3-step hedge-based valuations, the 2-step valuations and the hedge-based valuations are describing the same class of valuations:

$$\rho \text{ is a 2-step valuation } \Leftrightarrow \rho \text{ is a hedge-based valuation} \Leftrightarrow \rho \text{ is a 3-step hedge-based valuation.} \tag{4.12}$$

In this section, we introduced a new valuation for hybrid claims S. Moreover, similar to the 2-step and the hedge-based valuation, this new valuation is defined on $\mathcal{C}$. However, in our illustrations and examples, we only consider claims of the form (2.7). This approach is similar to Dhaene *et al*. (2017) and Barigou *et al*. (2021), who also define a general hybrid valuation principle but mainly focus on product claims of the form (2.7). In this case, conditioning on the vectors $\boldsymbol{Y}$ and $\boldsymbol{Z}$ results in a diversifiable claim, and we have an actuarial valuation together with a corresponding risk management strategy that we want to employ for such a diversifiable claim. However, if we have a claim $S\in\mathcal{C}$ that is not diversifiable after

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Table 1. The joint probabilities for the random vector (Y_1, Z, X_1).

Y_1	Z	X_1	Probabilities
1	0	1	0.04
1	0	0	0.10
1	1	1	0.20
1	1	0	0.14
0	0	1	0.05
0	0	0	0.15
0	1	1	0.30
0	1	0	0.02

conditioning on $\boldsymbol{Y}$ and $\boldsymbol{Z}$, the choice of $\rho^a[\,\cdot\,|\boldsymbol{Y},\boldsymbol{Z}]$ may not be intuitive clear. In this case, instead of first applying a conditional valuation, one may prefer to value the whole residual part $S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}$ with a single model-consistent valuation, which means we use a hedge-based valuation instead. The 3-step hedge-based valuations and the hedge-based valuations both provide a characterization for the class of fair valuations. The choice between the two valuations depends on the situation at hand.

4.3. Illustration

Consider a financial market where there is one tradeable asset and a risk-free bank account, that is, $\boldsymbol{Y}=(Y_0, Y_1)$. The market prices are denoted by $(1, y_1)$. The time-T asset value Y_1 can take values 0 or 1 and we assume the risk-free rate $r=0$. The claim S is defined as follows:

$$S = Y_1 \times \frac{\sum_{i=1}^{n^a} X_i}{n^a}. \tag{4.13}$$

The random variables $X_1, X_2, \ldots, X_{n^a}$ are assumed to be identical, but not independent. Moreover, we assume that $X_i=1$ corresponds with the survival of policyholder i at the maturity T, whereas $X_i=0$ otherwise. We assume that Z denotes the systematic longevity risk of an insurance portfolio. The joint probabilities of the random vector (Y_1, Z, X_1) are listed in Table 1. There are two longevity scenarios. In the first scenario, $Z=0$ and we experience a decrease in longevity, whereas $Z=1$ corresponds with the scenario where longevity is increasing. Indeed, one can verify that

$$\mathbb{P}\left[X_1=1|Z=0\right]=0.26, \quad \text{and} \quad \mathbb{P}\left[X_1=1|Z=1\right]=0.76.$$

Survival probabilities are larger if $Z=1$ compared to the situation where $Z=0$. Note that the random variable Z is dependent on the financial market. Indeed, we have that $\mathbb{P}\left[Y_1=1|Z=0\right]=0.41$, whereas $\mathbb{P}\left[Y_1=1|Z=1\right]=0.51$.

4.3.1. The hedgeable part

We assume that the hedge of a claim S is determined by the *mean-variance hedger* $\boldsymbol{\theta}_S$ (see also Theorem 5 of Dhaene *et al*. (2017)). Using the probabilities listed in Table 1, we then find that $\theta_S^{(1)}=\frac{\mathbb{E}[Y_1X_1]}{\mathbb{E}[Y_1]}=0.5$ and $\theta_S^{(0)}=0$. The hybrid claim S can be decomposed in two parts:

$$S=\theta_S^{(1)}Y_1+Y_1\left(\frac{1}{n^a}\sum_{i=1}^{n^a}\left(X_i-\theta_S^{(1)}\right)\right). \tag{4.14}$$

The first term is the hedgeable part of the hybrid claim and can be valuated using the corresponding hedging strategy, which consists of buying 0.5 units of the stock Y_1. The second term is the residual part, which is the part that remains after we take into account the income from the hedging strategy.

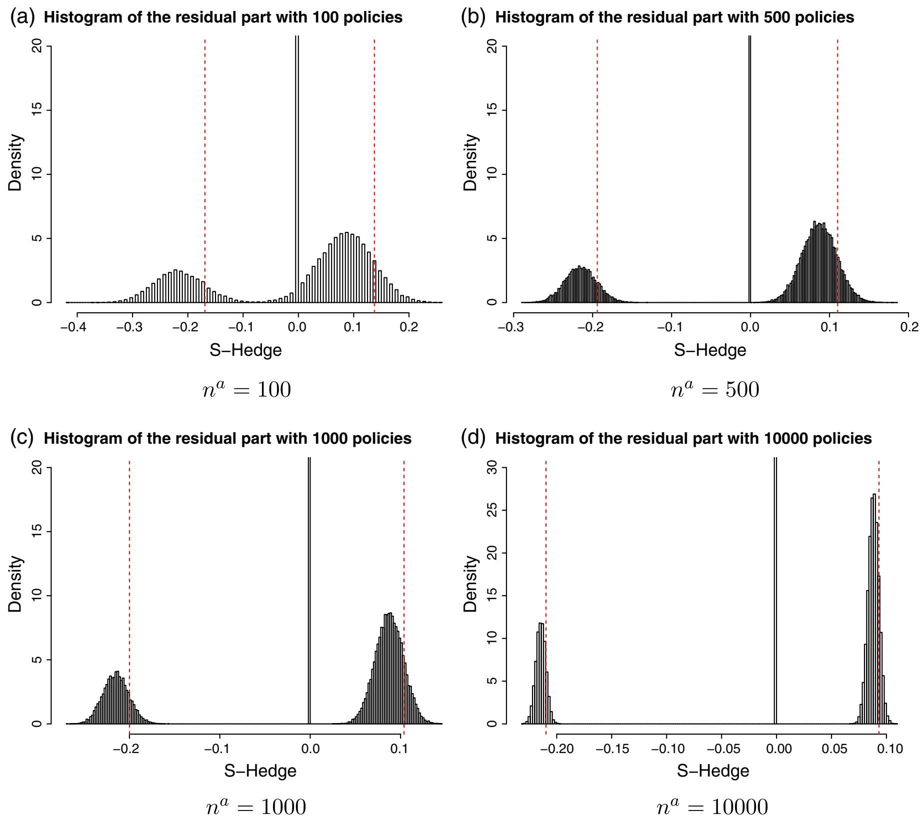

Figure 2. *Histogram of the residual part $S - \theta_S^{(1)} Y_1$ of the claim S given by (4.13). The red dotted lines correspond with the actuarial value of the residual part of the claim given that $Z = 0$ and $Z = 1$.*

Figure 2 shows a histogram of the residual part for different values of the portfolio size n^a. The residual part is not diversifiable since the random variables $X_1, X_2, \ldots, X_{n^a}$ are dependent and each term includes the asset value Y_1. The histogram shows that the variance does not converge to zero as we increase the portfolio size. Note also that the expected value is approximately zero.

4.3.2. 3-step valuation with the conditional standard deviation principle

The histogram shows there are three different ‘scenarios’. There is a positive probability mass in zero, which corresponds with the event where $Y_1 = 0$. There is a scenario where the residual part is positive, that is, we are underhedging the claim. This scenario corresponds with the situation where $Y_1 = 1$ and $Z = 1$. Indeed, in this case there is a strongly increasing longevity and therefore more policyholders survive than anticipated when determining the hedge $\theta_S^{(1)}$. The negative scenario corresponds with the situation where $Y_1 = 1$ and $Z = 0$. In this scenario, we are overhedging the claim since there are less survivors than initially anticipated. Note that $\mathbb{P}[Y_1 = 1, Z = 0] = 0.14$ whereas $\mathbb{P}[Y_1 = 1, Z = 1] = 0.34$, which explains why there is more probability mass in the positive part than in the negative part of the histograms.

Given a particular scenario of the asset and the longevity risk, one has a diversifiable claim. Indeed, consider, for example, the case where $Y_1 = 1$ and $Z = 1$, which corresponds with the positive part of the histograms shown in Figure 2. Given that we are in this positive scenario, applying an actuarial valuation

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principle is justified. For example, applying the conditional standard deviation principle gives

$$\rho^a\left[S-\theta_S^{(1)}Y_1 \mid Y_1=1, Z=1\right]=0.09+\beta\sqrt{\frac{0.24}{n^a}}. \tag{4.15}$$

For the negative scenario, applying a standard deviation principle results in

$$\rho^a\left[S-\theta_S^{(1)}Y_1 \mid Y_1=1, Z=0\right]=-0.21+\beta\sqrt{\frac{0.20}{n^a}}. \tag{4.16}$$

The vertical dashed lines in Figure 2 correspond with the location of the values (4.15) and (4.16) on the x-axis.

The 3-step hedge-based valuation takes into account that given the realization of Y_1 and Z, an actuarial valuation ρ^a is appropriate. Otherwise stated, given the financial and the systematic scenario, one can determine an appropriate risk management strategy based on diversification for the residual claim and therefore valuate the residual claim accordingly. A valuation ρ^s is used to cope with the uncertainty about the particular scenario that will eventually unfold.

We assume, for simplicity, that the safety loading for the conditional standard deviation principle is equal to one, that is, $\beta=1$. The 3-step hedge-based valuation for S is then given by

$$\rho[S]=\theta_S^{(1)}y_1+\rho^s[\epsilon],$$

where y_1 is the market price of the traded asset, ρ^s is a model-consistent valuation and the random variable ϵ is given by

$$\epsilon=\begin{cases}-0.21+\beta\sqrt{\frac{0.20}{n^a}} & \text{if } Y_1=1, Z=0,\\ 0 & \text{if } Y_1=0,\\ 0.09+\beta\sqrt{\frac{0.24}{n^a}} & \text{if } Y_1=1, Z=1.\end{cases}$$

4.3.3. Choice 1 for the systematic valuation: risk-free capital buffer

A possible choice for the valuation ρ^s is a prudent valuation, such as the Value-at-Risk, Tail Value-at-Risk or another $\mathbb{P}$-law invariant risk measure. This corresponds with a risk management strategy where one builds a sufficiently large capital buffer. In our particular example, we have that $\mathbb{P}\left[\epsilon<0.09+\beta\sqrt{\frac{0.24}{n^a}}\right]=0.66$, from which we find that

$$\rho^s[\epsilon]=\mathrm{VaR}_{0.95}[\epsilon]=0.09+\beta\sqrt{\frac{0.24}{n^a}}. \tag{4.17}$$

Otherwise stated, we prepare for the worst-case scenario. One can also use a cost-of-capital approach resulting in a value for $\rho^s[\epsilon]$ which corresponds with the cost of borrowing the amount $0.09+\beta\sqrt{\frac{0.24}{n^a}}$.

4.3.4. Choice 2 for the systematic valuation: Esscher transform

The random variable ϵ is dependent on the financial market. Therefore, one may consider a systematic valuation that takes into account the market prices of the traded assets. For example, if we use the valuation of Example 4, we consider a probability measure $\mathbb{Q}$ and define the following probabilities under this new probability measure

$$\mathbb{Q}[Y_1=0, Z=0]=q_1, \qquad \mathbb{Q}[Y_1=0, Z=1]=q_2$$

$$\mathbb{Q}[Y_1=1, Z=0]=q_3 \text{ and } \mathbb{Q}[Y_1=1, Z=1]=q.$$

The probability measure $\mathbb{Q}$ should satisfy the following conditions:

$$\mathbb{E}_{\mathbb{Q}}[Y_1]=y_1 \quad\text{and}\quad \mathbb{E}_{\mathbb{Q}}[Z]=\mathbb{E}[Z]+\gamma,$$

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for some $0 < \gamma < 1 - \mathbb{E}[Z]$. The first condition states that the valuation should take into account the market prices. The second condition states that the valuation of the longevity index should take into account a pre-specified longevity risk margin γ. Solving these equations leads to $q_3 = y_1 - q$, $q_2 = \mathbb{E}[Z] + \gamma - q$ and $q_1 = 1 - q_2 - q_3 - q$, for some $q \in (\mathbb{E}[Z] + \gamma + y_1 - 1, \min\{y_1, \mathbb{E}[Z] + \gamma\})$. We then find that

$$\rho^s[\epsilon] = (y_1 - q) \times \left(-0.21 + \beta\sqrt{\frac{0.20}{n^a}}\right) + q\left(0.09 + \beta\sqrt{\frac{0.24}{n^a}}\right), \tag{4.18}$$

for some $q \in (0, y_1)$. Note that this valuation principle is a linear combination between the best and the worst-case situation.

5. Independence between financial and non-financial risks

5.1. *A closed-form expression*

In this section, we assume that financial risks are independent of the non-financial risks, that is, the random vector $\boldsymbol{Y}$ is independent of both $\boldsymbol{Z}$ and $\boldsymbol{X}$. We will use the mean-variance hedger, which is a *fair* hedger (see Dhaene *et al*. (2017)), to determine the hedgeable part of a hybrid claim. The mean-variance hedger is defined as follows:

$$\boldsymbol{\theta}_S = \arg\min_{\boldsymbol{\mu}\in\Theta} \mathbb{E}\left[(S - \boldsymbol{\mu}_S \cdot \boldsymbol{Y})^2\right]. \tag{5.1}$$

Proposition 1. *Assume the random vector $\boldsymbol{Y}$ is independent of $\boldsymbol{Z}$ and $\boldsymbol{X}$. The mean-variance hedger $\boldsymbol{\theta}_S$ for the product claim (2.7) can be expressed as follows:*

$$\boldsymbol{\theta}_S = \boldsymbol{\theta}_{S^f} \times \mathbb{E}\left[g\right], \tag{5.2}$$

where $\boldsymbol{\theta}_{S^f}$ is the mean-variance hedge for the financial derivative S^f. The hedgeable part of the claim is given by

$$H_S^h = \mathbb{E}\left[g_1\right] \times H_{S^f}^h, \tag{5.3}$$

where $H_{S^f}^h = \boldsymbol{\theta}_{S^f} \cdot \boldsymbol{Y}$.

Proof. One can show that the mean-variance hedger can be determined by solving the following system of equations

$$\sum_{m=1}^{n^f} \mathrm{Cov}\left[Y_k, Y_m\right] \theta_S^{(m)} = \mathrm{Cov}\left[S, Y_k\right], \text{ for } k = 1, 2, \ldots, n^f, \tag{5.4}$$

$$\theta_S^{(0)} = \mathrm{e}^{-rT}\left(\mathbb{E}[S] - \sum_{i=1}^{n^f} \theta_S^{(i)} \mathbb{E}[Y_i]\right). \tag{5.5}$$

We have that $\mathrm{Cov}[S, Y_k] = \mathbb{E}[g]\mathrm{Cov}[S^f, Y_k]$. Denote by Σ the matrix containing all the covariances $\mathrm{Cov}[Y_i, Y_j]$, that is, $(\Sigma)_{i,j} = \mathrm{Cov}[Y_i, Y_j]$. Moreover, use the notation $\tilde{\boldsymbol{\theta}}_S = \left(\theta_S^{(1)}, \theta_S^{(2)}, \ldots, \theta_S^{(n^f)}\right)$ and $\boldsymbol{\beta} = \left(\mathrm{Cov}[S^f, Y_1], \ldots, \mathrm{Cov}[S^f, Y_{n^f}]\right)$. Then we can write (5.5) as $\Sigma\tilde{\boldsymbol{\theta}}_S{}' = \boldsymbol{\beta}'\mathbb{E}[g]$, from which we find that

$$\tilde{\boldsymbol{\theta}}_S{}' = \left(\Sigma^{-1}\boldsymbol{\beta}\right)\mathbb{E}[g].$$

Note that $\Sigma^{-1}\boldsymbol{\beta}$ corresponds with the mean-variance hedge of the financial derivative S^f. Indeed, we have that

$$\boldsymbol{\theta}_{S^f} = \left(\theta_{S^f}^{(0)}, \tilde{\boldsymbol{\theta}}_{S^f}\right).$$

It remains to prove that $\theta_S^{(0)} = \theta_{S^f}^{(0)}\mathbb{E}[g]$. We can write

$$\theta_S^{(0)} = e^{-rT}\left(\mathbb{E}[S] - \mathbb{E}[g]\sum_{i=1}^{n^f}\theta_{S^f}^{(i)}\mathbb{E}[Y_1]\right)$$
$$= e^{-rT}\left(\mathbb{E}[S] - \mathbb{E}[g]\left(\mathbb{E}[S^f] - \theta_{S^f}^{(0)}e^{rT}\right)\right)$$
$$= e^{-rT}\left(\mathbb{E}[S] - \mathbb{E}[g]\mathbb{E}[S^f] + \theta_{S^f}^{(0)}e^{rT}\right).$$

Taking into account that $\mathbb{E}[S] = \mathbb{E}[g]\mathbb{E}\left[S^f\right]$ then proves the result. □

In case the financial derivative in (2.7) is hedgeable, the hedgeable part can be expressed as follows $H_S^h = \mathbb{E}\left[g\right] \times S^f$. The hedger for the claim S invests in $\mathbb{E}\left[g\right]$ units of the hedgeable financial derivative. Note also that the mean-variance hedge does not depend on the portfolio size n^a. A similar result was derived in Møller (2001), where the authors determine the mean-variance hedger in the situation where there is a risky stock and a bank account. To be more precise, a hybrid claim where $S^f = Y_1$ is used in their example; see Section 3 in Møller (2001).

We can prove the following result which provides an explicit expression for the 3-step value of the product claim.

Proposition 2. *Assume the random vector* $\boldsymbol{Y}$ *is independent of* $\boldsymbol{Z}$ *and* $\boldsymbol{X}$. *The modified standard deviation principle, introduced in Example 5, for the product claim S given by (2.7) can be expressed as follows*

$$\rho^{3-step}[S] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \rho^s\left[S^f\left(\mathbb{E}\left[g|\,\boldsymbol{Z}\right] + \beta\sqrt{\frac{Var\left[g|\,\boldsymbol{Z}\right]}{n^a}}\right) - H_{S^f}^h\mathbb{E}\left[g\right]\right]. \tag{5.6}$$

Proof. The modified standard deviation principle of the claim S can be expressed using Expression (4.5) as follows:

$$\rho\left[S\right] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \rho^s\left[\rho^a\left[S - H_S^h|\,\boldsymbol{Y},\boldsymbol{Z}\right]\right], \tag{5.7}$$

where H_S^h is given by (5.3) and

$$\rho^a\left[S - H_S^h\right] = \mathbb{E}\left[S - H_S^h|\,\boldsymbol{Y},\boldsymbol{Z}\right] + \beta\sqrt{\mathrm{Var}\left[S - H_S^h|\,\boldsymbol{Y},\boldsymbol{Z}\right]}.$$

We can write:

$$\mathbb{E}\left[S - H_S^h|\,\boldsymbol{Y},\boldsymbol{Z}\right] = \mathbb{E}\left[S|\,\boldsymbol{Y},\boldsymbol{Z}\right] - H_S^h$$
$$= S^f\mathbb{E}\left[\sum_{i=1}^{n^a} g\left(\boldsymbol{Z},X_i\right) \mid \boldsymbol{Y},\boldsymbol{Z}\right] - H_S^h$$
$$= S^f\mathbb{E}\left[g|\,\boldsymbol{Z}\right] - H_S^h.$$

For the conditional variance, we find

$$\mathrm{Var}\left[S - H_S^h|\,\boldsymbol{Y},\boldsymbol{Z}\right] = \mathrm{Var}\left[S|\,\boldsymbol{Y},\boldsymbol{Z}\right]$$
$$= \left(S^f\right)^2\mathrm{Var}\left[\sum_{i=1}^{n^a} g\left(\boldsymbol{Z},X_i\right) \mid \boldsymbol{Y},\boldsymbol{Z}\right]$$
$$= \left(S^f\right)^2\frac{\mathrm{Var}\left[g|\,\boldsymbol{Z}\right]}{n^a}.$$

Plugging these expressions in (5.7) gives the desired result. □

The first term in (5.6) corresponds to the price we have to pay if we buy the mean-variance hedge. The amount we receive from the hedging portfolio is H_S^h. However, the hedging portfolio assumes the expected number of units of S^f needed is equal to $\mathbb{E}[g]$. If we have information about the longevity risk, this number will be updated. There are two sources that cause hedging errors: systematic and unsystematic deviations from the mean $\mathbb{E}\left[g\right]$. The systematic part is captured by the random variable $\mathbb{E}\left[g|\,\boldsymbol{Z}\right]$ whereas the unsystematic fluctuations are captured by $\frac{\mathrm{Var}\left[g|\,\boldsymbol{Z}\right]}{n^a}$.

5.1.1. Systematic valuation with the Esscher transform

In this subsection we consider the valuation using the 3-step hedge-based valuation when an Esscher transform is used; see Example 4. In order to determine the distortion φ, we follow the approach of Deelstra *et al*. (2020) and employ the Esscher transform

$$\text{Esscher distortion:}\ \ \varphi = \frac{\mathrm{e}^{-\sum_{i=1}^{n^s} w_i Z_i - \sum_{i=1}^{n^f} v_i Y_i}}{\mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{n^s} w_i Z_i - \sum_{i=1}^{n^f} v_i Y_i}\right]}, \tag{5.8}$$

for some constants $w_1, w_2, \ldots, w_{n^s}$ and $v_1, v_2, \ldots, v_{n^f}$. In order to determine the valuation (4.6) with Esscher transform (5.8), we need a joint calibration of the weights $w_1, w_2, \ldots, w_{n^s}$ and $v_1, v_2, \ldots, v_{n^f}$. In this section, we assume financial risks to be independent from $\boldsymbol{Y}$ and $\boldsymbol{Z}$. The valuation ρ^s can then be expressed as

$$\rho^s\left[S\right] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\varphi^f \times \varphi^s\right) S\right]. \tag{5.9}$$

From (5.9), we find that for financial derivatives S^f, we can write $\rho^s\left[S^f\right] = \mathrm{e}^{-rT}\mathbb{E}\left[\varphi^f S^f\right]$, which means that φ^f is the financial distortion. In this paper, we assume that the distortion φ^f is tuned in such a way that the valuation ρ^s is using risk neutral valuation for all financial claims. The risk neutral measure $\mathbb{Q}$ may be estimated from available market information. However, there are only finitely many traded assets and infinitely many possible risk neutral measures. The choice of the risk neutral measure is therefore not always straightforward. The distortion φ^s can be calibrated solely using the non-traded systematic claims. Indeed, for claims S^s which are functions of only the vector $\boldsymbol{Z}$, we find that $\rho^s[S^s] = \mathrm{e}^{-rT}\mathbb{E}\left[\varphi^s S^s\right]$. In Zeddouk and Devolder (2019), for example, the authors propose to calibrate the distortion φ^s based on a given risk margin imposed on longevity products. We conclude that in this special situation where non-traded systematic and financial risks are independent, we can determine the valuation defined in (5.9) in two steps. The traded assets will result in an estimate for φ^f whereas one can impose conditions on the risk margin of the non-traded systematic risks to calibrate the distortion φ^s.

It is straightforward to verify that the value of the 3-step hedge-based valuation when using the Esscher transform can be expressed as follows

$$\rho^{Esscher}[S] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S^f\right]\left(\rho^s\left[\mathbb{E}\left[g|\,\boldsymbol{Z}\right]\right] + \beta\rho^s\left[\sqrt{\frac{\mathrm{Var}\left[g|\,\boldsymbol{Z}\right]}{n^a}}\right]\right).$$

The 3-step hedge-based value considers the price of the financial derivative S^f, which has to be determined under a pricing measure $\mathbb{Q}$. The expression between brackets represents the number of financial derivatives needed, taking into account the information contained in $\boldsymbol{Z}$. Since the Esscher transform is linear, the two terms are valuated separately. The term between brackets only depends on the vector $\boldsymbol{Z}$, which means the distortion φ^s is used to determine $\rho^s\left[\mathbb{E}\left[g|\,\boldsymbol{Z}\right]\right]$ and $\rho^s\left[\sqrt{\frac{\mathrm{Var}\left[g|\,\boldsymbol{Z}\right]}{n^a}}\right]$. Note that in case we use the Esscher transform, the value is independent on the choice of the hedger. However, since the market is incomplete and the claim S^f may be not perfectly replicable, the 3-step value depends on the choice of the risk-neutral pricing measure. In case the claim S^f is hedgeable, the financial part can be determined in a model-free way using only prices of traded assets, that is, in this case we do not need to make any subjective choice about the hedger or the risk-neutral measure.

5.1.2. Systematic valuation with the VaR

A drawback of the systematic valuation explained in the previous subsection is the choice of the pricing measure $\mathbb{Q}$ in case the claim is not hedgeable. Therefore, in this subsection, we consider a prudent approach which determines an appropriate capital buffer:

$$\rho^{VaR}[S] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \mathrm{e}^{-rT}\mathrm{VaR}_p\left[S^f\left(\mathbb{E}\left[g|\,\boldsymbol{Z}\right] + \beta\sqrt{\frac{\mathrm{Var}\left[g|\,\boldsymbol{Z}\right]}{n^a}}\right) - H_{S^f}^h\mathbb{E}\left[g\right]\right]. \tag{5.10}$$

The amount $\rho^{VaR}[S]$ can be decomposed in two parts. The first term corresponds with the price to buy the hedging portfolio. The second term is the amount of cash invested in the risk-free bank account to cover the residual part. One can also include a cost-of-capital rate γ in the valuation, which implies the capital buffer is borrowed and the valuation only takes into account the cost to borrow this amount. In our case, only the real-world probability measure $\mathbb{P}$ is used to valuate the residual part of the claim. However, one now has to make a choice about the confidence level p. We use the VaR but one may decide to use another risk measure (e.g., Tail Value-at-Risk).

5.1.3. The valuation proposed in Møller (2002)

In Møller (2002) Section 5.2, the author considers the valuation of hybrid claims by transforming the standard deviation principle, under the assumption there are no systematic risks and the financial risks $\boldsymbol{Y}$ are independent of the actuarial risks $\boldsymbol{X}$. The financial market is assumed to be complete and there exists a pricing measure $\mathbb{Q}$. The distribution of $\boldsymbol{X}$ is not affected by the change of measure and the random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent under $\mathbb{Q}$. Consider the product claim (2.7) and assume that we start from the valuation ρ defined in Example 5 where $\boldsymbol{\theta}$ is the mean-variance hedger and $\rho^s[S] = \mathrm{e}^{-rT}\mathbb{E}[S]$. One can then write

$$\rho[S] = \mathrm{e}^{-rT}\left(\mathbb{E}_{\mathbb{Q}}[S] + \beta\mathbb{E}\left[\mathrm{Var}\left[S|Y\right]\right]\right), \tag{5.11}$$

where we used Proposition 1 together with the fact that completeness implies that S^f is hedgeable. In this particular case, our 3-step hedge-based valuation coincides with the valuation proposed by Møller (2002) in Section 5.2.2. Since $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent (under both the real world and the risk-neutral world) and the distribution of $\boldsymbol{X}$ is unchanged when moving to the $\mathbb{Q}$-measure, we can interpret the first term in (5.11) as the cost to buy the hedging portfolio for the claim S. The second term is an extra buffer because the claim is not perfectly hedgeable.

Instead of defining ρ^s as the real-world expectation, one can also define ρ^s as the $\mathbb{Q}$ expectation. Since $\mathbb{E}_{\mathbb{Q}}\left[S|\boldsymbol{Y}\right] = \mathbb{E}\left[S|\boldsymbol{Y}\right]$, we find that the modified standard deviation can then be expressed as follows

$$\rho[S] = \mathrm{e}^{-rT}\left(\mathbb{E}_{\mathbb{Q}}[S] + \beta\mathbb{E}_{\mathbb{Q}}\left[\mathrm{Var}\left[S|Y\right]\right]\right). \tag{5.12}$$

Comparing (5.11) and (5.12), we observe that both valuations start from the mean-variance hedge for the claim S and adjust its value by taking into account the conditional variance of the claim. The valuation (5.11) considers the real-world expectation, implying a capital buffer is used for covering future losses. The valuation (5.12) takes into account that the market is complete and therefore $\mathrm{Var}\left[S|Y\right]$ is hedgeable. We can rewrite (5.12) as $\rho[S] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}[S^f]\left(\mathbb{E}[g] + \beta\mathrm{Var}\left[\frac{g}{n^a}\right]\right)$. The mean-variance hedge for S invests in $\mathbb{E}[g]$ units of S^f. The valuation (5.12) increases the number of units invested in the claim S^f by $\beta\mathrm{Var}\left[\frac{g}{n^a}\right]$ to account for the hedging error.

In Section 5.1.1, we use the Esscher transform with distortion (5.9) for the valuation ρ^s. This combines the approaches discussed in (5.11) and (5.12) in that we use the risk-neutral measure for the financial part of the claim and the real-world measure for the actuarial part.

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5.2. Numerical illustration

We will provide a numerical illustration of the 3-step valuation under the assumption that X_i takes the value 1 if the policyholder survives until time T and zero otherwise. The policyholder i will pay an initial premium equal to P at time $t = 0$ and receives the amount S^f at time T provided the policyholder is alive. The claim S is then given by

$$S = S^f \frac{\sum_{i=1}^{n^a} X_i}{n^a}. \tag{5.13}$$

The premium is invested in a stock (or risky fund) and the amount S^f the policyholder receives at time T depends on the performance of this risky fund. If we denote the time-T value of this investment by Y_1, we assume that S^f can be expressed as follows:

$$S^f = 1 + \alpha \left(Y_1 - P(v+1)^T\right)_+, \tag{5.14}$$

where α is the bonus share and v is an internal rate of return. If the policyholder survives, he will always receive the amount 1 at maturity, however, this amount can be increased if the risky fund performs well.

We assume that the stock Y_1 follows a geometric Brownian motion, which implies that the time-T value has a lognormal distribution:

$$\log \frac{Y_1}{P} \stackrel{d}{=} N\left(\left(\mu^f - \frac{1}{2}\left(\sigma^f\right)^2\right)T, \left(\sigma^f\right)^2 T\right), \tag{5.15}$$

for some $\mu^f \in \mathbb{R}$ and $\sigma^f \geq 0$. The spot price of the stock is equal to P, since we invest the full premium in the risky asset. We assume here that we can only trade in one risky asset to hedge the claim. We assume that the conditional survival probabilities can be expressed as follows:

$$\mathbb{P}\left[X_1 = 1 | Z\right] = \mathrm{e}^Z. \tag{5.16}$$

The random variable Z models the longevity risk among the policyholders. If Z is large, the probability of surviving goes up for each policyholder, whereas the survival probability goes down if Z goes down. If we assume that the dynamics of the systematic longevity risk can be modeled using an Ornstein–Uhlenbeck process, we find a normal distribution for the random variable Z. However, from (5.16), we find that the random variable Z should be negative since e^Z is a probability. Therefore, define the random variable $\tilde{Z}$ as follows:

$$\tilde{Z} \stackrel{d}{=} N\left(\mu^s, (\sigma^s)^2\right), \tag{5.17}$$

for some $\mu^s \in \mathbb{R}$ and $\sigma^s \geq 0$. Note that μ^s and σ^s depend on the maturity T of the contract. More details are given in the online Appendix. The random variable Z is then defined as follows

$$Z = \begin{cases} \tilde{Z}, & \text{if } \tilde{Z} < 0 \\ 0, & \text{if } \tilde{Z} \geq 0. \end{cases} \tag{5.18}$$

The survival probability for a policyholder is given by $\mathbb{E}[X_1]$ and assuming (5.18) we can derive a closed-form expression for this probability.

Proposition 3. *Consider the random variable Z defined in (5.18). We have that*

$$\mathbb{E}[X_1] = e^{\mu^s + \frac{1}{2}(\sigma^s)^2} \Phi\left(-\left(\frac{\mu^s}{\sigma^s} + \sigma^s\right)\right) + \Phi\left(\frac{\mu^s}{\sigma^s}\right),$$

$$\mathbb{E}\left[\left(\mathbb{E}\left[X_1|Z\right]\right)^2\right] = e^{2\mu^s + 2(\sigma^s)^2} \Phi\left(-\left(\frac{\mu^s}{\sigma^s} + 2\sigma^s\right)\right) + \Phi\left(\frac{\mu^s}{\sigma^s}\right),$$

$$\mathbb{E}\left[Var\left[X_1|Z\right]\right] = \mathbb{E}[X_1] - \mathbb{E}\left[\left(\mathbb{E}\left[X_1|Z\right]\right)^2\right],$$

where Φ is the distribution function of a standard normal distribution.

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Proof. We have that $\mathbb{E}[X_1] = \mathbb{E}[\mathrm{e}^Z]$ and we can write

$$\mathbb{E}\left[\mathrm{e}^Z\right] = \int_{-\infty}^{0} \mathrm{e}^z f_{\tilde{Z}}(z)\mathrm{d}z + \mathbb{P}\left[\tilde{Z} \geq 0\right],$$

where $f_{\tilde{Z}}$ is the density of $\tilde{Z}$. The integral and the probability can be derived since $\tilde{Z}$ is normal distributed.

In order to prove the second equality, we note that $\mathbb{E}\left[(\mathbb{E}\left[X_1|Z\right])^2\right] = \mathbb{E}\left[\left(\mathrm{e}^Z\right)^2\right]$ and therefore we can write

$$\mathbb{E}\left[(\mathbb{E}\left[X_1|Z\right])^2\right] = \int_{-\infty}^{0} \mathrm{e}^{2z} f_{\tilde{Z}}(z)\mathrm{d}z + \mathbb{P}\left[\tilde{Z} \geq 0\right].$$

We can then derive the desired expression using the same approach as we did in the first part of this proof.

We have that $\mathrm{Var}\left[X_1|Z\right] = \mathrm{e}^Z\left(1 - \mathrm{e}^Z\right)$ which proves the last equality. □

5.2.1. Systematic valuation with the Esscher transform

We start with a linear valuation for the valuation ρ^s. To be more precise, the valuation ρ^s is given by (5.9) where $\varphi^s = \frac{\mathrm{e}^{-\theta^s Z}}{\mathbb{E}\left[\mathrm{e}^{-\theta^s Z}\right]}$, for some $\theta^s < 0$. The financial distortion φ^f is determined such that financial claims are valuated using a risk-neutral distribution $\mathbb{Q}$. Therefore, we have $\varphi^f = \mathrm{e}^{-\left(\frac{\mu^f - r}{\sigma^f}\right)^2 T - \frac{\mu^f - r}{\sigma^f} W}$, where $W = \frac{\log\frac{Y_1}{P} - \left(\mu^f - \frac{1}{2}\left(\sigma^f\right)^2\right)T}{\sigma^f}$.

If we use the modified standard deviation principle introduced in Example 5, the 3-step hedge-based valuation method gives the following value:

$$\rho^{Esscher}\left[S\right] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S^f\right]\left(\mathbb{E}\left[X_1\right]\mathrm{e}^{-\theta^s(\sigma^s)^2} + \beta\mathbb{E}\left[\mathrm{e}^{-\theta^s Z}\sqrt{\frac{\mathrm{Var}\left[X_1|\,\boldsymbol{Z}\right]}{n^a}}\right]\right). \tag{5.19}$$

A closed-form expression for $\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[S^f\right]$ can easily be obtained since the risky fund follows a lognormal distribution. The expectation $\mathbb{E}\left[X_1\right]$ is given in closed-form in Proposition 3. Note that in order to determine the 3-step value, one needs numerical simulation to determine the value of the diversifiable part of the claim. This term, however, tends to zero if the portfolio size increases. Therefore, in case one deals with a sufficiently large portfolio, this term may be neglected and the approximate 3-step value is given in closed-form. The parameter θ^s is assumed to be negative, since it implies that increasing longevity (and its volatility) leads to larger prices. Since we are valuating a unit-linked portfolio, the bad scenarios are those where longevity increases, that is, where there are more survivors than expected.

5.2.2. Systematic valuation with the VaR

The valuation using the VaR (see Section 5.1.2) depends on the choice of the hedger, which is assumed to be the mean-variance hedger. In order to determine the hedgeable part of the claim S, we use Proposition 1 and determine the hedge for the financial derivative S^f. Note that S^f is not perfectly hedgeable. The mean-variance hedge $\left(\theta^{(0)}_{S^f}, \theta^{(1)}_{S^f}\right)$ is given by

$$\theta^{(1)}_{S^f} = \frac{\mathrm{Cov}\left[S^f, Y_1\right]}{\mathrm{Var}\left[Y_1\right]} \tag{5.20}$$

$$\theta^{(0)}_{S^f} = \mathrm{e}^{-rT}\left(\mathbb{E}\left[S^f\right] - \theta^{(1)}_{S^f}\mathbb{E}\left[Y_1\right]\right). \tag{5.21}$$

If the financial derivative S^f is given by (5.14) and the stock Y_1 is described by (5.15), we have that $\mathbb{E}\left[Y_1\right] = P\mathrm{e}^{\mu T}$. Using standard techniques for lognormal distributions, we find

$$\mathbb{E}\left[S^f\right] = 1 + \alpha\mathbb{E}\left[Y_1\right]\Phi\left(d_1\right) - P(1+v)^T\Phi\left(d_2\right) \tag{5.22}$$

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$$\operatorname{Cov}\left[S^f, Y_1\right] = \alpha \mathbb{E}\left[Y_1\right]\left(\mathbb{E}\left[Y_1\right]\left(\mathrm{e}^{\left(\sigma^f\right)^2 T}\Phi\left(d_3\right) - \Phi(d_1)\right) - K\left(\Phi(d_1) - \Phi(d_2)\right)\right), \tag{5.23}$$

$$\operatorname{Var}\left[Y_1\right] = P^2 \mathrm{e}^{2\mu^f T}\left(\mathrm{e}^{\left(\sigma^f\right)^2 T} - 1\right), \tag{5.24}$$

and

$$d_1 = \frac{\log \frac{P}{K} + \left(\mu^f + \frac{1}{2}\left(\sigma^f\right)^2\right)T}{\sigma^f\sqrt{T}},\ d_2 = d_1 - \sigma^f\sqrt{T},\ \text{and } d_3 = d_1 + \sigma^f\sqrt{T},$$

where $K = P\left(1+v\right)^T$.

Proposition 4. *Consider the claim S given by (5.13). The dynamics of the risky fund are given by (5.15). The longevity risk is independent of the risky fund and modeled using (5.18). If the hedgeable part H_S^h of the hybrid claim S is determined using the mean-variance hedger, we find that*

$$H_S^h = \left(\boldsymbol{\theta}_{S^f} \cdot \boldsymbol{Y}\right) \times \ \mathbb{E}\left[X_1\right]$$

where $\boldsymbol{\theta}_{S^f}$ is given by (5.20), (5.21) and $\mathbb{E}\left[X_1\right]$ is given in Proposition 3.

Proposition 4 is important because it allows to determine the hedgeable part of the claim in closed-form while explicitly taking into account that the financial derivative S^f is not perfectly replicable. In order to determine the 3-step hedge-based value using Expression (5.10), we can determine the first term using Proposition 4. Indeed, we have that $\mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] = \mathbb{E}\left[X_1\right]\left(\theta_{S^f}^{(0)} + \theta_{S^f}^{(1)}P\right)$. The second term of (5.10), however, needs to be determined using Monte Carlo simulation.

5.2.3. The hedge-based valuation

We also determine values for our unit-linked portfolio when a hedge-based valuation is used where π is an actuarial valuation. The hedge-based valuation is given by $\rho^{HB}[S] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \pi\left[S - H_S^h\right]$, and we assume π is the standard-deviation principle with $\beta = 0.2$. We can express the hedge-based value as follows

$$\rho^{HB}[S] = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \mathrm{e}^{-rT}\beta\sqrt{A}, \tag{5.25}$$

where

$$A = \mathbb{E}\left[\left(S^f\right)^2\right]\left(\frac{\mathbb{E}\left[\operatorname{Var}\left[X_1 \mid Z\right]\right]}{n^a} + \mathbb{E}\left[\mathrm{e}^{2Z}\right]\right) + \operatorname{Var}\left[H_S^h\right] + \mathbb{E}[S]^2 - 2\mathbb{E}\left[X_1\right]\mathbb{E}\left[S^f Y_1\right].$$

Note that all terms of A are given in closed-form. Indeed, Proposition 3 can be used to determine $\mathbb{E}\left[\operatorname{Var}\left[X_1 \mid Z\right]\right]$, $\mathbb{E}\left[X_1\right]$ and $\mathbb{E}\left[\mathrm{e}^{2Z}\right]$. The closed-form expression for the covariance $\operatorname{Cov}\left[S^f, Y_1\right]$ shown in (5.23) leads to a closed-form expression for $\mathbb{E}\left[S^f Y_1\right]$. Taking into account $\mathbb{E}[S] = \mathbb{E}[S^f]\mathbb{E}[X_1]$, a closed-form expression for $\mathbb{E}[S]$ can be derived by combining Proposition 3 and (5.22). Finally, $\operatorname{Var}\left[H_S^h\right]$ can be determined in closed-form using (5.24) since H_S^h is a linear combination between the risk-free bond and the risky fund. An expression for $\rho^{HB}[S]$ was also derived in Exercise 12 of Dhaene (2022), but under the assumption the financial derivative S^f is hedgeable.

The numerical values and a discussion are provided in the online appendix.

6. The additive 3-step valuation

The idea of the 3-step hedge-based valuation is to determine a valuation which combines the risk neutral valuation principle, the conditional actuarial valuation principle ρ^a for diversifiable claims and the model-consistent valuation principle ρ^s for the remaining part. In this section, we take a different route to define 3-step valuations. We start by decomposing a hybrid claim in 3 different parts and valuate

https://doi.org/10.1017/asb.2023.8 Published online by Cambridge University Press

each part separately. This leads to an additive valuation. We show that this additive 3-step valuation is a subset of the more general 3-step hedge-based valuations.

We assume that a hedger for the hybrid claim $S \in \mathcal{C}$ is available and denoted by $\boldsymbol{\theta}_S$. We assume that $\boldsymbol{\theta}_S$ is a fair hedger (e.g., the mean-variance hedger). The claim H_S^h is defined as

$$H_S^h = \boldsymbol{\theta}_S \cdot \boldsymbol{Y}. \tag{6.1}$$

The claim H_S^h is a hedgeable claim, in the sense that there exists a trading strategy consisting of positions in the traded assets $\boldsymbol{Y}$ that can replicate the payoff of H_S^h. We thus have that $H_S^h \in \mathcal{C}^h$. The claim S can be decomposed in a hedgeable part and a residual part. The residual part $\left(S - H_S^h\right)$ is what remains of S after we apply the hedger $\boldsymbol{\theta}_S$. The claim H_S^s is defined as follows:

$$H_S^s = \mathbb{E}\left[S - H_S^h \vert\, \boldsymbol{Y}, \boldsymbol{Z}\right]. \tag{6.2}$$

The claim H_S^s only depends on the evolution of the traded assets and the non-traded systematic risks and we have that $H_S^s = h\left(\boldsymbol{Y}, \boldsymbol{Z}\right)$, for some function h. Note, however, that if the hedge H_S^h is adequate, then $S - H_S^h$ should not depend 'too strongly' on the traded risks $\boldsymbol{Y}$, meaning that H_S^s is mainly driven by the non-traded systematic risks $\boldsymbol{Z}$. Therefore, we also refer to H_S^s as the systematic part of the claim S. Roughly speaking, H_S^s is what remains in S after we first apply an appropriate hedge and then consider the average loss for different scenarios of the financial and non-traded systematic risks. We define the claim H_S^a as follows

$$H_S^a = \left(S - H_S^h - H_S^s\right). \tag{6.3}$$

The random variable H_S^a denotes the part of the claim S that remains after the hedgeable and the systematic parts are removed. Combining (6.1), (6.2) and (6.3), we can decompose the hybrid claim S as follows:

$$S = H_S^h + H_S^s + H_S^a. \tag{6.4}$$

Expression (6.4) decomposes the hybrid claim S into three parts. This decomposition was also introduced in Dhaene (2022) for product claims and Deelstra *et al*. (2020) for the situation where financial and actuarial risks are independent.

In order to value a hybrid claim S which can be decomposed in 3 different parts, one may wish to value the different parts separately using a different, appropriate valuation for each type of claim. This approach leads to an additive three-step valuation.

Definition 6.1 (Additive 3-step valuation). *Consider the model-consistent valuations ρ^s and ρ^a and a fair hedger $\boldsymbol{\theta}$. The additive 3-step valuation ρ^+ is defined as follows*

$$\rho^+[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^a\left[H_S^a\right] + \rho^s\left[H_S^s\right], \tag{6.5}$$

for any hybrid claim $S \in \mathcal{C}$, where H_S^h, H_S^s and H_S^a are given by (6.1), (6.2) and (6.3), respectively.

Theorem 6.1. *The additive valuation ρ^+ is a 3-step hedge-based valuation.*

Proof. It is straightforward to prove that ρ^+ is market and model consistent. From Theorem 4.2, we then find that the additive 3-step is a 3-step hedge-based valuation. □

The additive 3-step valuation ρ^+ is similar to the valuation proposed in Deelstra *et al*. (2020). Indeed, in Deelstra *et al*. (2020), the authors also consider an additive 3-step valuation for pricing hybrid claims by decomposing the hybrid claim in a hedgeable, diversifiable and residual part. However, the additive 3-step valuation introduced in this paper does not always coincide with their additive valuation. In Deelstra *et al*. (2020), the authors consider product claims of the following form:

$$S = \frac{1}{n^a} \sum_{i=1}^{n^a} g_i(X_i, Z) \times h_i(Y_1), \tag{6.6}$$

where g_i is a function of the policyholder specific risk X_i and the systematic longevity risk Z, whereas Y_1 is the time-T value of a traded stock or index. However, in Deelstra *et al*. (2020), the authors only consider a complete financial market, which implies that $h_i(Y_1)$ is hedgeable. The hedgeable part of the valuation in Deelstra *et al*. (2020) is given by

$$\text{Hedgeable part} = \frac{1}{n^a} \sum_{i=1}^{n^a} \mathbb{E}\left[g_i\right] \times h_i(Y_1). \tag{6.7}$$

In case the financial market is incomplete and the financial market consists of the traded stock and the risk-free bank account, the hedgeable part defined in (6.7) is not necessarily hedgeable, since it may not be possible to replicate the payoff $h_i(Y_1)$. In the online appendix, we provide two examples. Each example defines the hedgeable part (6.7) for a product claim. The first example assumes a complete market, whereas the second example considers an incomplete market. We also determine the hedgeable part as in (6.1) using the mean-variance hedge. The example shows that the claim defined in (6.7) is not necessarily hedgeable, whereas the claim defined in (6.1) is always hedgeable by construction.

7. Conclusion

In this paper, we introduced a new class of fair valuations, which we called the 3-step hedge-based valuations, for the valuation of hybrid claims that depend on hedgeable, diversifiable and non-traded systematic risks. This new valuation principle uses traded assets to construct a hedging portfolio for the hybrid claim. The value of this portfolio can be determined using the observable market prices. Equivalently, we express this price as a risk neutral expectation. We then employ a 2-step valuation for the residual part of the claim. The 3-step hedge-based valuation is a market-consistent valuation and can therefore be used to determine regulatory capital for complex insurance liabilities. Moreover, our valuation takes into account that systematic risks are fundamentally different from diversifiable risks and therefore require a different valuation principle. We follow Dhaene *et al*. (2017) and Deelstra *et al*. (2020) to decompose a hybrid claim in three different parts. We then define an additive 3-step valuation by applying an appropriate valuation to each of these parts. We show that this valuation is similar to the valuation defined in Deelstra *et al*. (2020). Moreover, we show that the additive 3-step valuation is a 3-step hedge-based valuation.

The idea of a fair valuation was introduced in Dhaene *et al*. (2017). A fair valuation finds a balance between pricing through hedging and pricing through modelling. Hedgeable claims should be priced using the replicating portfolio approach. Since we can observe the prices of the traded assets, pricing hedgeable claims through replication is model free. Market consistency of the valuation ensures that while valuating a hybrid claim, the hedgeable part is always consistent with the observable market prices. However, the incompleteness of the market requires a model to assess the risks of the unhedgeable part of the claim.The underlying model should be such that pure actuarial claims are priced using an actuarial valuation. We show that the class of 3-step hedge-based valuations coincides with the class of fair valuations. We can then conclude that the 3-step hedge-based valuations are equivalent with the hedge-based and the 2-step valuations.

Acknowledgement. The author thanks Prof. dr. Jan Dhaene, Prof. dr. Antoon Pelsser and dr. Karim Barigou, as well as the anonymous referees for their comments and suggestions which all greatly improved the paper.

Supplementary material. To view supplementary material for this article, please visit https://doi.org/10.1017/asb.2023.8.

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