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A note on the logarithmic derivative of the gamma function

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Introduction. The object of this note is to give simpler proofs of two formulae involving the function $\psi(z)$ which I have proved elsewhere by more complicated methods.[1]

The formulae are [2]

$$\psi(x+1) - \log x = 2\int_0^\infty \left\{\psi(t+1) - \log t\right\} \cos 2\pi xt \, dt \qquad (1)$$

for all real positive x, and

$$\sum_{n=1}^{\infty} \left\{\psi(1+nz) - \log nz - \frac{1}{2nz}\right\} + \frac{1}{2z}(\gamma - \log 2\pi z)$$
$$= \frac{1}{z} \sum_{n=1}^{\infty} \left\{\psi\left(1+\frac{n}{z}\right) - \log\frac{n}{z} - \frac{z}{2n}\right\} + \tfrac{1}{2}\left(\gamma - \log\frac{2\pi}{z}\right) \qquad (2)$$

for $|\arg z| < \pi$, where γ is Euler's constant.

Proof of (1). Now [3]

[1] *Journal London Math. Soc.*, 22 (1947), 14-18. $\psi(z)$ denotes $\Gamma'(z)/\Gamma(z)$.

[2] The first formula shows that $\psi(x+1) - \log x$ is self-reciprocal with respect to the Fourier cosine kernel $2\cos 2\pi x$. It is strange that this result should have been overlooked, but I can find no trace of it.

Cf. B. M. Mehrotra, *Journal Indian Math. Soc.* (New Series), 1 (1934), 209-27 for a list of self-reciprocal functions and references.

[3] C. A. Stewart, *Advanced Calculus*, (London, 1940), 495 and 497.

$$\psi(x+1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{e^t - 1}\right) dt \tag{3}$$

$$= \frac{1}{2x} + \log x - 2\int_0^\infty \frac{dt}{(t^2+x^2)(e^{2\pi t}-1)}. \tag{4}$$

Also

$$\int_0^\infty (e^{-t} - e^{-xt}) \frac{dt}{t} = \log x$$

by Frullani's integral,[1] and $\int_0^\infty \frac{dt}{t^2+x^2} = \frac{\pi}{2x}$.

Combining these results with (3) and (4) we have

$$\psi(x+1) - \log x = \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right) e^{-xt}\, dt$$

$$= 2\int_0^\infty \left(\frac{1}{2\pi t} - \frac{1}{e^{2\pi t}-1}\right) \frac{t\,dt}{t^2+x^2}. \tag{5}$$

Hence

$$2\int_0^\infty \left\{\psi(t+1) - \log t\right\} \cos 2\pi xt\, dt = 2\int_0^\infty \cos 2\pi xt\, dt \int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u-1}\right) e^{-ut}\, du$$

$$= 2\int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u-1}\right) du \int_0^\infty e^{-ut} \cos 2\pi xt\, dt$$

$$= 2\int_0^\infty \left(\frac{1}{u} - \frac{1}{e^u-1}\right) \frac{u\,du}{u^2+4\pi^2x^2}$$

$$= 2\int_0^\infty \left(\frac{1}{2\pi t} - \frac{1}{e^{2\pi t}-1}\right) \frac{t\,dt}{t^2+x^2}$$

$$= \psi(x+1) - \log x$$

by (5). The inversion of order of integration is justified by absolute convergence for u in the range (δ, ∞), $(\delta > 0)$ and the result follows on making δ tend to $+0$.

Proof of (2). Let $z = p/q$ where p and q are integers and $(p, q) = 1$. Consider the expression

$$S_N(p, q) = \frac{1}{q} \sum_{n=1}^{Nq} \left\{\psi\left(1 + \frac{np}{q}\right) - \log \frac{np}{q} - \frac{q}{2np}\right\}$$

$$- \frac{1}{p} \sum_{m=1}^{Np} \left\{\psi\left(1 + \frac{mq}{p}\right) - \log \frac{mq}{p} - \frac{p}{2mq}\right\}. \tag{6}$$

Now[2]
$$\psi(1+py) = \log p + \frac{1}{p} \sum_{r=1}^{p} \psi\left(y + \frac{r}{p}\right)$$

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[1] C. A. Stewart, loc cit. 457.

[2] C. A. Stewart, loc. cit. 504.

hence $$\psi\left(1+\frac{np}{q}\right)=\log p+\frac{1}{p}\sum_{r=1}^{p}\psi\left(\frac{np+rq}{pq}\right).$$

Substituting this result in both parts of (6) we have

$$S_N(p,q)=\frac{1}{q}\sum_{n=1}^{Nq}\left\{\frac{1}{p}\sum_{r=1}^{p}\psi\left(\frac{np+rq}{pq}\right)-\log\frac{n}{q}-\frac{q}{2np}\right\}$$
$$-\frac{1}{p}\sum_{m=1}^{Np}\left\{\frac{1}{q}\sum_{s=1}^{q}\psi\left(\frac{mq+sp}{pq}\right)-\log\frac{m}{p}-\frac{p}{2mq}\right\}. \quad (7)$$

Now $np+rq \quad (1\leqq n\leqq Nq, \quad 1\leqq r\leqq p)$

and $mq+sp \quad (1\leqq m\leqq Np, \quad 1\leqq s\leqq q)$

both run through the same sets of integers in the range $p+q$ to $(N+1)\,pq$, inclusive. Hence the repeated sums in (7) cancel, and

$$S_N(p,q)=-\frac{1}{q}\sum_{n=1}^{Nq}\left(\log\frac{n}{q}+\frac{q}{2np}\right)+\frac{1}{p}\sum_{m=1}^{Np}\left(\log\frac{m}{p}+\frac{p}{2mq}\right)$$
$$=N\log\frac{q}{p}+\frac{1}{p}\log(Np\,!)-\frac{1}{q}\log(Nq\,!)$$
$$+\frac{1}{2q}\sum_{m=1}^{Np}\frac{1}{m}-\frac{1}{2p}\sum_{n=1}^{Nq}\frac{1}{n}.$$

Now $$\log(M!)=(M+\tfrac{1}{2})\log M-M+\tfrac{1}{2}\log 2\pi+O(M^{-1}),$$

and $$\sum_{n=1}^{M}\frac{1}{n}=\log M+\gamma+O(M^{-1})$$

as $M\to\infty$. Hence, as $N\to\infty$

$$S_N(p,q)=N\log\frac{q}{p}+\frac{1}{p}\left\{(Np+\tfrac{1}{2})\log Np-Np+\tfrac{1}{2}\log 2\pi\right\}$$
$$-\frac{1}{q}\left\{(Nq+\tfrac{1}{2})\log Nq-Nq+\tfrac{1}{2}\log 2\pi\right\}$$
$$+\frac{1}{2q}(\gamma+\log Np)-\frac{1}{2p}(\gamma+\log Nq)+O(N^{-1})$$
$$=\frac{1}{2q}\left(\gamma-\log\frac{2\pi q}{p}\right)-\frac{1}{2p}\left(\gamma-\log\frac{2\pi p}{q}\right)+O(N^{-1})$$
$$=\frac{1}{q}\left\{\tfrac{1}{2}\left(\gamma-\log\frac{2\pi}{z}\right)-\frac{1}{2z}(\gamma-\log 2\pi z)\right\}+O(N^{-1}).$$

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Multiplying by q and re-arranging, we have

$$\sum_{n=1}^{Nq}\left\{\psi(1+nz)-\log nz-\frac{1}{2nz}\right\}+\frac{1}{2z}(\gamma-\log 2\pi z)$$

$$=\frac{1}{z}\sum_{n=1}^{Np}\left\{\psi\left(1+\frac{n}{z}\right)-\log\frac{n}{z}-\frac{z}{2n}\right\}+\tfrac{1}{2}\left(\gamma-\log\frac{2\pi}{z}\right)+O(N^{-1}). \qquad (8)$$

Further[1] if $|z|\to\infty$ in the region $|\arg z|\leqq\pi-\delta$, $(\delta>0)$ then

$$\psi(1+z)-\log z-\frac{1}{2z}\sim-\frac{1}{12z^2}.$$

Hence the two series in (8) converge absolutely as $N\to\infty$, and thus (2) holds for positive rational z. Further, both the series in (2) converge absolutely for $|\arg z|<\pi$, and they define analytic functions of z in this region. Hence (2) holds by analytic continuation for $|\arg z|<\pi$.

Extensions of (2). The method of the previous section can be used to prove that

$$z\sum_{n=1}^{\infty}\left\{\psi'(1+nz)-\frac{1}{nz}\right\}-\frac{1}{z}\sum_{n=1}^{\infty}\left\{\psi'\left(1+\frac{n}{z}\right)-\frac{z}{n}\right\}=\log z$$

for $|\arg z|<\pi$. For higher derivatives $(k\geqq 2)$ the corresponding results are

$$\sum_{n=1}\psi^{(k)}(1+nz)=z^{-k-1}\sum_{n=1}^{\infty}\psi^{(k)}\left(1+\frac{n}{z}\right). \qquad (9)$$

In these cases the results follow immediately on substituting[2]

$$\psi^{(k)}(1+ny)=(-1)^{k-1}k!\sum_{m=1}^{\infty}(m+ny)^{-k-1}$$

in each side of (9).

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[1] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927), 278.

[2] C. A. Stewart, loc. cit. 505.

Note added in proof.

Professor T. A. Brown tells me that he proved the self-reciprocal property of $\psi(1+x)-\log x$ some years ago, and that he communicated the result to the late Professor G. H. Hardy. Professor Hardy said that the result was also given in a progress report to the University of Madras by S. Ramanujan, but was not published elsewhere.

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