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TRIANGULARIZING SOLVABLE GROUPS OF UNIPOTENT MATRICES OVER A SKEW FIELD

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ABSTRACT. In this note we show that a solvable group of unipotent matrices over a skew field can be simultaneously triangularized.

It is well known (c.f. [1], p. 100) that a semigroup of unipotent matrices over a commutative field can be simultaneously triangularized. The corresponding question for a semigroup of unipotent matrices over a skew field is still unanswered. In this note we prove that the result holds for solvable groups of unipotent matrices over a skew field, and it follows that a group of unipotent matrices over a skew field can be triangularized if and only if it is solvable.

Before proving the main theorem we need a lemma about commuting unipotent matrices. A more general result is given in Theorem 2.1 of [2], but the proof is easier in the particular case given here:

LEMMA. *A set of commuting unipotent matrices over a skew field D can be simultaneously triangularized.*

Proof. Let Σ be a set of commuting unipotent $n\times n$ matrices. Denote by V the right D-space of n-dimensional column vectors. Then Σ acts on V by left multiplication in the natural way. We use induction on n to show that the lemma holds in case Σ leaves a non-trivial subspace of V invariant. If $n=1$, the lemma is clearly true, so assume $n>1$ and the result is true for sets of matrices of degree j whenever $n>j$. Suppose further that W is a non-trivial invariant subspace of dimension i. Let P be an invertible $n\times n$ matrix whose first i columns form a basis of W. Then for $M\in\Sigma$, $P^{-1}MP$ has the form

$$\begin{bmatrix} A_M & B_M \\ 0 & C_M \end{bmatrix},$$

where A_M is an $i\times i$ matrix. Then $\Sigma'=\{A_M \mid M\in\Sigma\}$ and $\Sigma''=\{C_M \mid M\in\Sigma\}$ are sets of commuting unipotent matrices of degree less than n, so by our induction hypothesis there are invertible matrices R, Q of the appropriate degrees such that $R^{-1}A_MR$ and $Q^{-1}C_MQ$ are upper triangular for all $M\in\Sigma$. Then

$$\begin{bmatrix} R^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} P^{-1}MP \begin{bmatrix} R & 0 \\ 0 & Q \end{bmatrix}$$

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is triangular for all M in Σ, and the lemma is proved. Thus we assume that Σ leaves no non-trivial subspaces of V invariant.

We now show that the absence of non-trivial invariant subspaces implies that $\Sigma=\{I\}$ and $n=1$. The result will follow. Let $M\in\Sigma$; M is unipotent, so $M=I+N$, where N is nilpotent. Since N is nilpotent, there is a non-zero vector $v\in V$ such that $Nv=0$. Thus $Mv=v$. Let $W=\{v\in V \mid Mv=v\}$. W is easily seen to be a non-zero subspace of V. If $A\in\Sigma$ and $w\in W$, then since Σ is commutative we have $MAw=AMw=Aw$. Thus $Aw\in W$, so W is a Σ-invariant subspace. Our assumption on non-trivial invariant subspaces implies $W=V$, and so by the way W was defined, $M=I$. But M was chosen to be any element of Σ, and we see that $\Sigma=\{I\}$; the assumption on non-trivial subspaces then shows us that $n=1$.

We can now prove the main

THEOREM. *Let D be a skew field, and let Γ be a solvable group of unipotent $n\times n$ matrices with entries in D. Then there exists an invertible matrix P with entries in D such that $P^{-1}MP$ is triangular for all M in Γ.*

Proof. Again, let V denote the right D-space consisting of column vectors. As in the lemma, an induction argument allows us to assume that Γ leaves no non-trivial subspaces of V invariant.

We shall now show that if Γ is a solvable group of unipotent matrices leaving no non-trivial subspace of V invariant then Γ is trivial and $n=1$. Γ is trivial if Γ is solvable of length 0; if on the other hand Γ is solvable of length $m>0$ then Γ^{m-1} is a non-trivial abelian normal subgroup of Γ. So to show Γ is trivial we need only show that it has no non-trivial abelian normal subgroups.

Let Δ be any abelian normal subgroup of Γ. By the lemma Δ can be upper triangularized; this fact and the fact that the matrices in Δ are unipotent imply that there is a non-zero vector $u\in V$ such that $Mu=u$ for all M in Δ. Let $W=\{v\in V \mid Mv=v \text{ for all } M\in\Delta\}$. Then W is a non-zero subspace of V. We want to show that Γ maps W into itself, so let $B\in\Gamma$, $w\in W$. By the definition of W, we must show that for any M in Δ, $MBw=Bw$. But for $M\in\Delta$, we have, since Δ is a normal subgroup of Γ, $MB=BM'$ for some M' in Δ. Then $MBw=BM'w=Bw$, by the definition of W and the fact that $M'\in\Delta$. Thus W is a non-trivial Γ-invariant subspace of V, so by assumption $W=V$. Then by definition of W, we see that $\Delta=\{I\}$, so Δ is trivial. But Δ was any abelian normal subgroup of the solvable group Γ, so $\Gamma=\{I\}$; then by our assumption on invariant subspaces $n=1$ and we are done.

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