AN INHOMOGENEOUS MINIMUM FOR NON-CONVEX STAR-REGIONS WITH HEXAGONAL SYMMETRY

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1. Introduction. Several authors have proved theorems of the following type:

Let x_0, y_0 be any real numbers. Then for certain functions $f(x, y)$, there exist numbers x, y such that

1.1 $$x \equiv x_0, \quad y \equiv y_0 \pmod 1,$$

and

1.2 $$|f(x, y)| \leqslant \max\,[|f(\tfrac{1}{2}, 0)|, |f(0, \tfrac{1}{2})|, |f(\tfrac{1}{2}, \tfrac{1}{2})|, |f(\tfrac{1}{2}, -\tfrac{1}{2})|].$$

The first result of this type, but with $|f(\frac{1}{2}, \frac{1}{2})|$, $|f(\frac{1}{2}, -\frac{1}{2})|$ replaced by $\min |f(\frac{1}{2}, \pm\frac{1}{2})|$, was given by Barnes **(3)** for the case when the function is an indefinite binary quadratic form. A generalisation of this was proved by elementary geometry by K. Rogers **(6)**. Bambah **(1)** proved the theorem for binary cubic forms with three real linear factors, and Chalk **(4)** proved the same result for binary cubic forms with only one real linear factor. Mordell **(5)** generalised Chalk's result and proved that for functions $f(x, y)$ satisfying certain conditions, including the condition

1.3 $$|f(x, y)| \leqslant k|f(2x, 2y)|,$$

for some k independent of x, y, one can find x, y to satisfy 1.1 and also

1.4 $$|f(x, y)| \leqslant k.\max\,[|f(1, 0)|, |f(0, 1)|, |f(1, 1)|, |f(1, -1)|].$$

Any function satisfying 1.3 and 1.2 also satisfies 1.4, and in fact one can modify Mordell's proof very slightly to get the theorem in the form 1.2 without imposing a condition 1.3. It is only when the function is not homogeneous that the results differ.

Since Mordell's and Rogers' papers were elementary generalisations to certain regions with one and two asymptotes respectively of the results of Chalk and Barnes, it might be interesting to see what properties of a region $f(x, y) \leqslant 1$ with three asymptotes through the origin are necessary in order that the result 1.2 may be proved. In this way, the essential property of the binary cubic required in Bambah's theorem is revealed, namely that the region can be transformed by a linear transformation into one with hexagonal symmetry.

2. Equivalent forms of the theorem, and some lemmas. Let l_1Ol_4, l_2Ol_5, l_3Ol_6 be three lines through the origin O such that Ol_2, Ol_3 make angles of 60°

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and 120° respectively with Ol_1. Bambah **(2)** defines a star-region $\mathfrak{R}$ as having *hexagonal symmetry* if

(i) $\mathfrak{R}$ is symmetric with respect to the lines l_1Ol_4, l_2Ol_5, l_3Ol_6 and their bisectors,

(ii) the boundary $\mathfrak{B}$ of $\mathfrak{R}$ either terminates in the lines $l_1Ol_4, \ldots, l_3Ol_6$ or has them as asymptotes,

(iii) the region external to $\mathfrak{R}$ and lying between Ol_1 and Ol_2 is convex,

(iv) each of the six branches of $\mathfrak{B}$ is a continuous curve.

The arithmetical form of the result to be proved is the following.

THEOREM 1. *Suppose the region $f(x, y) \leqslant 1$ has hexagonal symmetry, and let $\alpha, \beta, \gamma, \delta$ be fixed real numbers with $\alpha\delta - \beta\gamma \neq 0$. Then for any real u_0, v_0 there exist numbers $(u, v) \equiv (u_0, v_0)$ (mod 1), such that*

$$f(\alpha u + \beta v, \gamma u + \delta v) \leqslant \max\left[f\left(\frac{\alpha}{2}, \frac{\gamma}{2}\right), f\left(\frac{\beta}{2}, \frac{\delta}{2}\right), f\left(\frac{\alpha \pm \beta}{2}; \frac{\gamma \pm \delta}{2}\right)\right].$$

It is usually convenient to state the theorem geometrically. Let Λ be the lattice generated by the points $A(\alpha, \gamma)$ and $B(\beta, \delta)$, and let $C = A + B$. Let $\mathfrak{R}$ denote the region given by

$$f(x, y) \leqslant \max\left[f\left(\frac{\alpha}{2}, \frac{\gamma}{2}\right), f\left(\frac{\beta}{2}, \frac{\delta}{2}\right), f\left(\frac{\alpha \pm \beta}{2}, \frac{\gamma \pm \delta}{2}\right)\right].$$

Then $\mathfrak{R}$ is a region with hexagonal symmetry which contains the points $\pm\frac{1}{2}A$, $\pm\frac{1}{2}B$, $\pm\frac{1}{2}(A \pm B)$. We denote by $\mathfrak{R}+P$ the region obtained from $\mathfrak{R}$ by the translation which moves O to P. Then the above theorem is equivalent to the following.

THEOREM 2. *The parellelogram $OACB$, and hence the whole plane, is covered completely by the regions $\mathfrak{R} + P$, $P \in \Lambda$.*

Let Ω be any automorph of $\mathfrak{R}$, and write $\Omega\Lambda = \Lambda'$, $\Omega A = A'$, $\Omega B = B'$. Then $\mathfrak{R}$ contains $\pm\frac{1}{2}A'$, $\pm\frac{1}{2}B'$, $\pm\frac{1}{2}(A' \pm B')$, and the plane is covered by $\mathfrak{R} + \Lambda$ if and only if it is covered by $\mathfrak{R} + \Lambda'$. There is no loss of generality if we choose co-ordinates so that one asymptote is the x-axis. Then, since rotations through 60° or reflections in the axes are automorphisms of $\mathfrak{R}$, we can suppose by a suitable choice of Ω that one of the pairs $\Omega A, \Omega B$; $\Omega B, \Omega A$; $\Omega A, \Omega(-B)$; $\Omega(-B), \Omega A$, which we can still call (A, B), satisfies

(1) $OA \leqslant OB$,

(2) the angle AOB is acute,

(3) the rotation from OA to OB is anti-clockwise,

(4) A lies in the region $0 \leqslant y \leqslant x\sqrt{3}$.

Then we have only to prove the following:

Let Λ be a lattice generated by points A, B satisfying (1), (2), (3), (4). *Let $\mathfrak{R}$ be a region, say $f(x, y) \leqslant k$, with hexagonal symmetry which contains the points $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}(A \pm B)$, and for which the x-axis is an asymptote. Then the regions $\mathfrak{R} + \Lambda$ cover the parallelogram $OACB$ and hence the plane.*

For the proof we need three lemmas:

(a) *Let $PQRS$ be a parallelogram with PQ and PS parallel to Ol_i and Ol_{i+1} respectively ($i = 1, \ldots, 6$; $Ol_7 = Ol_1$). Define $\mathfrak{R}'$ by $f(x, y) \leqslant f(\alpha, \beta)$, where α, β are the co-ordinates of the point $\frac{1}{2}(R - P)$. Then $PQRS$ is covered by $\mathfrak{R}' + P$, $\mathfrak{R}' + R$.*

The region $\mathfrak{R}' + P$ has as part of its boundary an arc which passes through the point $\frac{1}{2}(R - P) + P = \frac{1}{2}(R + P)$ and has PQ, PS as asymptotes. For the region $\mathfrak{R}' + R$, the corresponding point and asymptotes are $-\frac{1}{2}(R - P) + R = \frac{1}{2}(P + R)$ and RS, RQ. The two arcs have a common tac-line at the point $\frac{1}{2}(P + R)$ and so do not cross in the parallelogram $PQRS$. Hence every point of the parallelogram is covered by one or other of the two regions, the small regions near Q, S in fact being covered twice.

(b) *Let PQR be an equilateral triangle with sides parallel to Ol_1, Ol_2, Ol_3 in some order. Let $\mathfrak{R}'$ be the region $f(x, y) \leqslant f(\frac{1}{2}\gamma\sqrt{3}, \frac{1}{2}\gamma)$, where γ is the distance of P from QR. Then PQR is covered by $\mathfrak{R}' + P$.*

Since the point $(\frac{1}{2}\gamma\sqrt{3}, \frac{1}{2}\gamma)$ lies on the bisector of the angle l_1Ol_2, a tac-line there to the boundary of $\mathfrak{R}'$ is parallel to Ol_3. Hence the boundary of $\mathfrak{R}' + P$ has an arc which passes through the foot of the perpendicular from P to QR, touches QR at this point, and has PQ, PR as asymptotes; and so PQR is covered.

(c) *If $0 \leqslant y_1 \leqslant x_1\sqrt{3}$, and $x_1 \leqslant x_2$, then $f(x_1, y_1) \leqslant f(x_2, y_1)$.*

This follows immediately from the convexity and the relationship of the region $\mathfrak{R}$ to the x-axis.

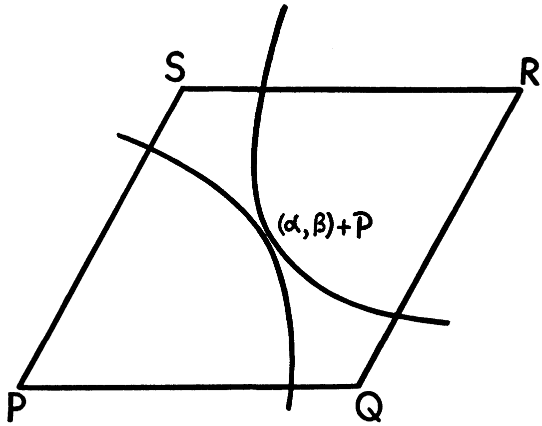

Figure 1

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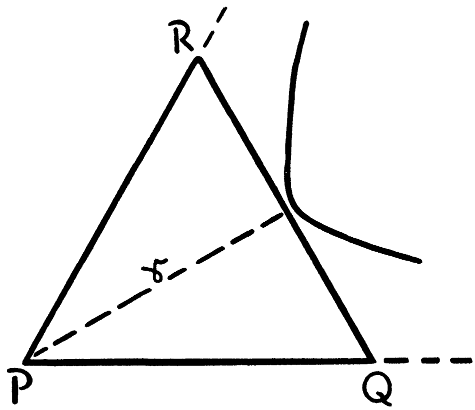

Figure 2

3. Proof of the theorem. The conditions (1) to (4) on A, B imply that A lies between Ol_1 and Ol_2, and B lies between Ol_i and Ol_{i+1}, where $i = 1, 2$ or 3. When B lies between Ol_2 and Ol_3, denote the point of intersection of Ol_2 with the line through B parallel to Ol_6 by F and that of Ol_2 with the line through A parallel to Ol_4 by G. Taking Ol_1 to be the positive x-axis, we have the following cases to consider.

(i) B lies between Ol_1 and Ol_2.

(ii) B lies between Ol_2 and Ol_3, and F lies above G, in the sense that the ordinate of F is not less than the ordinate of G.

(iii) B lies between Ol_2 and Ol_3, and F lies below G.

(iv) B lies between Ol_3 and Ol_4, and B is above A.

(v) B lies between Ol_3 and Ol_4, and B is below A.

Let the co-ordinates of A and B be (p, q) and (r, s) respectively.

(i)

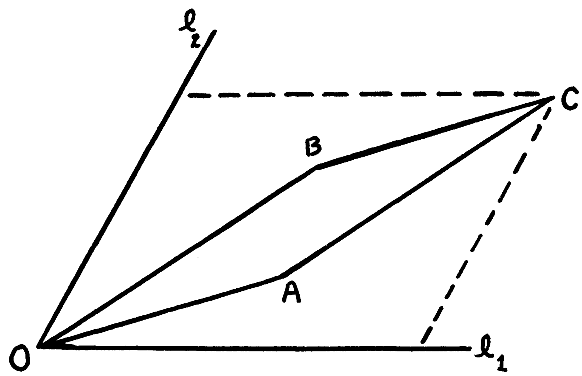

Figure 3

By (a), the parallelogram bounded by the lines through O and C parallel to Ol_1, Ol_2 is covered by $\mathfrak{N}$, $\mathfrak{N} + C$, and hence $OACB$ is covered.

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(ii)

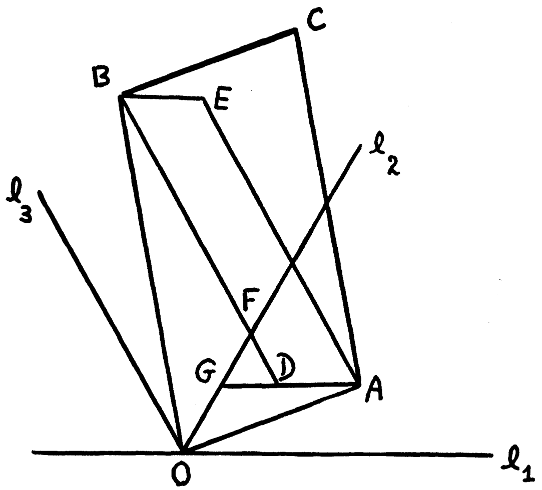

Figure 4

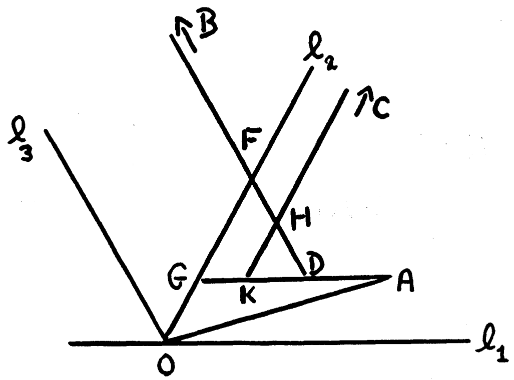

Figure 5

Since $OA \leqslant OB$, and since B lies in the second sector while A lies in the first, we have

$$y_A \leqslant OA \sin \tfrac{1}{3}\pi \leqslant OB \sin \tfrac{1}{3}\pi \leqslant y_B,$$

and so B lies above A. Draw lines ADG, AE parallel to Ol_4 and Ol_3, and lines BFD and BE parallel to Ol_6 and Ol_1 respectively. Since F lies above G, it is clear that D is in the first sector. The treatment differs according as C is in the second or first sector, the corresponding figures being respectively (4) and (5), but we do not separate into cases until it is necessary.

By (a), we see that $ADBE$ is covered by $\mathfrak{R} + A$, $\mathfrak{R} + B$, OAG by $\mathfrak{R}$, $\mathfrak{R} + A$, and OFB by $\mathfrak{R}$, $\mathfrak{R} + B$. Because of the symmetry of $OACB$ about its centre, it will now be enough to prove that the triangle DFG is covered by $\mathfrak{R}$, $\mathfrak{R} + A$, $\mathfrak{R} + B$, $\mathfrak{R} + C$.

Since the distances of DF from O and FG from A are $\frac{1}{2}(s + r\sqrt{3})$ and $\frac{1}{2}(p\sqrt{3} - q)$ respectively, it follows from (b) that DFG is covered by each of the sets

$$f(x, y) \leqslant f\{\tfrac{1}{4}\sqrt{3}(s + r\sqrt{3}), \tfrac{1}{4}(s + r\sqrt{3})\},$$

and

$$[f(x, y) \leqslant f\{\tfrac{1}{4}\sqrt{3}(p\sqrt{3} - q), \tfrac{1}{4}(p\sqrt{3} - q)\}] + A.$$

Now since B lies between Ol_2 and Ol_3, we have $0 \leqslant |r|\sqrt{3} \leqslant s$, and we take in turn the cases when r is negative and when r is positive:

(I) $0 \leqslant -r\sqrt{3} \leqslant s$: by reflection in Oy, then rotation through $\frac{1}{3}\pi$ in the clockwise direction, we deduce in turn that

$$\begin{aligned} f(\tfrac{1}{2}r, \tfrac{1}{2}s) &= f(-\tfrac{1}{2}r, \tfrac{1}{2}s) \\ &= f\left(\frac{-r + s\sqrt{3}}{4}, \frac{s + r\sqrt{3}}{4}\right) \\ &\geqslant f(\tfrac{1}{4}\sqrt{3}(s + r\sqrt{3}), \tfrac{1}{4}(s + r\sqrt{3})), \end{aligned}$$

by (c), since for $r \leqslant 0$ we have $s\sqrt{3} - r \geqslant s\sqrt{3} + 3r$. Hence, in this case DFG is covered by the region $f(x, y) \leqslant f(\frac{1}{2}r, \frac{1}{2}s)$ and so certainly by $\mathfrak{R}$.

(II) $0 \leqslant r\sqrt{3} \leqslant s$: by rotation through $\frac{1}{3}\pi$ in the clockwise direction, we see that

$$\begin{aligned} f(\tfrac{1}{2}r, \tfrac{1}{2}s) &= f\left(\frac{r + s\sqrt{3}}{4}, \frac{s - r\sqrt{3}}{4}\right) \\ &\geqslant f(\tfrac{1}{4}\sqrt{3}(s - r\sqrt{3}), \tfrac{1}{4}(s - r\sqrt{3})), \end{aligned}$$

the inequality following from (c), since for $r \geqslant 0$ we have $r + s\sqrt{3} \geqslant s\sqrt{3} - 3r$. Thus, *if we have* $s - r\sqrt{3} \geqslant p\sqrt{3} - q$, we can conclude that

$$f(\tfrac{1}{2}r, \tfrac{1}{2}s) \geqslant f(\tfrac{1}{4}\sqrt{3}(p\sqrt{3} - q), \tfrac{1}{4}(p\sqrt{3} - q)),$$

and so, by the remarks preceding (I), we see that DFG is covered by $\mathfrak{R} + A$.

Now suppose that $s - r\sqrt{3} < p\sqrt{3} - q$. This means that C is in the first sector, since the distance of A from the line through C parallel to Ol_5 is $\frac{1}{2}(s - r\sqrt{3})$, while the distance of A from Ol_2 is $\frac{1}{2}(p\sqrt{3} - q)$. The figure is as in Figure 5 and there are two possibilities:

Either, the triangle DFG lies completely between Ol_2 and the line through C parallel to Ol_5 and is consequently covered by $\mathfrak{R}$, $\mathfrak{R} + C$;

Or, the line through C parallel to Ol_5 cuts the lines DF, DG at points H, K, thus dividing DFG into the triangle DHK and the trapezium $FGKH$. Of these, $FGKH$ is covered by $\mathfrak{R}$, $\mathfrak{R} + C$, and DKH is covered by

$$[f(x, y) \leqslant f\{\tfrac{1}{4}\sqrt{3}(s - r\sqrt{3}), \tfrac{1}{4}(s - r\sqrt{3})\}] + A,$$

since $\frac{1}{2}(s - r\sqrt{3})$ is the distance of HK from A. Hence, by the inequality above, we see that DKH is covered by $\mathfrak{R} + A$. This completes the investigation of case (ii).

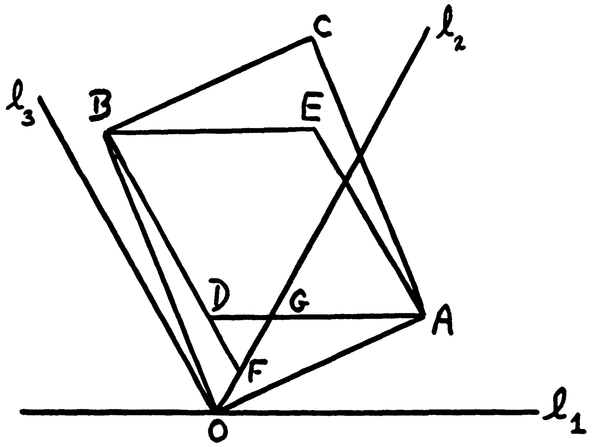

Figure 6

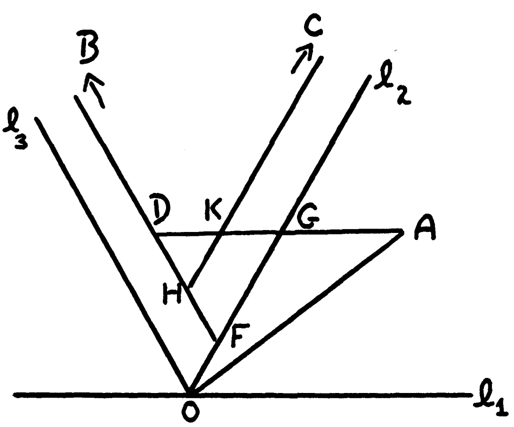

Figure 7

(iii) In this case, as F lies below G, D is in the second sector. Figures 6 and 7 represent the situation when the line through C parallel to Ol_5 does not meet or does meet the triangle DFG. The case when C is in the first sector is not drawn but will be considered. As in (ii), we have only to show that DFG is covered by $\mathfrak{N}$, $\mathfrak{N} + A$, $\mathfrak{N} + B$, $\mathfrak{N} + C$.

Since the perpendicular distances from O to DG and from B to GF are q and $\frac{1}{2}(s - r\sqrt{3})$ respectively, it is clear that DFG is covered by each of the sets

$$f(x, y) \leqslant f(\tfrac{1}{2}\sqrt{3}q, \tfrac{1}{2}q),$$

and

$$[f(x, y) \leqslant f\{\tfrac{1}{4}(s - r\sqrt{3})\sqrt{3}, \tfrac{1}{4}(s - r\sqrt{3})\}] + B$$

Now since A lies in the first sector, we have $O \leqslant q \leqslant p\sqrt{3}$, but in fact we divide into cases according as $q\sqrt{3}$ is not greater than or not less than p.

(I) $q\sqrt{3} \leqslant p$. In this case, (c) implies that

$$f(\tfrac{1}{2}p, \tfrac{1}{2}q) \geqslant f(\tfrac{1}{2}\sqrt{3}q, \tfrac{1}{2}q),$$

and hence DFG is covered by $\mathfrak{N}$.

(II) $q\sqrt{3} \geqslant p$. By reflection in the line $x = y\sqrt{3}$, the bisector of l_2Ol_1, we have

$$f(\tfrac{1}{2}p, \tfrac{1}{2}q) = f(\tfrac{1}{4}(p + q\sqrt{3}), \tfrac{1}{4}(p\sqrt{3} - q))$$
$$\geqslant f\left(\frac{\sqrt{3}(p\sqrt{3} - q)}{4}, \frac{(p\sqrt{3} - q)}{4}\right),$$

using (c) at the second stage, since $q\sqrt{3} \geqslant p$ implies that $p+q\sqrt{3} \geqslant 3p-q\sqrt{3}$. Hence, using the second region above which we showed covered DFG, we see that DFG is covered by $\mathfrak{N} + B$ if we make the further assumption that $p\sqrt{3} - q \geqslant s - r\sqrt{3}$. This leaves the cases when $p\sqrt{3} - q < s - r\sqrt{3}$. In this case B is nearer to the line through C parallel to Ol_5 than to OFG, so that certainly C is in the second sector. Figure 6 indicates the case when DFG lies entirely between Ol_2 and the line through C parallel to Ol_5; in this case, as before, DFG is covered by $\mathfrak{N}$, $\mathfrak{N} + C$. The case when the line through C parallel to Ol_5 meets DFG is shown in Figure 7, where it is seen that the triangle is divided into the trapezium $FGKH$ and the triangle DHK. The trapezium is covered by $\mathfrak{N}$, $\mathfrak{N} + C$, while DHK is covered by

$$\left[f(x, y) \leqslant f\left(\frac{\sqrt{3}(p\sqrt{3} - q)}{4}, \frac{p\sqrt{3} - q}{4}\right)\right] + B,$$

since the distance of B from HK is $\frac{1}{2}(p\sqrt{3} - q)$. By the inequality proved under (II), we infer that DHK is covered by

$$[f(x, y) \leqslant f(\tfrac{1}{2}p, \tfrac{1}{2}q)] + B,$$

and hence, a fortiori, is covered by $\mathfrak{N} + B$.

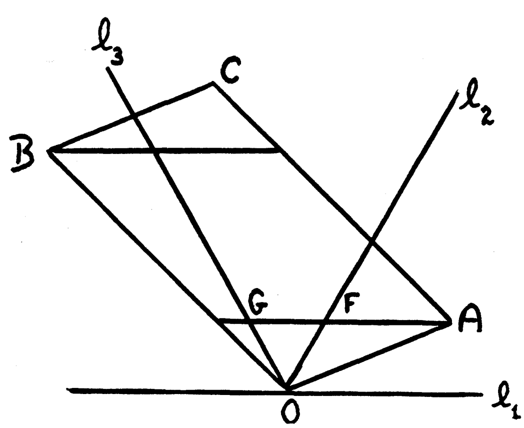

Figure 8

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(iv) By applications of (a), we have only to show that the triangle OFG is covered, as shown in Figure 8. Since the distance from B to OF is $\frac{1}{2}(s - r\sqrt{3})$, OFG is covered by

$$\left[f(x, y) \leqslant f\left(\frac{\sqrt{3}(s - r\sqrt{3})}{4}, \frac{s - r\sqrt{3}}{4}\right)\right] + B.$$

Hence, if $s - r\sqrt{3} \leqslant p\sqrt{3} - q$, then OFG is covered by

$$\left[f(x, y) \leqslant f\left(\frac{\sqrt{3}(p\sqrt{3} - q)}{4}, \frac{p\sqrt{3} - q}{4}\right)\right] + B;$$

and then, since the angle AOB is acute, A lies above the line $y = x/\sqrt{3}$, hence $q\sqrt{3} \geqslant p$, hence, as in (II) of (iii),

$$f\left(\frac{\sqrt{3}(p\sqrt{3} - q)}{4}, \frac{p\sqrt{3} - q}{4}\right) \leqslant f(\tfrac{1}{2}p, \tfrac{1}{2}q),$$

and therefore OFG is covered by $\mathfrak{R} + B$. Finally, suppose that $p\sqrt{3} - q < s - r\sqrt{3}$. Then C is in the second sector and, as before, we can show that OFG is covered by $\mathfrak{R}$, $\mathfrak{R} + C$, or by $\mathfrak{R}$, $\mathfrak{R} + C$, and

$$\left[f(x, y) \leqslant f\left(\frac{\sqrt{3}(p\sqrt{3} - q)}{4}, \frac{p\sqrt{3} - q}{4}\right)\right] + B.$$

Using the same inequality as above, we deduce the result.

Case (v) is similar to (iv). In fact, since the relation $OA \leqslant OB$ does not play any part in (iv), one can use the same proof after reflection in the y-axis.

This completes the proof of Theorem 1. We note that the argument also shows that, if the regions are strictly convex, then strict inequality can be obtained in the statement of the theorem, except possibly when (u_0, v_0) is congruent to one of $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$.

4. Concluding remarks. The example

$$f(x, y) = |xy(x - y)(x + y)|$$

shows that the results for regions with one, two or three asymptotes do not have an analogue for general regions with four or more asymptotes. This is clear from the fact that for the above function the right-hand side of (1.2) is zero.

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