IDEMPOTENTS IN COMPLETELY 0-SIMPLE SEMIGROUPS

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The structure theorem for completely 0-simple semigroups established by Rees [5] in 1940 has proved a very powerful tool in the investigation of such semigroups. In this paper the theorem is applied to an investigation of the subsemigroup of a completely 0-simple semigroup generated by its idempotents. Previous work on this problem has been carried out by Kim [4], but the present note offers a more direct approach.

1. Paths and values. The notations used will be those of [3]. A completely 0-simple semigroup S can, by Rees's Theorem [3, Theorem III.2.5], be identified with a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ in which G is a group, I and Λ are index sets and P is a $\Lambda \times I$ matrix $(p_{\lambda i})$ with entries in G^0 and with no row or column consisting of zeros. The non-zero elements of S are triples (a, i, λ) in $G \times I \times \Lambda$ multiplying according to the rule that

$$(a, i, \lambda)(b, j, \mu) = \begin{cases} (ap_{\lambda j}b, i, \mu) & \text{if } \ p_{\lambda j} \neq 0, \\ 0 \ \text{ if } \ p_{\lambda j} = 0. \end{cases}$$

For the present investigation it is convenient to assume that I and Λ are disjoint. Since they are merely index sets (in one-to-one correspondence respectively with the sets of $\mathcal{R}$-classes and $\mathcal{L}$-classes of S) there is no harm in doing so. With this assumption, consider the relation $\mathbf{K}$ on $I \cup \Lambda$ defined by the rule that $(i, \lambda) \in \mathbf{K}$ if and only if $i \in I$, $\lambda \in \Lambda$ and $p_{\lambda i} \neq 0$, and let $\mathcal{K}$ be the equivalence relation on $I \cup \Lambda$ generated by $\mathbf{K}$. Thus for x, y in $I \cup \Lambda$ we have that $(x, y) \in \mathcal{K}$ if and only if either $x = y$ or (for some $n \geq 2$) there exist $z_1, \ldots, z_n$ in $I \cup \Lambda$ such that

(i) $z_1 = x$ and $z_n = y$,

(ii) $z_r \in I \Rightarrow z_{r+1} \in \Lambda, \qquad z_r \in \Lambda \Rightarrow z_{r+1} \in I$,

(iii) $(z_r, z_{r+1}) \in \mathbf{K} \cup \mathbf{K}^{-1}$.

The sequence $(z_1, \ldots, z_n)$ will be called a *path* from x to y. Among the paths from x to x we shall include the *null path.*

The equivalence relation $\mathcal{K}$ will be called the *connectivity* relation, and we shall call the semigroup S *connected* if $\mathcal{K}$ is the universal relation on $I \cup \Lambda$. Notice that connectedness is a property of the semigroup and not merely of the matrix P. The isomorphism theorem associated with Rees's Theorem (see [3, Theorem III.2.8]) ensures that while the sandwich matrix P is not uniquely determined by S the pattern of non-zero entries in P is invariant. Hence the property of connectedness, which depends solely on this pattern, is either possessed by all representations of S as a Rees matrix semigroup or by none.

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Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let $(x, y) \in \mathcal{H}$ (and from now on we shall for simplicity write this as $x \sim y$), and let $p = (z_1, \ldots, z_n)$, where $z_1 = x$, $z_n = y$, be a path from x to y. The *value* $V(p)$ of the path p is the element of G defined by

$$V(p) = (z_1, z_2)\phi \,.\, (z_2, z_3)\phi \ldots (z_{n-1}, z_n)\phi,$$

where, for i in I and λ in Λ, we define

$$(i, \lambda)\phi = p_{\lambda i}^{-1}, \qquad (\lambda, i)\phi = p_{\lambda i}.$$

The value of the null path from x to x is defined to be e, the identity element of G. Thus, for example, the value of the path $(\lambda, i, \mu, j, \lambda)$ is the element $p_{\lambda i} p_{\mu i}^{-1} p_{\mu j} p_{\lambda j}^{-1}$ of G. Let $P_{x,y}$ be the set of all paths from x to y and let

$$V_{x,y} = \{V(p) : p \in P_{x,y}\},$$

the set of values of paths from x to y. By convention, define $V_{x,y} = \varnothing$ if $x \not\sim y$.

LEMMA 1. *If $x, y, z \in I \cup \Lambda$ and $x \sim y \sim z$ then*

$$\text{(i)}\ \ V_{y,x} = V_{x,y}^{-1}; \qquad \text{(ii)}\ \ V_{x,y} V_{y,z} = V_{x,z}.$$

Proof. Let $a \in V_{y,x}$. Then $a = V(p)$ where $p = (z_1, \ldots, z_n)$ is a path from y to x. Then $(z_n, \ldots, z_1)$ is a path from x to y whose value is a^{-1}. Thus

$$a = (a^{-1})^{-1} \in V_{x,y}^{-1},$$

and so $V_{y,x} \subseteq V_{x,y}^{-1}$. It follows that

$$V_{y,x}^{-1} \subseteq (V_{x,y}^{-1})^{-1} = V_{x,y};$$

hence, relabelling by interchanging x and y, we have $V_{x,y}^{-1} \subseteq V_{y,x}$. This establishes part (i).

Let $p = (x, z_2, \ldots, z_{m-1}, y) \in P_{x,y}$ and $q = (y, t_2, \ldots, t_{n-1}, z) \in P_{y,z}$. Then $(x, z_2, \ldots, z_{m-1}, y, t_2, \ldots, t_{n-1}, z) \in P_{x,z}$. Since the value of this last path is evidently $V(p)V(q)$, it is clear that

$$V_{x,y} V_{y,z} \subseteq V_{x,z}. \tag{1}$$

Conversely, if $a \in V_{x,z}$ then for every b in $V_{x,y}$ we have (using part (i) and formula (1))

$$a = bb^{-1}a \in V_{x,y} V_{y,x} V_{x,z} \subseteq V_{x,y} V_{y,z}.$$

Thus $V_{x,z} \subseteq V_{x,y} V_{y,z}$ as required.

THEOREM 1. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a completely 0-simple semigroup. Let E be the set of idempotents in S and $\langle E \rangle$ the subsemigroup of S generated by the idempotents. Then*

$$\langle E \rangle = \{(a, i, \lambda) \in S : i \sim \lambda \quad \textit{and} \quad a \in V_{i,\lambda}\} \cup \{0\}.$$

Proof. It is well-known, and in any event easy to verify, that the non-zero idempotents of S are the elements $(p_{\lambda i}^{-1}, i, \lambda)$ for which $p_{\lambda i} \neq 0$. Let $(a, i, \lambda) \in \langle E \rangle \backslash \{0\}$. Then there exist $i_2, \ldots, i_n$ in I and $\lambda_1, \ldots, \lambda_{n-1}$ in Λ such that

$$(a, i, \lambda) = (p_{\lambda_1 i}^{-1}, i, \lambda_1)(p_{\lambda_2 i_2}^{-1}, i_2, \lambda_2) \ldots (p_{\lambda i_n}^{-1}, i_n, \lambda) \neq 0.$$

Hence $i \sim \lambda_1 \sim i_2 \sim \lambda_2 \sim \ldots \sim i_n \sim \lambda$ and so $i \sim \lambda$. Also

$$a = p_{\lambda_1 i}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} \ldots p_{\lambda_{n-1} i_n} p_{\lambda i_n}^{-1},$$

the value of the path $(i, \lambda_1, i_2, \lambda_2, \ldots, i_n, \lambda)$ from i to λ, and so $a \in V_{i,\lambda}$.

Conversely, let $i \sim \lambda$ and $a \in V_{i,\lambda}$. Then there exists a path $(i, \lambda_1, i_2, \lambda_2, \ldots, i_n, \lambda)$ whose value

$$p_{\lambda_1 i}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} p_{\lambda_2 i_3} \ldots p_{\lambda_{n-1} i_n} p_{\lambda i_n}^{-1}$$

is equal to a. Hence

$$(a, i, \lambda) = (p_{\lambda_1 i}^{-1}, i, \lambda_1)(p_{\lambda_2 i_2}^{-1}, i_2, \lambda_2) \ldots (p_{\lambda i_n}^{-1}, i_n, \lambda) \in \langle E \rangle.$$

This completes the proof.

We shall say that $S = \mathcal{M}^0[G; I, \Lambda; P]$ is *replete* if it is connected and $V_{x,x} = G$ for some x in $I \cup \Lambda$. In the presence of connectedness this latter condition is in fact equivalent to the apparently stronger condition that $V_{y,z} = G$ for all y, z in $I \cup \Lambda$; if S is replete then $V_{y,x}$ and $V_{x,z}$ are both non-empty by connectedness and so

$$V_{y,z} = V_{y,x} V_{x,x} V_{x,z} = V_{y,x} G V_{x,z} = G.$$

A semigroup S with set of idempotents E is called *idempotent-generated* if $\langle E \rangle = S$. We now have the following obvious corollary to Theorem 1.

COROLLARY. *The completely* 0-*simple semigroup* $\mathcal{M}^0[G; I, \Lambda; P]$ *is idempotent-generated if and only if it is replete.*

2. The completely simple case. The case where S has no zero and is completely simple is easier, since the matrix P has no zero entries and connectedness is automatic. The results corresponding to Theorem 1 and its corollary do not require separate statement. One easy consequence of Theorem 1 is worth recording. A subsemigroup U of a semigroup S is called *unitary* if, for all u in U and all s in S,

$$us \in U \Rightarrow s \in U, \qquad su \in U \Rightarrow s \in U.$$

THEOREM 2. *In a completely simple semigroup S with set E of idempotents, the subsemigroup $\langle E \rangle$ generated by the idempotents is unitary.*

Proof. Let $S = \mathcal{M}[G; I, \Lambda; P]$ and suppose that $u = (a, i, \lambda) \in \langle E \rangle$, $s = (b, j, \mu) \in S$ and $us = (ap_{\lambda j}b, i, \mu) \in \langle E \rangle$. Then $a \in V_{i,\lambda}$ and $ap_{\lambda j}b \in V_{i,\mu}$, from which it follows that

$$b = p_{\lambda j}^{-1} a^{-1} a p_{\lambda j} b \in V_{j,\lambda} V_{\lambda,i} V_{i,\mu} = V_{j,\mu}.$$

Thus $s \in \langle E \rangle$. Similarly $su \in \langle E \rangle \Rightarrow s \in \langle E \rangle$, and so $\langle E \rangle$ is unitary.

We may remark that a closely analogous result exists for the completely 0-simple case. If S is a semigroup with zero element 0 then a subsemigroup U containing 0 is called

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0-*unitary* if, for all u in $U\backslash\{0\}$ and all s in $S\backslash\{0\}$,

$$us \in U\backslash\{0\} \Rightarrow s \in U\backslash\{0\}, \qquad su \in U\backslash\{0\} \Rightarrow s \in U\backslash\{0\}.$$

Then the following theorem can be proved. The details of the proof differ only slightly from those of the last proof and so may safely be omitted.

THEOREM 3. *In a completely* 0-*simple semigroup with set* E *of idempotents, the subsemigroup* $\langle E\rangle$ *generated by the idempotents is* 0-*unitary.*

Returning now to the completely simple case, we consider the simplifications that occur when we assume that the sandwich matrix P is *normal*. As remarked by Clifford [**2**], every completely simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ in which $P=(p_{\lambda i})$ is *normal*, in the sense that there exist k in I and ν in Λ such that $p_{\lambda k}=e$ (the identity element of G) for all λ in Λ and $p_{\nu i}=e$ for all i in I. To put it another way, P is normal if it contains at least one row and at least one column consisting entirely of e's.

Let us now suppose that $S=\mathcal{M}[G; I, \Lambda; P]$ and that P is normal, with $p_{\lambda k}=e$ for all λ and $p_{\nu i}=e$ for all i.

LEMMA 2. *With these assumptions,* $V_{x,y}=V_{z,t}$ *for all* x, y, z, t *in* $I\cup\Lambda$.

Proof. The first step is to show that $e\in V_{x,y}$ for all x, y in $I\cup\Lambda$. This is straightforward if we consider separately the four cases (i) $x, y\in I$, (ii) $x\in I$, $y\in\Lambda$, (iii) $x\in\Lambda$, $y\in I$, (iv) $x, y\in\Lambda$. In case (i) we have a path (x, ν, y) from x to y with value e and so $e\in V_{x,y}$. In case (ii) the path (x, ν, k, y) has value e. Cases (iii) and (iv) are similar.

The desired result now follows easily, since for all x, y, z, t in $I\cup\Lambda$,

$$V_{x,y}=eV_{x,y}e\subseteq V_{z,x}V_{x,y}V_{y,t}=V_{z,t},$$

and, similarly, $V_{z,t}\subseteq V_{x,y}$.

There is thus a fixed subset V of G equal to $V_{x,y}$ for every choice of x, y in $I\cup\Lambda$. An alternative description of V is as follows:

LEMMA 3. $V=\langle\{p_{\lambda i}:\lambda\in\Lambda, i\in I\}\rangle$, *the subgroup of* G *generated by the elements* $p_{\lambda i}$.

Proof. Since $V=V_{x,y}$ for arbitrarily chosen elements x, y in $I\cup\Lambda$, it is immediate that each element of V, being the value of a path from x to y, is a product of the entries of P and their inverses. Conversely, to show that V contains every such product we need only observe (a) that each $p_{\lambda i}\in V_{\lambda,i}=V$, (b) that each $p_{\lambda i}^{-1}\in V_{i,\lambda}=V$, and (c) that if $a\in V=V_{x,y}$ and $b\in V=V_{y,z}$ then $ab\in V_{x,y}V_{y,z}=V_{x,z}=V$.

The final easy consequence of Theorem 1 and Lemma 3 is the following theorem, which can of course be verified more directly. Part of this result is implicit in the proof of Theorem 1 in Benzaken and Mayr [**1**].

THEOREM 4. *Let* $S=\mathcal{M}[G; I, \Lambda; P]$ *be a completely simple semigroup in which* P *is normal. Then* $\langle E\rangle=V\times I\times\Lambda$, *where* V *is the subgroup of* G *generated by the entries of* P. *The semigroup* S *is idempotent-generated if and only if* $V=G$.

That this is untrue without normalisation is evident from the following elementary example. Let $S=\mathcal{M}[G;I,\Lambda;P]$, where $I=\{1,2\}$, $\Lambda=\{3,4\}$, $G=\mathbf{Z}_2=\{e,a\}$, $p_{31}=p_{32}=p_{41}=p_{42}=a$. Then the subgroup generated by the entries of P is G, but

$$\langle E\rangle=E=\{(a,1,3),(a,1,4),(a,2,3),(a,2,4)\}.$$

In fact $V_{1,3}=V_{1,4}=V_{2,3}=V_{2,4}=\{a\}$, in accord with Theorem 1.

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