On the tritangent planes of a quadri-cubic space curve

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1. With any twisted curve of order six is associated a system of planes, usually finite in number, which touch the curve at three distinct points. The curve with its system of tritangent planes possesses properties which recall the properties of a plane quartic curve and its system of bitangent lines; and this is specially true of the sextic which is the intersection of a cubic and a quadric surface. But whereas the properties of the plane curve were discovered by geometrical methods, such methods have only recently been applied with success to the space-curve; the earliest properties were obtained by Clebsch from his Theory of Abelian Functions. In the absence of any one place to which reference can conveniently be made, an account of these properties in their geometrical aspect will be useful. [The curve being of order six and genus four will be referred to as a C_6^4: the general plane quartic is a C_4^3].

2. A plane quartic curve has in general 28 bitangents. An infinite number of conics touch the curve at four separate points, and certain of these conics break up into two bitangent lines: it is scarcely possible to discuss the bitangents apart from the conics. It appears that the conics fall into 63 families, and that 6 members of each family factorise into two lines; also that any two bitangents conjointly form a conic belonging to one of the families.

The twisted sextic C_6^4 has 120 tritangent planes, and an infinity of quadric surfaces, known as *Contact-Quadrics*, touch the curve at six separate points. The Contact-Quadrics fall into 255 families, and 28 members of each family factorise into two planes. Any two tritangent planes conjointly form a contact-quadric belonging to one of the families.

But numbers such as these have little significance unless something is known of the grouping of the lines or planes. For the plane C_4^3 Hesse's notation (fully explained in Salmon's Higher Plane Curves) provides the information. For the C_6^4 a very similar method is available: see the final note of this paper.

3. The combinations of ten things taken two or three or four at a time number 45 or 120 or 210 respectively. Using ten letters $a, b, c, d, e, f, g, h, j, k,$ we may denote each tritangent plane by a set of three letters, and each of the 255 ($= 45 + 210$) families of quadrics by a set of two or four letters. Moreover this can be done in such a way that

(1) the two planes *acd*, *bcd* conjointly form a quadric of the *ab* family;

(2) the two planes *abe*, *cde* .... *abcd* family;

(3) the two planes *abc*, *def* .... *ghjk* family.

The ten letters being interchanged in every possible way, each family of quadrics will be found to include just 28 pairs of tritangent planes.

A further rule regarding possible changes of notation is necessary. Any family of quadrics having been selected, it is permissible to interchange the symbols (*i.e.* the sets of three letters) which denote each two planes forming a quadric of that family, the symbols denoting the other 64 tritangent planes remaining unaltered. This rule carries with it consequential interchanges among the symbols which denote families of contact-quadrics. If the selected family is one denoted by two letters, the rule merely affirms that those two letters may be everywhere interchanged—as already agreed. But if the selected family is one denoted by four letters, new facts are brought to light. With *abcd* as the family selected, pairs of tritangent planes such as those denoted by *abe* and *cde*, or by *efg* and *hjk*—and families of quadrics such as those denoted by *ae* and *bcde*, or by *aefg* and *ahjk*—have their representative symbols interchanged. We thus learn that there is no fundamental difference between a family represented by two letters and one represented by four.

With this notation and this rule the structure of the system of tritangent planes can be understood. The fundamental property of the quadrics which touch the C_6^4 at six points is that for two quadrics of the same family the twelve points of contact lie on another quadric, in addition to the quadric upon which the whole curve lies.

W. P. Milne's Investigations. The Contact-Quadrics of a Family.

4. A geometrical method of treatment that would have delighted Steiner and Hesse or Cayley was applied to the C_6^4 in 1922

by Milne[1]. Certain of his results had however been obtained otherwise by Roth.[2] Milne reasons as follows:

Let us suppose that the curve is the intersection of a quadric surface $\Gamma_2 = 0$ and a cubic surface $S = 0$, and that it is touched at six points by a quadric $Q = 0$. Then the quadrics $Q + k\,\Gamma_2 = 0$ touch the C_6^4 at these points whatever the value of k; and the cubic surfaces $S + P\Gamma_2 = 0$ all pass through the curve whatever the values of the four coefficients in the linear form P. Now a twisted cubic curve passes through the six points of contact, and by assigning proper values to k and P we may replace Q and S by surfaces Q' and S' which pass through the twisted cubic curve.

We now have a quadric surface $\Gamma_2 = 0$ and a cubic surface $S' = 0$ defining the C_6^4 by their intersection, and a quadric surface $Q'=0$ touching the C_6^4 at six points, with the further condition that S' and Q' pass through the C_3^0 defined by the six points. At any one of the six points the tangent lines of both these curves touch both S' and Q'; so that $S' = 0$ a cubic surface and $Q' = 0$ a quadric surface, both containing a twisted cubic curve, have the same tangent plane at six points of the curve. Cubic curves and the surfaces which pass through them are easily discussed: the tangent planes of the most general surfaces of orders two and three passing through the curve coincide at five points and no more. If, as here, they are known to coincide at six points, they must coincide at every point of the curve, and the surfaces must be specialised. In fact it is proved without serious difficulty that $Q' = 0$ must be a quadric cone whose vertex lies on the C_3^0 and that $S' = 0$ is a cubic surface having four double (conical) points which are also situated on the cubic curve.

Given the C_6^4 and one Contact-Quadric, a four-node cubic surface passing through the curve is thus determined. Many properties of such a surface are known and are easily proved. On applying these familiar properties of the surface to the curve cut out from the surface by an arbitrary quadric, we find that we have established the properties of a family of Contact-Quadrics of the sextic curve.

5. With the equation of a four-node cubic surface in its simplest form,

$$1/x_0 + 1/x_1 + 1/x_2 + 1/x_3 = \Sigma\,(1/x_n) = 0,$$

an equation $\Sigma\,(k_n/x_n) = 0$ defines a twisted cubic curve lying on the surface and passing through the nodes, which are the vertices of the tetrahedron of reference. One such curve can be made to pass through any two points of the surface; any two such curves intersect in one further point besides the nodes, and lie on a quadric. In fact the quadric

$$\Sigma (1/x_n) \times \Sigma (k_n k_n'/x_n) = \Sigma (k_n/x_n) \times \Sigma (k_n'/x_n)$$

contains the two curves $\Sigma (k_n/x_n) = 0$ and $\Sigma (k_n'/x_n) = 0$; and the quadric cone

$$\Sigma (1/x_n) \times \Sigma (k_n^2/x_n) = [\Sigma (k_n/x_n)]^2 \tag{5.0}$$

touches the cubic surface at each point of the curve $\Sigma (k_n/x_n) = 0$. A quadric passing through one twisted cubic on the surface cuts out a second such curve: if the quadric contains the cubic through the six points of contact of Q' in §4, it will cut out six further points of the C_6^4 which are the points of contact of another quadric of the family. It must however be borne in mind that, when we are considering only the curve of intersection of the four-node cubic surface with a quadric Γ_2, we may often add a multiple of Γ_2 to the equation of a quadric without affecting its relation to the C_6^4. For instance, the cones in (5.0) form a doubly infinite system; but the derived system of Contact-Quadrics of the C_6^4 is triply infinite.

In an unrestricted coordinate-system the four-node cubic surface is best approached by its tangent planes. These are represented by an equation of the form

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0, \tag{5.1}$$

a, b, c, f, g, h here denoting linear functions of the coordinates x_0, x_1, x_2, x_3. Rearranging the terms we may write this

$$x_0\phi_0 + x_1\phi_1 + x_2\phi_2 + x_3\phi_3 = 0, \tag{5.2}$$

ϕ_0, ϕ_1, ϕ_2, ϕ_3 being homogeneous quadratic functions of α, β, γ. To impose a linear relation upon α, β, γ,

$$L\alpha + M\beta + N\gamma = 0, \tag{5.3}$$

is to select from the tangent planes a set which touch a quadric cone like (5.0); its vertex may be called V.

The quadric Γ_2 which cuts out the C_6^4 from the four-node surface has a tangential equation of order two,

$$F(u_0, u_1, u_2, u_3,) = 0, \tag{5.4}$$

expressing the condition that the plane $\Sigma (u_n x_n) = 0$ should touch Γ_2; from this we derive the condition

$$F(\phi_0, \phi_1, \phi_2, \phi_3) = 0, \tag{5.5}$$

of order four in α, β, γ, that the plane (5.1) or (5.2) should touch Γ_2 as well as the four-node surface; and (5.5) combined with (5.3) defines four planes which touch the quadric cone with vertex V mentioned above and the second quadric cone with the same vertex which envelopes Γ_2. Now when a Contact-Quadric factorises into two tritangent planes the former cone meets Γ_2 in two plane conics. In these circumstances the four planes determined by (5.3) and (5.5) coincide two by two; so that if (α, β, γ) are regarded as coordinates of points in a plane the line (5.3) is a double tangent of the plane quartic curve (5.5). We thus arrive (by reasoning that appears simpler than that given by Milne) at the number of quadrics in the family which break up into two planes, viz. 28.

For a family of Contact-Quadrics as conceived by Clebsch the geometrical properties appear in a far clearer light when they are connected with the four-node cubic surface discovered by Roth and Milne. Each family is allied to one such surface. The notation explained in § 3 tells us a few facts concerning the relations between two (or more) of these surfaces; but it seems that nothing further has been published.

On a set of four tritangent planes and three allied four-node cubic surfaces.

6. When two contact-quadrics of the same family break up into planes, one into the pair x_0, x_1 and the other into the pair x_2, x_3, the twelve points of contact lie on a quadric (as well as on the quadric Γ_2 on which the whole curve lies). The pairs of planes x_0, x_2 and x_3, x_1 form contact-quadrics of a second family, and the pairs x_0, x_3 and x_1, x_2 contact-quadrics of a third family. It will be convenient to take these four tritangent planes for coordinate planes.

In the notation of (5.1), suppose that the cones $bc = f^2$ and $ca = g^2$, which arise when (5.3) takes the simple forms $\alpha = 0$ and $\beta = 0$, meet Γ_2 in the pairs of planes (x_0, x_1) and (x_2, x_3) respectively. Then must

$$bc - f^2 \equiv r\Gamma_2 + \rho x_0 x_1; \qquad ca - g^2 \equiv s\Gamma_2 + \sigma x_2 x_3; \tag{6.1}$$

r, s, ρ, σ being constants. We are at liberty to take for c

$$c \equiv x_0 + x_1 + x_2 + x_3 = \Sigma (x_n),$$

thus completely defining the coordinate system. For a, b, f, g, h we assume

$$a \equiv a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = \Sigma (a_n x_n) \text{ etc., etc.} \tag{6.21}$$

For Γ_2 it is convenient to assume the form

$$\Gamma_2 = \Sigma(x_n) \times \Sigma(k_n x_n) + \Sigma(k_{pq} x_p x_q), \qquad (p, q) = (0, 1, 2, 3). \qquad (6.22)$$

From (6.1), $a_n = sk_n + g_n^2$; $\qquad b_n = rk_n + f_n^2$; $\qquad$ (6.3)

and

$$\left.\begin{array}{l} a_p + a_q - 2g_p g_q = s(k_p + k_q + k_{pq}) \ldots . [+\sigma] \text{ if } (p, q) = (2, 3) \\ b_p + b_q - 2f_p f_q = r(k_p + k_q + k_{pq}) \ldots . [+\rho] \text{ if } (p, q) = (0, 1) \end{array}\right\}.$$

Hence $$\left\{\begin{array}{l} (g_p - g_q)^2 = s \times k_{pq} \ldots . [+\sigma] \text{ if } (p, q) = (2, 3) \\ (f_p - f_q)^2 = r \times k_{pq} \ldots . [+\rho] \text{ if } (p, q) = (0, 1) \end{array}\right\}. \qquad (6.4)$$

Omitting the values (2, 3) and (0, 1) of (p, q), we take square roots of the remaining eight equations; $\sqrt{r}$, $\sqrt{s}$, $\sqrt{k_{pq}}$, are all capable of assuming either of two values and on the first appearance of each we suppose that its value has been chosen to suit the formula. Thus we may assert that

$$g_0 - g_2 = \sqrt{s} \times \sqrt{k_{02}}; \quad g_0 - g_3 = \sqrt{s} \times \sqrt{k_{03}}; \quad g_1 - g_2 = \sqrt{s} \times \sqrt{k_{12}};$$
$$g_1 - g_3 = \sqrt{s} \times \sqrt{k_{13}}; \quad f_0 - f_2 = \sqrt{r} \times \sqrt{k_{02}};$$

but in the remaining formulae a $\pm$ sign should at first be prefixed to each right hand member. The ambiguities can however be settled. If $f_0 - f_3$ had a $+$ sign on the right, it would follow that

$$(g_2 - g_3)^2 : (f_2 - f_3)^2 :: s : r,$$

and this would make $\sigma = 0$, which cannot be. The last three equations must be

$$f_0 - f_3 = -\sqrt{r}\sqrt{k_{03}}; \quad f_1 - f_2 = -\sqrt{r}\sqrt{k_{12}}; \quad f_1 - f_3 = +\sqrt{r}\sqrt{k_{13}}.$$

We are able to simplify these results by modifying the values of α, β, γ. We have made use of the cones $ca = g^2$ and $bc = f^2$ for which α and β vanish, but we are free to replace α and β by any multiples of α and β. We may thus absorb $\sqrt{r}$ and $\sqrt{s}$ into β and α, which amounts to putting $\sqrt{r}$ and $\sqrt{s}$ equal to 1. Hence

$$\left.\begin{array}{l} g_0 - g_2 = +\sqrt{k_{02}}; \; g_0 - g_3 = +\sqrt{k_{03}}; \; g_1 - g_2 = +\sqrt{k_{12}}; \; g_1 - g_3 = +\sqrt{k_{13}}; \\ f_0 - f_2 = +\sqrt{k_{02}}; \; f_0 - f_3 = -\sqrt{k_{03}}; \; f_1 - f_2 = -\sqrt{k_{12}}; \; f_1 - f_3 = +\sqrt{k_{13}}; \\ \text{whence} \qquad \sqrt{k_{02}} + \sqrt{k_{13}} = 0; \quad \sqrt{k_{03}} + \sqrt{k_{12}} = 0. \end{array}\right\} \qquad (6.5)$$

Again, we may modify γ by adding to it any multiples of α and β. The effect of this is to increase f and g by arbitrary multiples of c, and we shall choose those multiples which make

$$f_0 + f_1 + f_2 + f_3 = 0, \qquad g_0 + g_1 + g_2 + g_3 = 0.$$

7. The results may now be summed up in a fairly simple form; to avoid square roots we express them in terms of three constants e_1, e_2, e_3, such that

$$e_1 + e_2 + e_3 = 0; \quad e_2 = \sqrt{k_{02}} = -\sqrt{k_{13}}; \quad e_3 = \sqrt{k_{03}} = -\sqrt{k_{12}}.$$

We find that

$$\left.\begin{array}{c} g_0 = -g_1 = -f_2 = f_3 = \tfrac{1}{2}(e_2 + e_3); \;\; f_0 = -f_1 = -g_2 = g_3 = \tfrac{1}{2}(e_2 - e_3) \\ k_{01} = k_{23} = e_1^2; \;\; k_{02} = k_{13} = e_2^2; \;\; k_{03} = k_{12} = e_3^2. \end{array}\right\} \qquad (7.1)$$

For the quadric $\Gamma_2 = 0$ upon which the C_6^4 lies we have

$$\begin{aligned} \Gamma_2 \equiv (x_0 + x_1 + x_2 + x_3)(k_0 x_0 + k_1 x_1 + k_2 x_2 + k_3 x_3) \\ + e_1^2(x_0 x_1 + x_2 x_3) + e_2^2(x_0 x_2 + x_3 x_1) + e_3^2(x_0 x_3 + x_1 x_2). \end{aligned} \qquad (7.2)$$

Further we find that $\rho = \sigma = -4e_2 e_3$;

$$\left.\begin{array}{l} g = \tfrac{1}{2}e_2(x_0 - x_1 - x_2 + x_3) + \tfrac{1}{2}e_3(x_0 - x_1 + x_2 - x_3); \\ f = \tfrac{1}{2}e_2(x_0 - x_1 - x_2 + x_3) - \tfrac{1}{2}e_3(x_0 - x_1 + x_2 - x_3); \end{array}\right\} \qquad (7.3)$$

$$\left.\begin{array}{l} a = (k_0 x_0 + k_1 x_1 + k_2 x_2 + k_3 x_3) + (g_0^2 x_0 + g_1^2 x_1 + g_2^2 x_2 + g_3^2 x_3); \\ b = (k_0 x_0 + k_1 x_1 + k_2 x_2 + k_3 x_3) + (f_0 x_0 + f_1^2 x_1 + f_2^2 x^2 + f_3^2 x_3). \end{array}\right\} \qquad (7.4)$$

The value of c, $\Sigma(x_n)$, was assumed at the outset, and all that has been discovered is shown above. Nothing can be learnt regarding the four coefficients of h, the last term in (5.1); it has never appeared in our investigations. The coefficients in the linear form k which occurs in Γ_2 and in a and b are also unrestricted; when we set about verifying the formulae upon which our work was based (see (6.1))

$$bc - f^2 = \Gamma_2 - 4e_2 e_3 x_0 x_1; \quad ca - g^2 = \Gamma_2 - 4e_2 e_3 x_2 x_3; \qquad (7.5)$$

they obviously appear on either side and may be left out of account.

The equation of the four-node cubic surface is $\Delta = 0$, where Δ is the three-rowed determinant we associate with (5.1):

$$\Delta = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \qquad (7.6)$$

The cones $bc = f^2$ and $ca = g^2$ touch $\Delta = 0$ at points lying on the twisted cubic curves $b/f = f/c = h/g$ and $a/g = g/c = h/f$ respectively;

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the points of contact of x_0 and x_1 lie upon the former, those of x_2 and x_3 upon the latter. All the twelve points of contact therefore lie upon

$$\Gamma_2 = 0 \quad \text{and} \quad fg - ch = 0, \tag{7.7}$$

and upon the quartic curve C_4^1 in which these two quadrics intersect. The C_4^1 meets the planes (x) in four further points, one in each plane; these may be shown to lie in the plane $c = 0$ and to have coordinates

$$(0,\ e_1,\ e_2,\ e_3),\ (e_1,\ 0,\ e_3,\ e_2),\ (e_2,\ e_3,\ 0,\ e_1),\ (e_3,\ e_2,\ e_1,\ 0).$$

In fact the plane $c = 0$ cuts Γ_2 in a conic which touches the coordinate planes (x) at these points.

By a property of determinants

$$\begin{aligned} c \times \Delta &= (bc - f^2)(ca - g^2) - (fg - ch)^2 \\ &= (\Gamma_2 - 4e_2 e_3 x_0 x_1)(\Gamma_2 - 4e_2 e_3 x_2 x_3) - (fg - ch)^2. \end{aligned} \tag{7.8}$$

Putting $\Gamma_2 = 0$ in this formula we see that the conic in $c = 0$ and the C_6^4 form the complete intersection of Γ_2 with a surface of the type

$$\Sigma^2 - x_0 x_1 x_2 x_3 = 0, \tag{7.9}$$

(Σ being a quadric), a quartic surface having the planes (x) as singular tangent planes. In general the section of this surface by a quadric is a curve of order eight which has the planes (x) as fourfold tangent planes. Exceptionally, as we see here, the section may break up into a conic touching each plane, and a sextic of the type that we are investigating having each as a tritangent plane.

8. We have obtained general equations for a C_6^4 referred to a set of four tritangent planes x_0, x_1, x_2, x_3 under the conditions that two contact-quadrics of one family break up into the pairs (x_0, x_1) and (x_2, x_3). There are at our disposal the values of the constants e, and the coefficients h_n and k_n that appear in the linear functions h and k (see (6.2) and (7.4)). It is easy to derive equations applicable to the problem when the four planes are paired off differently—(x_0, x_2) and (x_3, x_1), or else (x_0, x_3) and (x_1, x_2)—: a cyclical interchange of suffixes (2, 3, 1 or 3, 1, 2 for 1, 2, 3) is all that is required. But, as was pointed out at the beginning of § 6, it must be possible to adapt the new formulae so that they apply to the *same* curve that has been discussed in § 7. The quadric $\Gamma_2 = 0$ on which the curve lies must remain the same; therefore the constants e and k must be unchanged. The C_4^1 of (7.7) must also remain unchanged, so that the new quadrics

analogous to $fg - ch$ must belong to the pencil of quadrics derived from (7.7), a fact that will lead us to the values of h. But first the results that we have obtained will be expressed more simply by means of a new coordinate system:

Let

$$\left.\begin{aligned} x_0 + x_1 + x_2 + x_3 &= X_0;\\ x_0 + x_1 - x_2 - x_3 &= X_1;\\ x_0 - x_1 + x_2 - x_3 &= X_2;\\ x_0 - x_1 - x_2 + x_3 &= X_3;\end{aligned}\right\} \text{ so that } \left.\begin{aligned} &g + f = e_2 X_3; \qquad g - f = e_3 X_2;\\ &a - b = e_2 e_3 X_1; \qquad c = X_0;\\ &a + b = 2K + \tfrac{1}{2}(e_2^2 + e_3^2) X_0.\end{aligned}\right\} \tag{8.1}$$

[K and H are used for k and h when they are expressed in terms of the new coordinates X].

$$4\Gamma_2 \equiv 4KX_0 + e_2 e_3 (X_1^2 - X_0^2) + e_3 e_1 (X_2^2 - X_0^2) + e_1 e_2 (X_3^2 - X_0^2) \tag{8.2}$$

$$\begin{aligned} 4\Delta &\equiv \begin{vmatrix} a+b+2h, & a-b, & g+f \\ a-b, & a+b-2h, & g-f \\ g+f, & g-f, & c \end{vmatrix} = \begin{vmatrix} a+b+2h, & e_2 e_3 X_1, & e_2 X_3 \\ e_2 e_3 X_1, & a+b-2h, & e_3 X_2 \\ e_2 X_3, & e_3 X_2, & X_0 \end{vmatrix} \\ &\equiv (e_2 e_3)^2 X \begin{vmatrix} X_0 + P, & X_1, & X_3 \\ X_1, & X_0 + Q, & X_2 \\ X_3, & X_2, & X_0 \end{vmatrix} = (e_2 e_3)^2 X \begin{vmatrix} X_0, & X_3, & X_2 \\ X_3, & X_0 + P, & X_1 \\ X_2, & X_1, & X_0 + Q \end{vmatrix} \end{aligned} \tag{8.3}$$

where $$\left.\begin{aligned} e_2^2 (X_0 + P) &= a + b + 2h = 2K + 2H + \tfrac{1}{2}(e_2^2 + e_3^2) X_0;\\ e_3^2 (X_0 + Q) &= a + b - 2h = 2K - 2H + \tfrac{1}{2}(e_2^2 + e_3^2) X_0.\end{aligned}\right\} \tag{8.4}$$

We now consider the linear function h of (5.1). In the new coordinates the C_4^1 on which all the points of contact lie is the intersection of

$$\Gamma_2 = 0 \quad \text{and} \quad e_2^2 X_3^2 - e_3^2 X_2^2 - 4X_0 H = 0,$$

H being h expressed in the new coordinates. A cyclical change of suffixes 1, 2, 3 into 2, 3, 1 or 3, 1, 2 must leave this curve unaffected. Suppose that H is changed into H' and H'' by the process; then the two quadrics just given, with the two

$$e_3^2 X_1^2 - e_1^2 X_3^2 - 4X_0 H' = 0, \quad \text{and} \quad e_1^2 X_2^2 - e_2^2 X_1^2 - 4X_0 H'' = 0$$

belong to a pencil of quadrics. From the last three we infer that

$$e_1^2 H + e_2^2 H' + e_3^2 H'' \equiv 0,$$

and therefore we are justified in writing

$$H = e_2 e_3 \phi + e_1 (e_2 - e_3)\psi;\ H' = e_3 e_1 \phi + e_2 (e_3 - e_1)\psi;\ H'' = e_1 e_2 \phi + e_3 (e_1 - e_2)\psi; \tag{8.5}$$

ϕ and ψ being linear functions of the coordinates which are not affected by the cyclical interchange of the suffixes. On substituting in Γ_2 the values of X_2^2 and X_3^2 given by the last two equations, an identical result should be reached. We find that X_1^2 disappears and that we must have

$$4\psi(e_2 e_3 + e_3 e_1 + e_1 e_2) = 4K - (e_2 e_3 + e_3 e_1 + e_1 e_2) X_0. \qquad (8.6)$$

When these values of H and ψ are introduced, P and Q assume unexpectedly simple values, viz.

$$\left.\begin{aligned} e_2 P &= 2K(e_1 - e_2)/(e_2 e_3 + e_3 e_1 + e_1 e_2) + 2e_3\phi; \; [= (\lambda_2 - \lambda_1)] \\ -e_3 Q &= 2K(e_3 - e_1)/(e_2 e_3 + e_3 e_1 + e_1 e_2) + 2e_2\phi. \; [= (\lambda_1 - \lambda_3)] \end{aligned}\right\} \qquad (8.7)$$

Obviously we must take these two expressions on the right as a basis for formulae; K and ϕ are readily expressed in terms of them. But it is even better to express them as differences of three quantities λ_1, λ_2, λ_3 as suggested in the square brackets. We are concerned only with differences of the quantities λ and may as well assume that their sum vanishes, though this is not essential. What is important is the very simple value of K,

$$4K = e_1\lambda_1 + e_2\lambda_2 + e_3\lambda_3, \qquad (8.8)$$

and the almost equally simple form of the determinant Δ, viz.

$$4\,\Delta = (e_2 e_3)^2 \times \begin{vmatrix} X_0, & X_3, & X_2 \\ X_3, & X_0 + (\lambda_2 - \lambda_1)/e_2, & X_1, \\ X_2, & X_1, & X_0 + (\lambda_3 - \lambda_1)/e_3 \end{vmatrix}. \qquad (8.9)$$

A cyclical change of the suffixes 1, 2, 3 gives the other four-node cubic surfaces: but the rows and columns of the determinant may be moved (as was done in (8.3)) so as to bring X_1, X_2, X_3 back to the places they occupy in (8.9). When this has been done the terms of the leading diagonal are

$X_0 + (\lambda_1 - \lambda_2)$, X_0, $X_0 + (\lambda_3 - \lambda_2)$ and $X_0 + (\lambda_1 - \lambda_3)$, $X_0 + (\lambda_2 - \lambda_3)$, X_0.

9. The final form of the equations is so simple and verification of the theorems is so easy that it may be worth while to collect all the formulae, expressing them in a notation that is not hampered by suffixes. A few trivial alterations are made.

FORMULAE.

Notation. Four coordinates:—w, x, y, z.

Three constants:—a, b, c, such that $a + b + c = 0$.

Three linear functions of w, x, y, z:—α, β, γ, such that $\alpha+\beta+\gamma=0$.

Let $D(\theta) = (abc) \times \begin{vmatrix} w + (a-\theta)/a, & z, & y, \\ z, & w + (\beta-\theta)/b, & x, \\ y, & x, & w + (\gamma - \theta)/c \end{vmatrix}$

$$= (a - \theta)(\beta - \theta)(\gamma - \theta) + S\theta + T;$$

then $S = bc(x^2 - w^2) + ca(y^2 - w^2) + ab(z^2 - w^2) + w(aa + b\beta + c\gamma)$.

The C_6^4 is the curve of intersection of the quadric $S = 0$ with the cubic surface $T = 0$. Three four-node cubic surfaces $D(a) = 0$, $D(\beta) = 0$, $D(\gamma) = 0$, clearly pass through the curve, and lead to three families of contact-quadrics. Let us consider one of these surfaces, $D(\gamma) = 0$; for the moment we shall write

$$w + (a - \gamma)/a = u \text{ and } w + (\beta - \gamma)/b = v,$$

so that the elements of the determinant are u, v, w, x, y, z; their minors may as usual be denoted by U, V, W, X, Y, Z. It may be seen that

$$b^2 U + a^2 V + 2abZ = S - ab(w + x + y + z)(w + x - y - z);$$
$$b^2 U + a^2 V - 2abZ = S - ab(w - x + y - z)(w - x - y + z);$$

again

$$(b^2U + a^2V + 2abZ)(b^2U + a^2V - 2abZ) - (b^2U - a^2V)^2 = 4a^2b^2(UV - Z^2) = 4a^2b^2wD(\gamma),$$

a result which proves the four planes

$$w + x + y + z = 0;\ w + x - y - z = 0;\ w - x + y - z = 0;\ w - x - y + z = 0;$$

to be tritangent planes.

Certain properties stated in the course of this paper can be verified, and it is to be hoped that these properties may be extended.

REFERENCES.

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2. P. Roth. *Monatshefte für Math. und Phys.*, **22** (1911), 64-88.

The notation used in § 3 is taken from two papers by Bath and Richmond: *Journal London Math. Soc.*, **7** (1932), 183-192, and *Proc. London Math. Soc.* (2), **38** (1934), 49-71, where references to previous work will be found. See also Bath, *Journal London Math. Soc.*, **3** (1928), 84-86.