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SOME REMARKS ON EISENSTEIN SERIES FOR METAPLECTIC COVERINGS

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Introduction. In recent years the harmonic analysis of n-fold ($n > 2$) metaplectic coverings of GL_2 has played an increasingly important role in certain aspects of algebraic number theory. In large part this has been inspired by the pioneering work of Kubota (see [**3**] for example); as an application one could cite the solution by Heath-Brown and Patterson [**3**] to a question of Kummer's on the distribution of the arguments of cubic Gauss sums. In that paper, Eisenstein series on the 3-fold metaplectic cover of $GL_2(\mathbf{A})$ play a crucial role.

The object of this note is to point out that the theory of Eisenstein series can be made to work for a wide class of finite central coverings. Indeed, once the assumptions are made, the usual theory carries over readily, and one obtains a spectral decomposition of the appropriate L^2-space of functions; this is done in Section 2 of this paper. In Section 1 we define the kind of coverings we are interested in, and draw some elementary conclusions from the definitions. Finally, in Section 3 we discuss the extent to which the assumptions are satisfied.

This note originated sometime ago (late 1977) following conversations with S. J. Patterson; subsequent discussions with P. Deligne were also helpful. The referee has pointed out that a number of other authors have already independently used one or another of the observations in this paper in their work. Examples, in chronological order, include [**5**], [**4**], and [**3**]; see also Séminaire Bourbaki, No. 539 by P. Deligne.

1. Finite central coverings. Let F be a global field, $\mathbf{A}_F$ (or more briefly $\mathbf{A}$) the associated ring of adeles. If F is a function field we shall let q denote the cardinality of its field of constants. In general, notation is adapted from [**8**] and [**10**].

Now suppose $\widetilde{G}$ is a locally compact (Hausdorff) topological group, equipped with a continuous projection $\pi:\widetilde{G} \to G(\mathbf{A})$ where G is a (connected) reductive group defined over F. We suppose π is surjective

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and open, with a kernel μ which is a finite subgroup of the centre $Z_{\widetilde{G}}$ of $\widetilde{G}$; we shall summarize all this by the diagram

$$0 \to \mu \to \widetilde{G} \to G(\mathbf{A}) \to 0$$

and refer to such a situation as a *finite central covering* (or *extension*). Occasionally we shall refer to the analogous situation when G is defined over a local field k, and $G(\mathbf{A})$ is replaced by $G(k)$.

Given H, a subgroup of $G(\mathbf{A})$, we denote its inverse image in $\widetilde{G}$ by $\widetilde{H}$.

We now make some assumptions on the covering

$$0 \to \mu \to \widetilde{G} \to G(\mathbf{A}) \to 0.$$

First, if $P = NM$ is a standard parabolic subgroup (always supposed defined over F), we set $\widetilde{P} = \pi^{-1}(P(\mathbf{A}))$

MI The central covering splits "naturally" over $N(\mathbf{A})$.

MII $\widetilde{P} = \mathrm{Norm}_{\widetilde{G}}(N(\mathbf{A}))$.

MIII There is a positive integer n, such that

$$(Z_M(\mathbf{A})^n)^{\sim} \subseteq Z_{\widetilde{M}}.$$

Finally, we assume

(A) The central extension splits over $G(F)$.

We defer to Section 3 the extent to which these axioms are satisfied, contenting ourselves instead with a couple of remarks of more immediate pertinence.

Firstly the word "naturally" in MI means that if $P \supset Q$, with $N_Q \supset N_P$, then the splittings provided by MI are compatible.

Secondly, as we shall point out in more detail later, the axiom MIII can be weakened.

Choose a (maximal) compact subgroup K of $G(\mathbf{A})$ such that $G(\mathbf{A}) = P(\mathbf{A})K$; such K always exist. We then have immediately

LEMMA. $\widetilde{G} = \widetilde{P}\widetilde{K}$.

From MI and MII follows

LEMMA. $\widetilde{P} = N(\mathbf{A})\widetilde{M}$ (*semidirect product*).

If $\widetilde{H}$ is a subgroup of $\widetilde{G}$, then axiom (A) enables us to speak of $\widetilde{H}(F) = \widetilde{H} \cap G(F)$. Thus for example,

$$\widetilde{P}(F) = \widetilde{P} \cap G(F) = P(F),$$

but $Z_{\widetilde{M}}(F) \neq Z_M(F)$.

With regard to the second example, the following lemma is useful.

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LEMMA. *Given a central covering, S a subset of $\tilde{G}$, $Z_{\tilde{G}}(S)$ its centralizer. Then $\pi(Z_{\tilde{G}}(S))$ is the subgroup of $Z_G(\pi(S))$ given by*

$$\{z \in Z_G(\pi(S)) \mid \beta(z^{-1}, sz)\beta(z, z^{-1})\beta(s, z) = 1, \text{ each } s \in \pi S\}.$$

Here $\beta: G(\mathbf{A}) \times G(\mathbf{A}) \rightarrow \mu$ is the 2-cocycle corresponding to the central covering.

The proof of this is entirely straightforward.

We note that $\pi(Z_{\tilde{G}}(S))$ is a closed subgroup, since the 2-cocycle β is continuous.

Since we may speak of $\tilde{H}(F)$, we can give a definition of quasicharacter suitable for our purposes: a quasicharacter on $\tilde{H}$ is a homomorphism

$$\chi: \tilde{H} \rightarrow \mathbf{C}^*$$

which is trivial on $\tilde{H}(F)$. The set of quasicharacters on H forms a group, containing the subgroup of characters; i.e., those quasicharacters whose range is in the unit circle. If χ is a quasicharacter then Re χ, defined by Re $\chi(h) = |\chi(h)|$ is a real valued quasicharacter.

We specialize this discussion to $\tilde{M}$, and $Z_{\tilde{M}}$. Suppose that ξ is a character of the centre $Z_{\tilde{G}}$. Let $D_{\tilde{M}}(\xi)$ be the set of quasicharacters on $Z_{\tilde{M}}$ which prolong ξ, and following Langlands, write $X_M(\mathbf{R})$ for the set of homomorphisms

$$\chi: Z(\mathbf{A})Z_M(F)\backslash Z_M(\mathbf{A}) \rightarrow \mathbf{R}^*_+.$$

This has a natural structure as finite dimensional real vector space.

There is an obvious injection

$$Z_{\tilde{G}}Z_{\tilde{M}}(F)\backslash Z_{\tilde{M}} \rightarrow Z(\mathbf{A})Z_M(F)\backslash Z_M(\mathbf{A})$$

as well as a homomorphism

$$H_M: Z_M(\mathbf{A}) \rightarrow \text{Mor}\,(X_M(\mathbf{R}), \mathbf{R}).$$

Explicitly, in the number field case, H_M is given by

$$z \rightarrow (\chi \rightarrow \exp(\langle H_M(z), \chi\rangle))$$

and in the function field case by

$$z \rightarrow (\chi \rightarrow \exp(\log q\langle H_M(z), \chi\rangle)).$$

Consequently there is a map

$$H_{\tilde{M}}: Z_{\tilde{M}} \rightarrow \text{Mor}\,(X_M(\mathbf{R}), \mathbf{R})$$

factoring through $Z_{\tilde{G}}Z_{\tilde{M}}(F)\backslash Z_{\tilde{M}}$. The image of $Z_{\tilde{M}}$ is either a vector group (number field case), or a free $\mathbf{Z}$ module of finite type (functional field case). Our axiom MIII ensures that these modules have the same rank as $H_M(Z_M(\mathbf{A}))$; in either case write it as $Z_{\tilde{M},\infty}$. In the number field case $Z_{\tilde{M},\infty}$ splits $Z_{\tilde{M}}$, but in the function field case this is not so, nor are the ranks of the two modules above equal.

Put $X_M(\mathbf{C}) = X_M(\mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C}$. One can make this complex vector space act on the set $D_{\tilde{M}}(\xi)$ via

$$\zeta \to \zeta \cdot q^{\langle H_{\tilde{M}}(\ ),\chi\rangle} = \zeta_\chi$$

where in the number field case $q = e = 1 + 1 + 1/2! + 1/3! + \dots$

The stabilizer of ζ is trivial in the number field case, and is equal to the lattice $\mathrm{i}L^*_{Z_M}$, where $\mathrm{i} = \sqrt{-1}$, and

$$L^*_{Z_{\tilde{M}}} = \{\nu \in X_M(\mathbf{R}) \mid \langle H_{\tilde{M}}(z), \nu\rangle \in 2\pi\mathbf{Z}/\log q, \text{ all } z\}$$

in the function field case (that it is a lattice follows from MIII again).

Consequently one has a structure of complex analytic manifold on $D_{\tilde{M}}(\xi)$, characterized by the fact that each orbit under the above action is an open and closed submanifold. We shall write $D^{\circ}_{\tilde{M}}(\xi)$ for the set of characters; i.e., those ζ for which Re ζ is trivial. In the number field case it is a non compact differentiable manifold; in the function field case it is compact, because of the presence of $\mathrm{i}L^*_{Z_{\tilde{M}}}$.

The kernel of $H_{\tilde{M}}$ is denoted by $Z^{\circ}_{\tilde{M}}$; it contains $Z_{\tilde{G}}Z_{\tilde{M}}(F)$, and modulo this, is a compact subgroup.

Note that any element of $X_M(\mathbf{R})$ prolongs uniquely to a quasicharacter of $M(\mathbf{A})$, thence to one of $\tilde{M}$. Write

$$\tilde{M}^{\circ} = \{m \in \tilde{M} | \chi(m) = 1, \text{ all } \chi \in X_M(\mathbf{R})\}.$$

This group contains any compact subgroup of $\tilde{M}$, and all unipotent radicals which are in $\tilde{M}$.

We conclude with a few observations necessary for Section 2. First of all, if $\zeta \in D_{\tilde{M}}(\xi)$, we have the real quasicharacter $|\zeta|$; I claim $|\zeta| \in X_M(\mathbf{R})$. This follows from the following

LEMMA. *Let $X_{\tilde{M}}(\mathbf{R})$ be defined analogously to $X_M(\mathbf{R})$ so that $X_M(\mathbf{R}) \subset X_{\tilde{M}}(\mathbf{R})$. Then $X_{\tilde{M}}(\mathbf{R}) = X_M(\mathbf{R})$.*

The proof of this follows immediately from MIII.

Thus $X_{\tilde{M}}(\mathbf{R}) = X_M(\mathbf{R})$ and we can speak of the Weyl chamber C_P with respect to P in $X_{\tilde{M}}(\mathbf{R})$. If (in additive notation) $\zeta_1 - \zeta_2 \in C_P$, we say $\zeta_1 > \zeta_2$.

Secondly, we shall say $\widetilde{R}$ and $\widetilde{P}$ are *associate* if their respective Levi components are conjugate by an element of $G(F)$: it is equivalent to say M_R and M_P are conjugate by an element of $G(F)$. We write $\{\widetilde{P}\}$ for an equivalence class of parabolic subgroups.

Finally, we remind the reader that $\widetilde{G}$, as a locally compact group, can be equipped with a Haar measure; moreover, $\widetilde{G}$ is unimodular.

2. Eisenstein series. We shall proceed by imitating the framework of [**8**], appendix II. Suppose given a character ξ of the centre $Z_{\widetilde{G}}$ of $\widetilde{G}$ (to start, it is convenient to assume that ξ is a character, and not merely a quasicharacter, but this is definitely not essential). We define the space $\mathfrak{L}_{\widetilde{G}}(\xi)$ to be the space of measurable functions.

$$\phi : G(F)\backslash\widetilde{G} \rightarrow \mathbf{C}$$

which satisfy

$$\text{(i) } \phi(zg) = \xi(z)\phi(g), \quad z \in Z_{\widetilde{G}},\ g \in \widetilde{G}$$

(i.e., ϕ transforms by ξ)

$$\text{(ii) } \int_{Z_{\widetilde{G}}G(F)\backslash\widetilde{G}} |\phi(g)|^2 dg < \infty.$$

The latter condition puts the structure of a Hilbert space on $\mathfrak{L}_{\widetilde{G}}(\xi)$, on which $\widetilde{G}$ acts by right translations; the problem we set ourselves is to break up this space into simpler $\widetilde{G}$-invariant subspaces.

The axioms MI and MII evidently ensure that one can speak of cuspforms [**8**] on G. In particular, we write $\mathfrak{L}(\{\widetilde{G}\}, \xi)$ for the space of cusp forms of $\mathfrak{L}_{\widetilde{G}}(\xi)$.

The usual arguments show that $\mathfrak{L}(\{\widetilde{G}\}, \xi)$ decomposes discretely c.f. [**8**] 3.2, [**10**] part I, I.5.8.

If $\widetilde{P} = N(\mathbf{A})\widetilde{M}$, one can define $\mathfrak{L}(\{\widetilde{M}\}, \xi)$ analogously, and $\widetilde{M}$ acts on this space by right translations. There is then an induced representation $\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\mathfrak{L}(\{\widetilde{M}\}, \xi)$, which by definition acts on the space of functions

$$\phi : : N(\mathbf{A})\ \widetilde{P}(F)\backslash\widetilde{G} \rightarrow \mathbf{C}$$

such that

$$\text{(i) } m \rightarrow \phi(mg) \in \mathfrak{L}(\{\widetilde{M}\}, \xi) \text{ for each } g \in \widetilde{G}$$

$$\text{(ii) } \int_{Z_{\widetilde{G}}N(A)\widetilde{P}(F)\backslash\widetilde{G}} |\phi(mg)|^2 dg < \infty, \text{ each } m \in \widetilde{M}.$$

Writing $\mathfrak{E}(\widetilde{P}, \xi)$ for this space, we define the subspace $\mathfrak{E}_0(\widetilde{P}, \xi)$ to consist of the continuous functions ϕ such that

(i) ϕ is right $\widetilde{K}$-finite

(ii) the support of $m \to \phi(mg)$ is compact mod $\widetilde{M}^{\circ}$, each $g \in \widetilde{G}$

(iii) there exists an invariant subspace V of cusp forms on $\widetilde{M}^{\circ}$ transforming by ξ which is the sum of a finite number of irreducible subspaces such that for each $g \in \widetilde{G}$, the function $m \to \phi(mg)$ lies in V.

Given an element $\phi \in \mathscr{C}_{\circ}(\widetilde{P}, \xi)$, one constructs the series

$$\phi^{\wedge}(g) = \sum_{P(F)\backslash G(F)} \phi(\gamma g)$$

c.f. [**8**] appendix II, [**10**]. This has the usual good properties. We write $\mathfrak{L}(\widetilde{P}, \xi) \subset \mathfrak{L}_{\widetilde{G}}(\xi)$ for the closure of the space of functions $\phi^{\wedge}$, $\phi \in \mathfrak{C}_{\circ}(\widetilde{P}, \xi)$ and put

$$\mathfrak{L}(\{P\}, \xi) = \bigoplus_{P \in \{P\}} \mathfrak{L}(\widetilde{P}, \xi).$$

One then has the following proposition, established just as in [**8**] 4.6, [**10**].

PROPOSITION. $\mathfrak{L}_{\widetilde{G}}(\xi) = \bigoplus_{\{P\}} \mathfrak{L}(\{P\}, \xi)$.

To go further, one must introduce Eisenstein series, a special case of which already plays a role in the proof of the above proposition. They can be defined exactly as in the usual case.

First, one can introduce the space

$$\mathfrak{L}(\widetilde{G}, \{\widetilde{P}\}, \xi) = \int_{D_{\widetilde{M}}^{\circ}(\xi)}^{\oplus} \mathfrak{L}(\widetilde{G}, \{\widetilde{P}\}, \zeta) d\zeta$$

as in [**8**] appendix II. On replacing $\widetilde{G}$ by $\widetilde{M}$ and modifying the definitions slightly one can then form

$$\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}} (\mathfrak{L}(\widetilde{M}, \{\widetilde{R}(\widetilde{M})\}, \xi))$$

where $\{\widetilde{R}(\widetilde{M})\}$ is a class of associate parabolics in $\widetilde{M}$ of the form $\widetilde{R} \cap \widetilde{M}$. This representation acts on a space of functions $\mathfrak{C}(\widetilde{P}, \{\widetilde{R}(\widetilde{M})\}, \xi)$, and we may consider the subspace $\mathfrak{C}_{\circ}(\widetilde{P}, \{\widetilde{R}(\widetilde{M})\}, \xi)$ in the same way as before.

There are also spaces $\mathfrak{C}_{\circ}(\widetilde{P}, \{\widetilde{R}(\widetilde{M})\}, \zeta)$, $\zeta \in D_{\widetilde{M}}(\xi)$ so that in particular

$$\mathfrak{C}_{\circ}(\widetilde{P}, \{\widetilde{R}(\widetilde{M})\}, \xi) = \int_{Re\zeta = \zeta_{\circ}}^{\oplus} \mathfrak{C}_{\circ}(\widetilde{P}, \{\widetilde{R}(\widetilde{M})\}, \zeta) d\zeta.$$

As ζ ranges over $D_{\tilde{M}}(\xi)$ this collection of spaces forms a complex analytic vector bundle on $D_{\tilde{M}}(\xi)$.

Given $\Phi \in \mathfrak{C}_0(\tilde{P}, \{\tilde{R}(\tilde{M})\}, \zeta\delta_P)$ with $\mathrm{Re}\,\zeta > \delta_P$ one shows that the formal series (δ_P is the modular character of P)

$$E(g, \Phi) = \sum_{P(F)\backslash G(F)} \Phi(\gamma g)$$

converges uniformly on subsets of $\tilde{G}$ compact modulo $Z_{\tilde{G}}G(F)$. It can be analytically continued to a meromorphic function over the entire vector bundle, all of whose singularities lie on hyperplanes, none of which meet $\mathrm{Re}\,\zeta = 0$.

Now let

$$\phi = \int_{D^{\circ}_{\tilde{M}}(\xi)} \Phi(\zeta)d\zeta$$

where

$$\phi \in \mathfrak{C}_0(\tilde{P}, \{\tilde{R}(\tilde{M})\}, \xi),$$

$$\Phi(\zeta) \in \mathfrak{C}_0(\tilde{P}, \{\tilde{R}(\tilde{M})\}, \zeta\delta_P).$$

The integral, like those preceding it, is the Fourier decomposition of the function ϕ viewed on $Z_{\tilde{M}}(F)\backslash Z_{\tilde{M}}$.

Then

$$T\phi(g) = \text{l.i.m.} \int_{D^{\circ}_{\tilde{M}}(\xi)} E(g, \Phi(\zeta))d\zeta$$

exists, where l.i.m. is of course "limit in the mean", and (in the number field case) is taken over an exhaustive family of compact subsets of $D^{\circ}_{\tilde{M}}(\xi)$. The linear map $\phi \to T\phi$ extends to a linear map (continuous in a suitable topology) from $\mathfrak{C}_0(\tilde{P}, \{\tilde{R}(\tilde{M})\}, \xi)$ to $\mathfrak{L}_G(\xi)$ and then in the obvious way to a linear map on

$$\bigoplus_{P\in\{P\}} \bigoplus_{\mathfrak{p}} \mathfrak{C}(\tilde{P}, \{\tilde{R}(\tilde{M})\}, \xi)$$

where $\mathfrak{p}$ runs over all associate classes of the form $\{\tilde{R}(\tilde{M})\}$. The operator T intertwines the action of $\tilde{G}$ and its image is by definition $\mathfrak{L}(\{\tilde{P}\}, \{\tilde{R}\}, \xi)$. One then has

THEOREM.

$$\mathfrak{L}(\{\tilde{R}\}, \xi) = \bigoplus_{\{\tilde{P}\}\geqq\{\tilde{R}\}} \mathfrak{L}(\{\tilde{P}\}, \{\tilde{R}\}, \xi).$$

Here $\{\tilde{P}]\geqq\{\tilde{R}\}$ *if for some* $\tilde{P} \in \{\tilde{P}\}$, $\tilde{R} \in \{\tilde{R}\}$, *one has* $\tilde{P} \geqq \tilde{R}$.

One can also define intertwining operators $M(w, \zeta)$ just as in **[8]**, **[10]**, analytically continue them to sections over all of $D_{\widetilde{M}}(\xi)$, and show that they (and the associated Eisenstein series) satisfy the appropriate functional equations.

In this way it follows that T is the composition of an orthogonal projection, and an isometric injection.

We have not offered any proofs of the above assertions. In fact, once the axioms above are formulated, the above results follow in much the same fashion as the ordinary case (this is one of the virtues of the Selberg-Langlands approach). A few remarks are in order, however.

(i) The notion of a Weyl chamber depends only on $X_M(\mathbf{R})$.

(ii) In the number field case, differential operators play an indispensable role (c.f. the definition of automorphic form in **[2]**); this has been carefully disguised here. In the case at hand, at the infinite places one is dealing with a finite covering of linear Lie groups, and the infinitesimal theory is exactly the same.

(iii) The usual theory of reduction suffices, because the relevant calculations are done modulo the centre $Z_{\widetilde{G}}$, the covering is central, and one has MIII.

Finally, note that one should be able to adapt the work of Arthur on the Selberg trace formula to this framework, although I have not done so. For steps in this direction see [**3**] and [**7**]. To end on a local note the cusp form philosophy is also valid in the p-adic version as well. The true difficulties of this situation lie elsewhere.

3. Complements. The results of this section appear to be part of the folklore, but I first learned of them from Deligne.

Suppose for the moment that k is an arbitrary field, G is defined over k, and that we are given a finite central covering of abstract groups

$$0 \to \mu \to \widetilde{G} \xrightarrow{\pi} G(k) \to 0.$$

Suppose $P = NM$ is a standard parabolic subgroup of G (thus we are presupposing a choice of a fixed minimal parabolic subgroup P_0). Then $M = Z_G(T)$ where T is the maximal k-split torus in $Z(M)$. The group $M(k)$ acts on $N(k)$ by inner automorphisms of course; it also acts on $\widetilde{N}$. Indeed if $m \in M(k)$, lift it to $\widetilde{m} \in \widetilde{M}$, then $\widetilde{m}\,\widetilde{n}\,\widetilde{m}^{-1} \in \widetilde{N}$, and only depends on m, as one sees from the definitions and an easy cocycling argument. It is obvious that π intertwines this action of $M(k)$, and the resulting action of $P(k)$.

LEMMA. *Suppose G is semisimple and simply connected, and that* $|k| \neq 4$, 9. *There is a unique splitting of the sequence over* $N(k)$ *which intertwines the action of* $P(k)$.

Sketch of the proof. We may as well suppose G is almost simple and simply connected. We can also suppose $P = NM$ is a minimal parabolic, so that N, and the split component of $Z(M)$ are as large as possible.

The uniqueness follows from the fact that $N(k)$ is generated by commutators $(t, n) = tnt^{-1}n^{-1}$, $t \in T(k)$, $n \in N(k)$. For any splitting normalized by $P(k)$,

$$(t, n)^{\sim} = ((tnt^{-1})n^{-1})^{\sim} = (t\tilde{n}t^{-1})\tilde{n}^{-1}$$

and exactly as above this depends only on n, t. Thus if there is a splitting with the cited property, it is unique.

Now for existence. There is a sequence of k-algebraic groups

$$N = N_0 > N_1 > N_2 \ldots\ldots > N_r = \{e\}$$

such that each N_1 is characteristic in N, and N_i/N_{i+1} is a vector group. In fact, if $\sum^+$ denotes the set of positive roots for T with respect to a given set of simple roots Δ, let $\sum^+ = \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_r\}$ where α_i are written in increasing order with respect to some total order. Then N_i is the group generated by the N_{α_j} with $r \geqq j \geqq i$. From this one sees that N_i/N_{i+1} breaks into a sum of eigenspaces under T with weights given by certain of the roots. To lift $N(k)$ preserving the action of $P(k)$ amounts to lifting the filtration setwise, preserving appropriate relations. So one is reduced to lifting the N_i/N_{i+1} while preserving the relations between them. But then one is reduced in essence to the arguments of [**11**] p. 123-124, which is where one requires that $|k| \neq 4$, 9.

The lemma implies that, in this case, MI and MII hold. Thus in a certain sense MI and MII are very mild.

We now turn to assumption MIII. Here it was assumed that $(Z_M(\mathbf{A})^n)^{\sim} \subseteq Z_{\tilde{M}}$ for some positive integer n. In fact a careful examination shows that it would be enough to assume the images of $Z_{\tilde{M}}$ and $Z_M(\mathbf{A})$ under $H_{\tilde{M}}$, H_M (respectively) have the same rank. True, we required that $Z_{\tilde{M}}(F)Z_{\tilde{G}}\backslash Z^{\circ}_{\tilde{M}}$ be compact, but this did not depend on MIII. Indeed $Z_{\tilde{M}}(F)Z_{\tilde{G}}\backslash Z^{\circ}_{\tilde{M}}$ embeds into $Z(\mathbf{A})Z_M(F)\backslash Z^{\circ}_M$ as a closed subgroup and that is enough.

We conclude with some remarks concerning the behaviour of finite central coverings when restricted to compact subgroups.

LEMMA. *Let H be a profinite group, with*

$$0 \to \mu \to \widetilde{H} \to H \to 0$$

a finite central covering. Then there is an open normal subgroup $H' < H$ over which the sequence splits.

Proof. This follows straight from the definitions: an extension of the above type corresponds to a 2-cocycle $\beta \in H^2(H, \mu)$ (with H acting on μ trivially). Thus

$$\beta: H \times H \to \mu$$

is a continuous cocycle, and the result follows.

Suppose now that k is a non archimedean local field, G reductive defined over k, and

$$0 \to \mu \to \widetilde{G} \to G(k) \to 0$$

a finite central covering. We write r for the ring of integers of k, κ for the residue field, p for its characteristic.

The above result tells us that if K is any open compact subgroup of $G(k)$, then the extension splits on an open compact subgroup $K' < K$. Indeed K is totally disconnected, and is hence profinite since one can always find a base of neighbourhoods at 1 consisting of normal open compact subgroups (if $K' < K$, just take $\cap_{x \in K} K'^x$: this is of finite index in K', and normal in K).

The same kind of arguments apply if we have

$$0 \to \mu \to \widetilde{G} \to G(\mathbf{A}^f) \to 0$$

where $\mathbf{A}^f$ is the ring of finite adeles associated to a global field F. For example, if $K = \prod_v K_v$ then the sequence will split over a compact open subgroup of K; in fact for almost v, it splits over K_v.

Returning to the local case one can also ask when a finite central covering splits over a given compact subgroup. One approach is to lean on the structure theory of Bruhat-Tits; as a general reference we cite [**12**].

Suppose that K is the group of units of a group scheme $\mathfrak{G}$ defined over r, then there is a short exact sequence of profinite groups obtained by reduction mod p:

$$0 \to N \to K \to \bar{G}(\kappa) \to 0.$$

Here $\bar{G}$ is the algebraic group obtained from $\mathfrak{G}$ by tensoring with κ. There is a spectral sequence of profinite groups

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$$H^p(\bar{G}(\kappa), H^q(N, \mu)) \to H^{p+q}(K, \mu).$$

Suppose that

$$H^q(N, \mu) = 0, q > 0.$$

Then we find that $H^2(\bar{G}(\kappa), \mu) \cong H^2(K, \mu)$, since the spectral sequence will collapse. For example, if $\bar{G}$ is simply connected semisimple we find $H^2(K, \mu) = 0$, by the results of Steinberg [**11**].

Two examples of this come to mind.

(i) K is a hyperspecial ([**12**]) compact subgroup such that $\bar{G}$ is simply connected. Whenever K is hyperspecial, the group N is a pro-p-group ([**11**], 1.10.2, 3.4.2-3); if $(|\mu|, p) = 1$ as is the generic case, then $H^q(N, \mu) = 0$, $q > 0$ and the argument above implies that the sequence splits over K.

(ii) K an Iwahori subgroup. Here K is such that $\mathfrak{G} \otimes \kappa$ can be viewed as a Borel subgroup of a suitable reductive group defined over κ. Thus $\mathfrak{G} \otimes = T \cdot U$, T a torus, U unipotent, and T normalizing U. One finds $K = T(\kappa) \cdot N$ (semidirect product) where N is a pro-p-group. At any rate $H^2(N, \mu) = 0$ if $(|\mu|, p) = 1$ so that the sequence splits over N (which is of finite index in K).

As a special case of all this, suppose G is split and simply connected over F, a global field. Let $K = K_\infty \times K^f$ where K_∞ is a suitable maximal compact at the infinite places, and

$$K^f = \prod_v G(r_v).$$

Then for all v such that $(|\mu|, p) = 1$ (p = residue characteristic of F_v) one finds from the above discussion that the exact sequence splits above $K_v = G(r_v)$. Since μ is finite this leaves only a finite number of places (the troublesome ones); c.f. also [**12**], 3.9.1.

Finally we remark that arguments of a related nature occur in Section 11 of [**9**].

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