# A NOTE ON THE JOINT OPERATOR NORM OF HERMITIAN OPERATORS ON BANACH SPACES

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Let $X$ be a complex Banach space and $H$ be a hermitian operator on $X$. Then in [7] Sinclair proved that $r(H) = \|H\|$, where $r(H)$ and $\|H\|$ are the spectral radius and the operator norm of $H$, respectively.

For a commuting $n$-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $X$, we denote the (Taylor) joint spectrum of $\mathbf{T}$ by $\sigma(\mathbf{T})$ (see [9]) and define the joint operator norm $\|\mathbf{T}\|$ and the joint spectral radius $r(\mathbf{T})$ by

$$\|\mathbf{T}\| = \sup_{\|x\|=1} \left(\sum_{i=1}^{n} \|T_i x\|^2\right)^{1/2}$$

and

$$r(\mathbf{T}) = \sup\{|z| : z \in \sigma(\mathbf{T})\},$$

respectively.

When $X$ is a Hilbert space, it holds that $r(\mathbf{T}) = \|\mathbf{T}\|$ for a doubly commuting $n$-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of hyponormal operators (see [3]). We asked in [2] whether the equality $r(\mathbf{H}) = \|\mathbf{H}\|$ holds for a commuting $n$-tuple $\mathbf{H}$ of hermitian operators on any Banach space. In this paper we will give an answer to this problem.

Example. Let $X = B(\mathbb{C}^3)$ (the set of all 3 by 3 matrices with the operator norm). Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

For $S \in X$ let $D_S$ denote the derivation on $X$ defined by $D_S(T) = ST - TS$. Then $\mathbf{H} = (D_A, D_B)$ is a commuting pair of hermitian operators satisfying $r(\mathbf{H}) < \|\mathbf{H}\|$.

*Proof.* It is easy to see that $\mathbf{H}$ is a commuting pair of hermitian operators (see [6]). Also we have

$$\sigma(\mathbf{H}) = \left\{(0,0), \left(\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right), \left(-\frac{3}{2}, \pm\frac{\sqrt{3}}{2}\right), (0, \pm\sqrt{3})\right\}.$$

Hence we have $r(\mathbf{H}) = \sqrt{3}$.

Next let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Since then $\|T\| = 1$,

$$D_A(T) = \begin{bmatrix} 0 & 0 & \frac{3}{2} \\ -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D_B(T) = \begin{bmatrix} 0 & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \end{bmatrix},$$

we have $\|D_A(T)\|^2 + \|D_B(T)\|^2 = \frac{9}{4} + 3 = \frac{21}{4} > 4$. Hence it holds that

$$r(\mathbf{H}) = \sqrt{3} < 2 < \|\mathbf{H}\|.$$

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