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Weak Sequential Completeness of $\mathcal{K}(X, Y)$

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Abstract. For Banach spaces X and Y, we show that if X^* and Y are weakly sequentially complete and every weakly compact operator from X to Y is compact, then the space of all compact operators from X to Y is weakly sequentially complete. The converse is also true if, in addition, either X^* or Y has the bounded compact approximation property.

1 Introduction

For Banach spaces X and Y, let $\mathcal{L}(X, Y)$, $\mathcal{W}(X, Y)$, and $\mathcal{K}(X, Y)$ denote the spaces of all bounded linear, all weakly compact, and all norm compact operators from X to Y, respectively. When Lewis [10] discussed the weak sequential completeness of the injective tensor product of X and Y, he obtained a result about the weak sequential completeness of $\mathcal{K}(X, Y)$ as follows. Assume that X^* or Y has the metric approximation property. Then $\mathcal{K}(X, Y)$ is weakly sequentially complete if and only if X^* and Y are weakly sequentially complete and every weakly compact operator from X to Y is compact. In this paper, we use a result of Kalton [9] to improve Lewis' result by replacing the assumption of the metric approximation property on X^* or Y by the assumption of the bounded compact approximation property on X^* or Y (Theorem 2.3). Moreover, without the assumption of the metric approximation property on X^* or Y, we show that $\mathcal{K}(X, Y)$ is still weakly sequentially complete if X^* and Y are weakly sequentially complete and every weakly compact operator from X to Y is compact (Theorem 2.2).

The reflexivity of $\mathcal{K}(X, Y)$ was discussed by several authors, including Ruckle [14], Holub [8], Kalton [9], Baker [3], and Godefroy and Saphar [6]. In this paper, we also use a result of Kalton [9] to give a sufficient condition for a subset of $\mathcal{K}(X, Y)$ being conditionally weakly compact (Lemma 2.4), and we show that if both X and Y are reflexive, then $\mathcal{K}(X, Y)$ is reflexive if and only if $\mathcal{K}(X, Y)$ is weakly sequentially complete (Theorem 2.5). We then give several examples of $\mathcal{K}(X, Y)$ with the weak sequential completeness but without reflexivity.

2 Main Results

For a Banach space X, let X^* denote its dual space and B_X denote its closed unit ball. To avoid confusion, we emphasize here that throughout the paper, the weak topology on $\mathcal{K}(X, Y)$ refers to the weak topology of $\mathcal{K}(X, Y)$ as a Banach space rather than the

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weak operator topology of $\mathcal{K}(X, Y)$ as an operator space. For $T \in \mathcal{L}(X, Y)$, let T^* denote its adjoint operator and let $T[X]$ denote the range of T. It is known that if $T \in \mathcal{W}(X, Y)$, then $T^{**}[X^{**}] \subseteq Y$.

Following Kalton's paper [9], we define the w' topology on $\mathcal{L}(X, Y)$ by the family of seminorms $T \mapsto |\langle x^{**}, T^*(y^*)\rangle|$, where $x^{**} \in X^{**}$ and $y^* \in Y^*$. Let $K = (B_{X^{**}}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$. Given any $T \in \mathcal{L}(X, Y)$, by $\chi(T)$, we denote the scalar-valued function on K defined by $\chi(T)(x^{**}, y^*) = \langle x^{**}, T^*(y^*)\rangle$. It is easy to see that, under this map, the w'-topology in $\mathcal{L}(X, Y)$ corresponds to the topology of pointwise convergence on K. Also, it follows from [9, Lemma 1] that the restriction of χ to $\mathcal{K}(X, Y)$ is an isometry onto a subspace of $C(K)$.

Kalton showed in [9, Corollary 3] that if $T_n, T \in \mathcal{K}(X, Y)$ is such that $\lim_n \langle x^{**}, T_n^*(y^*)\rangle = \langle x^{**}, T^*(y^*)\rangle$ for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$, then $\lim_n T_n = T$ weakly in $\mathcal{K}(X, Y)$. We can now use the techniques of Kalton's proof to obtain the following result.

Lemma 2.1 *Let $\{T_n\}_1^\infty$ be a sequence in $\mathcal{K}(X, Y)$ such that $\{\langle x^{**}, T_n^*(y^*)\rangle\}_1^\infty$ is a scalar-valued Cauchy sequence for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$. Then $\{T_n\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{K}(X, Y)$.*

Proof For each $n \in \mathbb{N}$, let $f_n = \chi(T_n)$ in $C(K)$. Since $\{T_n\}_1^\infty$ is w'-Cauchy, the sequence $\{f_n\}_1^\infty$ is pointwise Cauchy in $C(K)$, and hence converges pointwise to some (not necessarily continuous) function f. Also since $\{T_n\}_1^\infty$ is w'-Cauchy, it is w'-bounded. That is, the scalar sequence $\{\langle x^{**}, T_n^*(y^*)\rangle\}_1^\infty$ is bounded for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$. It follows from the Uniform Boundedness Principle that $\{T_n\}_1^\infty$ is norm bounded. Therefore, $\{f_n\}_1^\infty$ is uniformly bounded. For every finite regular Borel measure μ on K, by the Lebesgue Dominated Convergence Theorem, $\lim_n \int_K f_n d\mu = \int_K f d\mu$. It folows that $\{f_n\}_1^\infty$ is weakly Cauchy in $C(K)$ and hence, $\{T_n\}_1^\infty$ is weakly Cauchy in $\mathcal{K}(X, Y)$, since χ is an isometry on $\mathcal{K}(X, Y)$. ■

Lewis [10] showed that under the assumption of the metric approximation property on X^* or Y, $\mathcal{K}(X, Y)$ is weakly sequentially complete if X^* and Y are weakly sequentially complete and $\mathcal{W}(X, Y) = \mathcal{K}(X, Y)$. We will use [9, Corollary 3] to improve Lewis' result by removing the assumption of the metric approximation property from X^* or Y.

Theorem 2.2 *Let X and Y be Banach spaces such that X^* and Y are weakly sequentially complete and every weakly compact operator from X to Y is compact. Then $\mathcal{K}(X, Y)$ is weakly sequentially complete.*

Proof Take a weakly Cauchy sequence $\{T_n\}_1^\infty$ in $\mathcal{K}(X, Y)$. For every $x^{**} \in X^{**}$ and every $y^* \in Y^*$, define a linear functional ϕ on $\mathcal{K}(X, Y)$ by $\phi(T) = \langle x^{**}, T^*(y^*)\rangle$ for every $T \in \mathcal{K}(X, Y)$. Then $\phi \in \mathcal{K}(X, Y)^*$ so that the scalar sequence $\{\phi(T_n)\}_1^\infty = \{\langle x^{**}, T_n^*(y^*)\rangle\}_1^\infty$ is Cauchy. Thus $\{T_n^{**}(x^{**})\}_1^\infty$ and $\{T_n^*(y^*)\}_1^\infty$ are weakly Cauchy sequences in Y and X^* respectively and hence, weakly convergent sequences in Y and X^* respectively. Define $S\colon X^{**} \to Y$ by $S(x^{**}) = \text{weak}\lim_n T_n^{**}(x^{**})$ for every $x^{**} \in X^{**}$ and define $R\colon Y^* \to X^*$ by $R(y^*) = \text{weak}\lim_n T_n^*(y^*)$ for every $y^* \in Y^*$. Since $\{T_n\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{K}(X, Y)$, it is bounded, and hence $\|T_n\| \le c$

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for all $n \in \mathbb{N}$, where c is a positive constant. Thus for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$,

$$|\langle S(x^{**}), y^*\rangle| = \lim_n |\langle T_n^{**}(x^{**}), y^*\rangle| \le \|x^{**}\| \cdot \|y^*\| \cdot \sup_n \|T_n^{**}\| \le c \cdot \|x^{**}\| \cdot \|y^*\|,$$

which implies that $\|S\| \le c$ and hence $S \in \mathcal{L}(X^{**}, Y)$. Similarly, $R \in \mathcal{L}(Y^*, X^*)$.

Now for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$,

$$\langle x^{**}, R(y^*)\rangle = \lim_n \langle x^{**}, T_n^*(y^*)\rangle = \lim_n \langle T_n^{**}(x^{**}), y^*\rangle = \langle S(x^{**}), y^*\rangle,$$

which implies that $R^* = S$ and R is weak* to weak continuous. Thus $R \in \mathcal{W}(Y^*, X^*)$ and there is $T \in \mathcal{L}(X,Y)$ such that $T^* = R$. It follows that $T \in \mathcal{W}(X,Y) = \mathcal{K}(X,Y)$. Moreover, for every $x^{**} \in X^{**}$ and every $y^* \in Y^*$,

$$\lim_n \langle T_n^{**}(x^{**}), y^*\rangle = \langle S(x^{**}), y^*\rangle = \langle T^{**}(x^{**}), y^*\rangle.$$

By [9, Corollary 3], $\lim_n T_n = T$ weakly in $\mathcal{K}(X,Y)$, and hence, $\mathcal{K}(X,Y)$ is weakly sequentially complete. ■

Recall that a Banach space X is said to have the *compact approximation property* (CAP) (see [2]) if for every compact subset K of X and for every $\varepsilon > 0$ there is $T \in \mathcal{K}(X,X)$ such that $\|T(x) - x\| \le \varepsilon$ for all $x \in K$. A Banach space X is said to have the *bounded compact approximation property* (BCAP) (see [2]) if there exists $\lambda \ge 1$ so that for every compact subset K of X and for every $\varepsilon > 0$ there is $T \in \mathcal{K}(X,X)$ such that $\|T(x) - x\| \le \varepsilon$ for all $x \in K$ and $\|T\| \le \lambda$. Every Banach space with the bounded approximation property has the BCAP. But the converse is not true (see [17]).

Lewis [10] characterized the weakly sequential completeness of $\mathcal{K}(X,Y)$ under the assumption of the metric approximation property on X^* or Y. We will use Lemma 2.1 to improve Lewis' result by replacing the assumption of the metric approximation property by the assumption of the bounded compact approximation property.

Theorem 2.3 *Let X and Y be Banach spaces such that either X^* or Y has the BCAP. Then $\mathcal{K}(X,Y)$ is weakly sequentially complete if and only if X^* and Y are weakly sequentially complete and every weakly compact operator from X to Y is compact.*

Proof Since X^* and Y are isometrically isomorphic to subspaces of $\mathcal{K}(X,Y)$ respectively, both X^* and Y are weakly sequentially complete if $\mathcal{K}(X,Y)$ is weakly sequentially complete. By Theorem 2.2, we only need to show that if X^* or Y has the BCAP and if $\mathcal{K}(X,Y)$ is weakly sequentially complete then $\mathcal{W}(X,Y) = \mathcal{K}(X,Y)$.

Case 1: Y has the BCAP. Suppose that there would exist $T \in \mathcal{W}(X,Y)$ but $T \notin \mathcal{K}(X,Y)$. Then there is a sequence $\{x_n\}_1^\infty$ in B_X such that $\{T(x_n)\}_1^\infty$ has no Cauchy subsequence in Y. Let F be the closed subspace of Y generated by the weakly compact subset $\{T(x) : x \in B_X\}$. By [1, p. 43], there exists a norm one projection P of F onto some closed separable subspace Z of F that contains the closed linear span of

$\{T(x_n)\}_1^\infty$. Let $\{z_i\}_1^\infty$ be a dense sequence in Z. Since Y has the BCAP, there exist $\lambda \geq 1$ and a sequence $\{T_n\}_1^\infty$ in $\mathcal{K}(Y,Y)$ with $\|T_n\| \leq \lambda$ such that

$$\|T_n(z_i) - z_i\| < \frac{1}{n}, \quad i = 1, 2, \dots, n, \quad n = 1, 2, \dots .$$

Then $\lim_n T_n(z_i) = z_i$ in Y for each $i \in \mathbb{N}$. For any $z \in Z$ and any $i, n \in \mathbb{N}$ with $i < n$,

$$\begin{aligned}\|T_n(z) - z\| &\leq \|T_n(z) - T_n(z_i)\| + \|T_n(z_i) - z_i\| + \|z_i - z\| \\ &\leq \|T_n(z_i) - z_i\| + (\lambda + 1) \cdot \|z_i - z\|.\end{aligned}$$

This implies that $\lim_n T_n(z) = z$ in Y and hence, $\lim_n T_n(z) = z$ weakly in Y for all $z \in Z$.

Note that $P \circ T \in \mathcal{W}(X,Z)$. Then $(P \circ T)^{**}[X^{**}] \subseteq Z$. For every $x^{**} \in X^{**}$ and every $y^* \in Y^*$, since $(P \circ T)^{**}(x^{**}) \in Z$,

$$\lim_n \langle (T_n \circ P \circ T)^{**}(x^{**}), y^* \rangle = \lim_n \langle T_n\big((P \circ T)^{**}(x^{**})\big), y^* \rangle = \langle (P \circ T)^{**}(x^{**}), y^* \rangle,$$

which implies that the scalar sequence $\{\langle (T_n \circ P \circ T)^{**}(x^{**}), y^* \rangle\}_1^\infty$ is Cauchy. Note that $T_n \circ P \circ T \in \mathcal{K}(X,Y)$. It follows from Lemma 2.1 that $\{T_n \circ P \circ T\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{K}(X,Y)$ and hence, a weakly convergent sequence in $\mathcal{K}(X,Y)$. Therefore $P \circ T = \text{weak}\lim_n T_n \circ P \circ T \in \mathcal{K}(X,Y)$. But the sequence $\{(P \circ T)(x_n)\}_1^\infty = \{T(x_n)\}_1^\infty$ has no Cauchy subsequence in Y. This contradiction shows that $\mathcal{W}(X,Y) = \mathcal{K}(X,Y)$.

Case 2: X^* has the BCAP. Note that every compact operator from Y^* to X^* is the adjoint operator of a compact operator from X to Y. So $\mathcal{K}(Y^*, X^*)$ is isometrically isomorphic to $\mathcal{K}(X,Y)$ and hence, is also weakly sequentially complete. It follows from Case 1 that $\mathcal{W}(Y^*, X^*) = \mathcal{K}(Y^*, X^*)$. Note that a bounded linear operator is (weakly) compact if and only if its adjoint operator is (weakly) compact. Thus $\mathcal{W}(X,Y) = \mathcal{K}(X,Y)$. ■

Recall that a subset of a Banach space is *relatively weakly compact* if and only if it is *relatively weakly sequentially compact*, that is, every sequence in it has a weakly convergent subsequence. Recall that a subset of a Banach space is called *conditionally weakly compact* if every sequence in it has a weakly Cauchy subsequence.

Lemma 2.4 *Let X and Y be Banach spaces. A subset M of $\mathcal{K}(X,Y)$ is conditionally weakly compact if*

(i) *$\{T^{**}(x^{**}) : T \in M\}$ is a relatively weakly compact set in Y for every $x^{**} \in X^{**}$, and*

(ii) *$\{T^*(y^*) : T \in M\}$ is a conditionally weakly compact set in X^* for every $y^* \in Y^*$.*

Proof Take any sequence $\{T_n\}_1^\infty$ in M. Since each T_n is compact, the range $T_n^{**}[X^{**}]$ is a separable subspace of Y. Without loss of generality, we may assume that Y is separable. Thus there is a countable subset D of Y^* such that the linear span of D is dense in Y^* with respect to the weak* topology. By (ii) and by using a diagonal

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method, we may assume that $\{T_n^*(d^*)\}_1^\infty$ is a weakly Cauchy sequence in X^* for each $d^* \in D$. It follows that

$$\lim_n \langle x^{**}, T_n^*(d^*)\rangle \text{ exists}, \qquad \forall x^{**} \in X^{**},\ \forall d^* \in D. \tag{2.1}$$

Take any $x^{**} \in X^{**}$, any $y^* \in Y^*$, and any $\varepsilon > 0$. Since the linear span of D is dense in Y^* with respect to the weak* topology, it is dense in Y^* with respect to the Mackey topology. (In fact, the *weak** topology and the Mackey topology on Y^* have the same closed subspaces, see [16, p. 111, Corollary 6].) Thus by (i) there is z^* in the linear span of D such that

$$|\langle T_n^{**}(x^{**}), y^* - z^*\rangle| < \varepsilon/3, \quad n = 1, 2, \dots. \tag{2.2}$$

Note that z^* is a linear combination of elements of D. By (2.1) there exists $N \in \mathbb{N}$ such that for all $m,\ n \geq N$,

$$\langle x^{**}, T_m^*(z^*) - T_n^*(z^*)\rangle| < \varepsilon/3. \tag{2.3}$$

By (2.2) and (2.3) for all $m,\ n \geq N$,

$$|\langle T_m^{**}(x^{**}) - T_n^{**}(x^{**}), y^*\rangle| < \varepsilon,$$

which implies that the scalar sequence $\{\langle T_n^{**}(x^{**}), y^*\rangle\}_1^\infty$ is Cauchy. It follows from Lemma 2.1 that $\{T_n\}_1^\infty$ is a weakly Cauchy sequence in $\mathcal{K}(X,Y)$. ■

If both X and Y are reflexive Banach spaces, then Lemma 2.4 shows that every bounded subset of $\mathcal{K}(X,Y)$ is conditionally weakly compact and hence, relatively weakly compact if $\mathcal{K}(X,Y)$ is weakly sequentially complete. This fact yields the following result.

Theorem 2.5 *If both X and Y are reflexive Banach spaces then* $\mathcal{K}(X,Y)$ *is reflexive if and only if it is weakly sequentially complete.*

There are several examples of $\mathcal{K}(X,Y)$ with weak sequential completeness but without reflexivity. We list them as follows.

(a) It is known that weakly compact subsets in ℓ_1 are norm compact. Thus by Theorem 2.2, if X^* is weakly sequentially complete, then so is $\mathcal{K}(X,\ell_1)$ (see also [10, Corollary 2.2]).

(b) By [13, p. 206, Theorem A.2], every continuous linear operator from ℓ_p $(2 < p < \infty)$ to $L^1(\mu)$ is compact. Thus by Theorem 2.2, $\mathcal{K}(\ell_p, L^1(\mu))$ $(2 < p < \infty)$ is weakly sequentially complete (also see [10, Corollary 2.3]).

(c) It is known that if a Banach space Y is weakly sequentially complete, then it contains no copy of c_0, and that a Banach space Y contains no copy of c_0 if and only if every continuous linear operator from c_0 to Y is compact. Thus by Theorem 2.2, if Y is weakly sequentially complete, then so is $\mathcal{K}(c_0, Y)$.

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(d) It is known that if a Banach space X is a Grothendieck space, then its dual X^* is weakly sequentially complete. (In fact, every dual Banach space is weak* sequentially complete, see [11, p. 230, Corollary 2.6.21].) By [12, Corollary 4.4], every bounded linear operator from ℓ_∞ to ℓ_p factors through ℓ_2 and hence, is compact if $1 \leq p < 2$. Thus by Theorem 2.2, $\mathcal{K}(\ell_\infty, \ell_p)$ $(1 \leq p < 2)$ is weakly sequentially complete.

(e) Let T^* be the original Tsirelson space and let T be the dual of T^* (see [4]). By [7, Lemma 13], $\mathcal{L}(\ell_\infty, T) = \mathcal{K}(\ell_\infty, T)$. Thus by Theorem 2.2, $\mathcal{K}(\ell_\infty, T)$ is weakly sequentially complete.

From Theorems 2.2 and 2.5 we obtain a result of Kalton in [9, Corollary 2]; that is, if both X and Y are reflexive and $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$, then $\mathcal{L}(X, Y)$ is reflexive. It follows from [6, Corollary 1.6] that if a reflexive Banach space X has the CAP, then both X and X^* have the BCAP. Thus by Theorem 2.3 we obtain a result of Godefroy and Saphar in [6, Corollary 1.3], that is, if X and Y are reflexive Banach spaces such that either X or Y has CAP then $\mathcal{L}(X, Y)$ is reflexive if and only if $\mathcal{L}(X, Y) = \mathcal{K}(X, Y)$. Moreover, by [5, Theorem 2] we obtain the following corollary, which is a generalization of a result of Feder and Saphar in [5, Corollaries 2.1 and 2.2] and a result of Schatten in [15].

Corollary 2.6 *Let X and Y be reflexive Banach spaces such that either X or Y has CAP. If $\mathcal{L}(X, Y) \neq \mathcal{K}(X, Y)$ then $\mathcal{K}(X, Y)$ is non-conjugate. In particular, if X is a infinitely dimensional reflexive Banach space with CAP, then $\mathcal{K}(X, X)$ is non-conjugate.*

As a consequence of Lemma 2.4 it is interesting to mention the following results about the embedding of ℓ_1 to $\mathcal{K}(X, Y)$. Recall that the classical Rosenthal's ℓ_1-Theorem states that a Banach space X contains no copy of ℓ_1 if and only if every bounded sequence in X has a weakly Cauchy subsequence, or every bounded subset of X is conditionally weakly compact. Thus by Lemma 2.4 we obtain the following corollary.

Corollary 2.7 *If X^* contains no copy of ℓ_1 and if Y is reflexive, then $\mathcal{K}(X, Y)$ contains no copy of ℓ_1.*

If Y^* is separable, then in the proof of Lemma 2.4, D is dense in Y^* with respect to the norm topology. Thus Lemma 2.4(i) can be replaced by the condition that $\{T^{**}(x^{**}) : T \in M\}$ is a bounded subset in Y for every $x^{**} \in X^{**}$. We obtain another corollary about embedding of ℓ_1 to $\mathcal{K}(X, Y)$ as follows.

Corollary 2.8 *If X^* contains no copy of ℓ_1 and if Y^* is separable, then $\mathcal{K}(X, Y)$ contains no copy of ℓ_1.*

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