Canad. Math. Bull. Vol. 32 (3), 1989

BIJECTIVE PROOFS OF SOME n-COLOR PARTITION IDENTITIES

BY
A. K. AGARWAL

ABSTRACT. Using a technique of Agarwal and Andrews (1987), bijective proofs of some n-color partition identities discovered recently by the author, are given.

1. **Introduction, Definitions and the Main Result.**

Recently in [3] the following q-identitites of Rogers [5]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2n+1})(1-q^{20n+4})(1-q^{20n+16})} \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} = \prod_{\substack{n=1 \\ n\not\equiv \pm 3, \pm 4, \pm 7, 10 \,(\mathrm{mod}\, 20)}}^{\infty} (1-q^n)^{-1} \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \prod_{\substack{n=1 \\ n\not\equiv \pm 1, \pm 8, \pm 9, 10 \,(\mathrm{mod}\, 20)}}^{\infty} (1-q^n)^{-1} \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q)_{2n+1}} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2n+1})(1-q^{20n+8})(1-q^{20n+12})}, \tag{1.4}$$

where $(a;q)_n$ is a rising q-factorial which in general is defined by

$$(a;q)_n = \sum_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})},$$

if n is a positive integer, then obviously

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}),$$

Received by the editors November 2, 1987.
1985 Mathematics Subject Classification: 05A17, 11P80.

https://doi.org/10.4153/CMB-1989-047-1 Published online by Cambridge University Press

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ldots,$$

were interpreted combinatorially as follows:

THEOREM 1.1. *Let $A_1(\nu)$ denote the number of partitions of ν such that the anti-hook differences on diagonal* 0 *are* 0 *or* 1. *Let $B_1(\nu)$ denote the number of partitions of ν in which each part is either odd or congruent to* $\pm 4 \pmod{20}$. *Then $A_1(\nu) = B_1(\nu)$ for all ν.*

THEOREM 1.2. *Let $A_2(\nu)$ denote the number of partitions of ν such that the anti-hook differences on diagonal* -1 *are* 0 *or* 1. *Let $B_2(\nu)$ denote the number of partitions of ν into parts* $\not\equiv \pm 3, \pm 4, \pm 7, 10 \pmod{20}$. *Then $A_2(\nu) = B_2(\nu)$ for all ν.*

THEOREM 1.3. *Let $A_3(\nu)$ denote the number of partitions of ν such that the anti-hook differences on diagonal* -1 *are* 1 *or* 2. *Let $B_3(\nu)$ denote the number of partitions of ν into parts* $\not\equiv \pm 1, \pm 8, \pm 9, 10 \pmod{20}$. *Then $A_3(\nu) = B_3(\nu)$ for all ν.*

THEOREM 1.4. *Let $A_4(\nu)$ denote the number of partitions of ν such that the anti-hook differences on diagonal* -2 *are* 1 *or* 2. *Let $B_4(\nu)$ denote the number of partitions of ν in which each part is either odd or congruent to* $\pm 8 \pmod{20}$. *Then $A_4(\nu) = B_4(\nu)$ for all ν.*

NOTE. Theorems 1.3 and 1.4 were incorrectly stated in [3].

Later in [2] following the method of [1], n-color partition theoretic interpretations for the same q-identities (1.1)–(1.4) were given in the following form:

THEOREM 1.5. *Let $C_1(\nu)$ denote the number of partitions of ν with "n copies of n" such that*

(1.5.a) *even parts appear with even subscripts and odd with odd, and*

(1.5.b) *each pair of parts has nonnegative weighted difference.*

Then $C_1(\nu) = B_1(\nu)$ for all ν.

THEOREM 1.6. *Let $C_2(\nu)$ denote the number of partitions of ν with "$n+1$ copies of n" such that*

(1.6.a) *even parts appear with odd subscripts and odd with even,*

(1.6.b) *each pair of parts has nonnegative weighted difference,*

(1.6.c) *for some i, i_{i+1} is a part, and*

(1.6.d) *the parts are nonnegative.*

Then $C_2(\nu) = B_2(\nu)$ for all ν.

THEOREM 1.7. *Let $C_3(\nu)$ denote the number of partitions of ν with "n copies of n" such that*

(1.7.a) *even parts appear with even subscripts and odd with odd subscripts greater than 1, and*

(1.7.b) *the weighted difference of each pair of parts m_i, m_j is either nonnegative or* -2.

https://doi.org/10.4153/CMB-1989-047-1 Published online by Cambridge University Press

Then $C_3(\nu) = B_3(\nu)$ *for all* ν.

THEOREM 1.8. *Let* $C_4(\nu)$ *denote the number of partitions of* ν *with "*$n+2$ *copies of* n*" such that*

(1.8.a) *even parts appear with even subscripts and odd with odd,*

(1.8.b) *each pair of parts has a nonnegative weighted difference,*

(1.8.c) *for some* i, i_{i+2} *is a part,*

and

(1.8.d) *the parts are nonnegative.*

Then $C_4(\nu) = B_4(\nu)$ *for all* ν.

Using a technique of [4] we give here a bijective proof of the following:

THEOREM 1.9. *For* $1 \leqq \kappa \leqq 4$,

$$A_\kappa(\nu) = C_\kappa(\nu).$$

Before we give the proof of this theorem we recall the definitions of anti-hook differences from [3] and those of partitions with "$n+l$ copies of n" and the weighted difference from [4].

DEFINITION 1. *Let* Π *be a partition whose Ferrers graph is embedded in the fourth quadrant. Each node* (i, j) *of the fourth quadrant which is not in the Ferrers graph of* Π *is said to possess an anti-hook difference* $\rho_I - \kappa_j$ *relative to* Π, *where* ρ_i *is the number of nodes on the* i*-th row of the fourth quadrant to the left of node* (i, j) *that are not in the Ferrers graph of* Π *and* κ_j *is the number of nodes in the* j*-th column of the fourth quadrant that lie above node* (i, j) *and are not in the Ferrers graph of* Π.

DEFINITION 2. *A partition with "*$n + l$ *copies of* n*"* $l \geqq 0$ *is a partition in which a part of size* n, $n \geqq 0$, *can come in* $n + l$ *different colors denoted by subscripts:* $n_1, n_2, \ldots, n_{n+l}$. *Thus for example, the partitions of* 2 *with "*$n+1$ *copies of* n*" are:*

$$2_1, 2_1 + 0_1, 1_1 + 1_1, 1_1 + 1_1 + 0_1$$
$$2_2, 2_2 + 0_1, 1_2 + 1_1, 1_2 + 1_1 + 0_1$$
$$2_3, 2_3 + 0_1, 1_2 + 1_2, 1_2 + 1_2 + 0_1.$$

Note that zeros are permitted if and only if l *is greater than or equal to* 1.

DEFINITION 3. *The weighted difference of two parts* m_i *and* n_j, $m \geqq n$ *is defined by* $m - n - i - j$ *and is denoted by* $((m_i - n_j))$.

2. PROOF OF THE THEOREM 1.9.

Each of these four cases is proved in a similar way. We provide the details for $\kappa = 1$ and sketch the main steps to treat the remainder.

Let Π be a partition enumerated by $A_1(\nu)$. Let

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix},$$

where $a_1 > a_2 > \ldots > a_r \geqq 0$, $b_1 > b_2 > \ldots > b_r \geqq 0$, and $a_1 + a_2 + \cdots + a_r + b_1 + b_2 + \cdots + r = \nu$, be the corresponding Frobenius' notation. Then the anti-hook difference conditions of Theorem 1.1 are equivalent to

$$(2.1.a) \qquad a_i \geqq b_i, \text{ and}$$

$$(2.1b) \qquad b_i \geqq a_{i+1} + 1.$$

We now establish a 1-1 correspondence between the ordinary partitions enumerated by $A_1(\nu)$ and the partitions with n copies of n enumerated by $C_1(\nu)$. We do this by mapping each column $\genfrac{}{}{0pt}{}{a}{b}$ of the Frobenius Symbol to a single part m_i of a partition with n copies of n. The mapping ϕ is

$$(2.2) \qquad \phi : \binom{a}{b} \longrightarrow \begin{cases} (a+b+1)_{b-a} & \text{if } a < b \\ (a+b+1)_{a-b+1} & \text{if } a \geqq b, \end{cases}$$

the inverse mapping ϕ^{-1} is then given by

$$(2.3) \qquad \phi^{-1} : m_1 \longrightarrow \begin{cases} \binom{(m-i-1)/2}{(m+i-1)/2} & \text{if } m \not\equiv i \,(\text{mod}\, 2) \\ \binom{(m+i-2)/2}{(m-i)/2} & \text{if } m \equiv i \,(\text{mod}\, 2). \end{cases}$$

Now for any two adjacent columns $\genfrac{}{}{0pt}{}{a}{b}\ \genfrac{}{}{0pt}{}{c}{d}$ in the Frobenius symbol with $\phi\binom{a}{b} = m_i$ and $\phi\binom{c}{d} = n_j$ (defined by (2.2)) we have

$$(2.4) \qquad ((m_i - n_j)) = \begin{cases} 2b - 2c - 2 & \text{if } a \geqq b, c \geqq d \\ 2a - 2c - 1 & \text{if } a < b, c \geqq d \\ 2b - 2d - 1 & \text{if } a \geqq b, c < d \\ 2a - 2d & \text{if } a < b, c > d \end{cases}$$

Clearly (2.1.a) and (2.2) imply (1.5.a) and then (2.1.b) and only the first line of (2.4) will imply (1.5.b).

To see the reverse implication we note that by (1.5.a) $m = i, n \equiv j$ (mod 2) and so under ϕ^{-1}

$$(2.5) \qquad a - c = \frac{1}{2}((m_i - n_j)) + i$$

$$(2.6) \qquad b - d = \frac{1}{2}((m_i - n_j)) + j$$

$$(2.7) \qquad a - b = i - 1$$

$$(2.8) \qquad b - c - 1 = \frac{1}{2}((m_i - n_j))$$

Now (2.5) and (2.6) by (1.5.b) guarantee that $a_i > a_{i+1}$ and $b_i > b_{i+1}$. (2.7) implies (2.1.a) and (2.8) by (1.5.b) implies (2.1.b). This completes the proof of $A_1(\nu) = C_1(\nu)$.

For $\kappa = 2$, the anti-hook difference conditions are equivalent to

$$(2.9) \qquad b_{i+1} + 2 \leqq a_i \leqq b_i + 1.$$

The map ϕ is

$$\phi\binom{a}{b} \longrightarrow \begin{cases} (a+b+1)_{b-a+2} & \text{if } a \leqq b+1 \\ (a+b+1)_{a-b-1} & \text{if } a > b+1. \end{cases} \tag{2.10}$$

and ϕ^{-1} is given by

$$\phi^{-1} : m_i \longrightarrow \begin{cases} \binom{(m+i)/2}{(m-i-2)/2} & \text{if } m \not\equiv i+1 \,(\text{mod}\, 2) \\ \binom{(m-i+1)/2}{(m+i-3)/2} & m \equiv i+1 \,(\text{mod}\, 2) \end{cases} \tag{2.11}$$

For $\kappa = 3$, the anti-hook difference conditions are equivalent to

$$a_{i+1} \leqq b_i < a_i. \tag{2.12}$$

The map ϕ is

$$\phi\binom{a}{b} \longrightarrow \begin{cases} (a+b+1)_{b-a} & \text{if } a < b \\ (a+b+1)_{a-b+1} & \text{if } a > b \end{cases} \tag{2.13}$$

and the inverse map ϕ^{1} is given by

$$\phi^{-1} : m_i \longrightarrow \begin{cases} \binom{(m-i-2)/2}{(m+i-1)/2} & \text{if } m \not\equiv i \,(\text{mod}\, 2) \\ \binom{(m+i-2)/2}{(m-i)/2} & \text{if } m \equiv i \,(\text{mod}\, 2), i \neq 1 \end{cases} \tag{2.14}$$

Lastly, in the case $\kappa = 4$, we see that the anti-hook difference conditions are equivalent to

$$b_{i+1} + 3 \leqq a_i \leqq b_i + 2, a_i \neq 1. \tag{2.15}$$

The map ϕ is

$$\phi\binom{a}{b} \longrightarrow \begin{cases} (a+b+1)_{b-a+3} & \text{if } a \leqq b+2, a \neq 1 \\ (a+b+1)_{a-b-2} & \text{if } a > b+2 \end{cases} \tag{2.16}$$

and ϕ^{-1} is given by

$$\phi^{-1} : m_i \longrightarrow \begin{cases} \binom{(m+i+1)/2}{(m-i-3)/2} & \text{if } m \not\equiv i+2 \,(\text{mod}\, 2) \\ \binom{(m-i+2)/2}{(m+i-4)/2} & \text{if } m \equiv i+2 \,(\text{mod}\, 2), m \neq i. \end{cases} \tag{2.17}$$

https://doi.org/10.4153/CMB-1989-047-1 Published online by Cambridge University Press

In Theorems 1.6 and 1.8 the reason why i_{i+l} ($l = 1$ in Th. 1.6 and $l = 2$ in Th. 1.8) is required as a part and that the parts are nonnegative can be given in a similar way as given in [4, p. 46].

To illustrate the bijections we have constructed we give an example for $\kappa = 1$, $\nu = 7$ shown in the following table:

Partitions enumerated by $A_1(7)$	Frobenius Symbol for partitions enumerated by $A_1(7)$	Image under ϕ i.e., partitions enumerated by $C_1(7)$
7	$\binom{6}{0}$	7_7
$6+1$	$\binom{5}{1}$	7_5
$5+1+1$	$\binom{4}{2}$	7_3
$4+1+1+1$	$\binom{3}{3}$	7_1
$4+2+1$	$\begin{pmatrix}3 & 0\\ 2 & 0\end{pmatrix}$	$6_2 + 1_1$
$3+3+1$	$\begin{pmatrix}2 & 1\\ 2 & 0\end{pmatrix}$	$5_1 + 2_2$
$5+2$	$\begin{pmatrix}4 & 0\\ 1 & 0\end{pmatrix}$	$6_4 + 1_1$

REFERENCES

1. A. K. Agarwal, *Partitions with "N copies of N"*, Lecture Notes in Mathematics, 1234, Proceedings of the Colloque De Combinatorie Enumerative, University of Quebec at Montreal, (1985), Springer-Verlag, 1–4.

2. ——— *Rogers-Ramanujan identities for n-color partitions*, J. Number Theory, **28** (3), (1988), pp. 299–305.

3. A. K. Agarwal and G. E. Andrews, *Hook-differences and lattice paths*, J. Statist. Plann. Inference, **14** (1986), 5–14.

4. ——— *Rogers-Ramanunujan identities for partitions with "N copies of N"*, J. Combin. Theory Ser A, Vol. 45, No. 1, (1987), 40–49.

5. L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25** (1884), 318–343.

Department of Mathematics
The Pennsylvania State University
(Mont Alto Campus)
Mont Alto, PA 17237, U.S.A.