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DECOMPOSITION PROBLEM OF PROBABILITY MEASURES RELATED TO MONOTONE REGULARLY VARYING FUNCTIONS

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1. Introduction

This paper deals with a decomposition problem for some classes of distributions. Let $\mathbf{D}$ be a given class of distribution on $\mathbf{R}^1$, which we are interested in. After showing that the class $\mathbf{D}$ is closed under convolution, our purpose is to give an answer to the inverse problem: if the convolution of two distributions μ_1 and μ_2 belongs to $\mathbf{D}$, then do μ_1 and μ_2 belong to $\mathbf{D}$?

Such an inverse problem is solved affirmatively for the class of Gaussian distributions, the class of Poisson distributions and the class of convolutions of Gaussian and Poisson ([5]). In this paper, we study this decomposition problem for several classes characterized by regular variation. A positive measurable function f is said to be regularly varying (r.v.) with index $\rho(\in \mathbf{R}^1)$ if $\lim_{x\to\infty} f(kx)/f(x) = k^{\rho}$ for each $k > 0$. In particular, f is called slowly varying (s.v.) if $\rho = 0$. It is well-known that the domain of attraction of Gaussian distribution (denoted by $\mathbf{D}_2$) is identical with the class of distributions whose truncated variances $\int_{|t|<x} t^2\mu(dt)$ are s.v. Concerning the inverse problem for $\mathbf{D}_2$, the author shows in [7] that there exist two distributions μ_1 and μ_2 such that neither μ_1 nor μ_2 belongs to $\mathbf{D}_2$ but the convolution of μ_1 and μ_2 belongs to $\mathbf{D}_2$. The proof depends on the fact that there is a non-decreasing s.v. function that is represented as the sum of positive non-decreasing functions that are not s.v.

We investigate the class $\mathbf{D}(\alpha)$ of distributions on $[0, \infty)$ with r.v. tails with index $-\alpha$ for $\alpha \geq 0$ and the class $\mathbf{C}$ of distributions on $[0, \infty)$ with s.v. truncated means. These classes are related to various limit theorems: the domain of attraction of stable laws, relative stability, the ratio of maximum to sum of an i.i.d.

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sequence and so on. We extend the argument for non-decreasing s.v. functions to general monotone r.v. functions. Since the decomposition for the classes of the corresponding functions is essential in solving the decomposition problem for the classes of distributions, we study it in detail in Section 3. We say that a non-negative non-decreasing (resp. non-increasing) f is decomposed into f_1 and f_2, if both f_1 and f_2 are non-negative non-decreasing (resp. non-increasing) and $f = f_1 + f_2$. In this case f_1 and f_2 are said to be components of f. We are interested in whether components of a r.v. function are r.v. or not. There are non-decreasing s.v. functions such that all their positive components are s.v. However, we will show that if f is a non-increasing s.v. function convergent to 0 or a monotone r.v. function with non-zero index, then f always has positive components that are not r.v. and f can be written as the sum of such components. Further, we study properties of the components of f. Especially, properties of the components that are not r.v. are interesting. When a component g of f is not r.v., the property of g differs between zero index case (s.v. case) and non-zero index case. If f is s.v., then g is occasionally small, i.e. $\liminf_{x\to\infty} g(x)/f(x) = 0$. But, in the case of non-zero index, g does not necessarily have this property. This is because of the difference in the manner of losing the regular variation. Let f be a non-decreasing r.v. function with index $\rho \geq 0$. If the component g is not r.v., then either $\limsup_{x\to\infty} g(kx)/g(x) > k^\rho$ (occasional rapid increase) or $\liminf_{x\to\infty} g(kx)/g(x) < k^\rho$ (occasional insufficient increase) must occur for some $k > 1$. If f is s.v. then only the first case can occur and then g is ocassionally small. But, if the index is positive, then both cases are possible. Further, the rapid increase of g and the insufficient increase of the complementary component $f - g$ can occur simultaneously on an interval and the two properties compensate for each other; in this way both of the components can lose the regular variation without being occasionally small.

In Section 4, decomposition problems of probability measures are considered. First, we give some relations between a distribution in $\mathbf{D}(\alpha)$ or $\mathbf{C}$ and its factors. These relations and general facts on regular variation imply that these classes are closed under convolution. Second, we answer the inverse problem by using the results in Section 3. It is easy to see that, for each $\alpha \geq 0$, there exist two distributions such that one of them belongs to $\mathbf{D}(\alpha)$, the other does not and their convolution does. For, it is known that, if $\mu \in \mathbf{D}(\alpha)$ and $\nu \in \mathbf{D}(\beta)$ with $\alpha \leq \beta$, then $\mu * \nu \in \mathbf{D}(\alpha)$ ([8]). Therefore our interest is in construction of two distributions μ_1 and μ_2 such that neither μ_1 nor μ_2 belongs to $\cup_{0\leq\beta<\infty}\mathbf{D}(\beta)$ but the convolution $\mu_1 * \mu_2$ belongs to $\mathbf{D}(\alpha)$. We also construct two distributions μ_1 and μ_2 such that μ_1 and $\mu_1 * \mu_2$ belong to $\mathbf{D}(\alpha)$, μ_2 does not belong to it, and $\limsup_{x\to\infty} \mu_2(x, \infty)/\mu_1(x, \infty) > 0$. Considering the same problem for $\mathbf{C}$, in addition, we will give a sufficient condition for a distribution in $\mathbf{C}$ to have the property that all non-trivial factors of it belong to $\mathbf{C}$. The situation of $\mathbf{D}(\alpha)$ $(\alpha > 0)$ is exceedingly different from the cases of $\mathbf{D}_2$, $\mathbf{C}$ and $\mathbf{D}(0)$, owing to the difference, which we mentioned, between the decompositon of monotone r.v. functions with non-zero index and that of monotone s.v. functions.

2. Preliminaries

The totality of all probability measures on the real line $\mathbf{R}^1$ is denoted by $\mathbf{P}(\mathbf{R}^1)$. We call them distributions (or laws). Delta distributions are called trivial distributions. A distribution μ_1 is called a factor of a distribution μ, if $\mu = \mu_1 * \nu$ with some $\nu \in \mathbf{P}(\mathbf{R}^1)$. Here $\mu_1 * \nu$ denotes the convolution of μ_1 and ν. We call $\mu(x, \infty)$ and $\mu(-\infty, -x)$ the right-tail and the left-tail of μ, respectively. Two functions f_1 and f_2 are said to be asymptotically equal and expressed as $f_1 \sim f_2$ if $\lim_{x\to\infty} f_1(x)/f_2(x) = 1$. The composite function $f_1(f_2(x))$ is denoted by $f_1 \circ f_2$. We state some fundamental facts on regular variation in the first half of this section and the probabilistic meaning of the classes of distributions with which we deal in this paper in the latter half. All the facts in this section are proved in [1], [2], [3], [4] and [6].

Slowly varying functions have the following representation.

THEOREM 2.1. *A function f is s.v. if and only if it can be written in the form*

$$f(x) = c(x)\exp\left(\int_A^x \varepsilon(t)t^{-1}\,dt\right),\ x \geq A$$

for some $A > 0$, where $c(x)$ and $\varepsilon(t)$ are measurable functions such that $\lim_{x\to\infty} c(x) = c$ $(0 < c < \infty)$ and $\lim_{t\to\infty}\varepsilon(t) = 0$.

A s.v. function f is called normalized if the function $c(x)$ in the above representation of f can be chosen to be a constant function.

Through this paper we use the following notations. $(\uparrow)$ is the set of positive non-decreasing functions. $(\downarrow)$ is the set of positive non-increasing functions and $(\downarrow)_0$ is the set of positive non-increasing functions convergent to 0. $\mathbf{SV}$ is the set of s.v. functions. $\mathbf{NS}$ is the set of normalized s.v. functions. $\mathbf{RV}$ is the set of r.v. functions. $\mathbf{RV}_\rho$ is the set of r.v. functions with index ρ. $\mathbf{SV}(\uparrow) = \mathbf{SV} \cap (\uparrow)$, $\mathbf{SV}(\downarrow) = \mathbf{SV} \cap (\downarrow)$, $\mathbf{RV}(\uparrow) = \mathbf{RV} \cap (\uparrow)$, $\mathbf{RV}(\downarrow) = \mathbf{RV} \cap (\downarrow)$, $\mathbf{RV}_\rho(\uparrow) = \mathbf{RV}_\rho \cap (\uparrow)$ and $\mathbf{RV}_\rho(\downarrow) = \mathbf{RV}_\rho \cap (\downarrow)$.

THEOREM 2.2. *A positive measurable function f is in* **NS** *if and only if, for every $\rho > 0$, $x^{\rho} f(x)$ is ultimately increasing and $x^{-\rho} f(x)$ is ultimately decreasing.*

If a monotone function f satisfies $\lim_{x \to \infty} f(2x)/f(x) = 1$, then f is s.v. On the other hand, the following theorem shows that any r.v. function with non-zero index is asymptotically equal to a monotone one.

THEOREM 2.3. *Let $f \in \mathbf{RV}_{\rho}$ and locally bounded on $[A, \infty)$ for some A. If $\rho > 0$, then*

$$f(x) \sim \sup\{f(t) \,;\, A \leq t \leq x\} \sim \inf\{f(t) \,;\, t \geq x\}.$$

If $\rho > 0$, then

$$f(x) \sim \sup\{f(t) \,;\, t \geq x\} \sim \inf\{f(t) \,;\, A \leq t \leq x\}.$$

The characteristic function of $\mu \in \mathbf{P}(\mathbf{R}^1)$ is denoted by $\hat{\mu}(z)$. A distribution μ is called stable if, for any $a > 0$ and $b > 0$, there exist $c > 0$ and $d \in \mathbf{R}^1$ such that

$$\hat{\mu}(az)\hat{\mu}(bz) = \hat{\mu}(cz)e^{idz}.$$

It is well-known that except in the case of delta distribution this c is uniquely determined by a and b, and there exists α with $0 < \alpha \leq 2$ such that $c^{\alpha} = a^{\alpha} + b^{\alpha}$. This α is called the index of the stable law.

A canonical representation of stable law μ with index α $(0 < \alpha < 2)$ is as follows:

$$\hat{\mu}(z) = \exp\Big[i\gamma z + c_1 \int_{-\infty}^{0} \Big(e^{iuz} - 1 - \frac{iuz}{1+u^2}\Big) \frac{du}{(-u)^{1+\alpha}} + c_2 \int_{0}^{\infty} \Big(e^{iuz} - 1 - \frac{iuz}{1+u^2}\Big) \frac{du}{u^{1+\alpha}}\Big], \tag{2.1}$$

where $\gamma \in \mathbf{R}^1$, c_1, $c_2 \geq 0$ and $c_1 + c_2 > 0$. A stable law μ is called spectrally positive if $c_1 = 0$ in (2.1).

Hereafter, in this section, let $X_1, X_2, \ldots, X_n, \ldots$ be $\mathbf{R}^1$-valued i.i.d. (independent and identically distributed) random variables with distribution ν and denote $S_n = \sum_{j=1}^{n} X_j$ (random walk) and $L_n = \max_{1 \leq j \leq n} X_j$.

If, for suitably chosen constants $B_n > 0$ and $A_n \in \mathbf{R}^1$, the distribution of

$$B_n^{-1} \sum_{j=1}^{n} X_j - A_n$$

converges to a distribution μ as $n \to \infty$, then we say that ν is attracted to μ. The totality of distributions attracted to μ is called the domain of attraction of μ.

Let $\mathbf{D}(\alpha)$ $(\alpha \geq 0)$ denote the class of distributions on $[0, \infty)$ with r.v. tails with index $-\alpha$.

THEOREM 2.4. *Let $0 < \alpha < 2$. Assume that $\nu(-\infty, 0) = 0$. Then ν belongs to the domain of attraction of a spectrally positive stable law with index α if and only if $\nu \in \mathbf{D}(\alpha)$.*

THEOREM 2.5. *Suppose that $\nu(x, \infty)$ is positive for all x. In order that, with suitably chosen normalizing constants B_n, the distribution of L_n/B_n converges to a non-trivial distribution μ, it is necessary and sufficient that $\nu(x, \infty) \in \mathbf{RV}_{-\alpha}$ with some $\alpha > 0$. In this case,*

$$\mu(-\infty, x] = \exp(-cx^{-\alpha}) \quad \textit{for} \quad x > 0 \quad \textit{with} \quad c > 0$$

and $\mu(-\infty, 0] = 0$.

THEOREM 2.6. *If ν is in $\mathbf{D}(0)$, then*

$$\lim_{n\to\infty} \mathbf{P}(n \cdot \nu(S_n, \infty) \geq x) = e^{-x}, x > 0.$$

If, for suitably chosen constants $B_n > 0$, the distribution of S_n/B_n converges to 1 in probability as $n \to \infty$, then we say that ν, or the random walk S_n, is relatively stable. For $\mu \in \mathbf{P}(\mathbf{R}^1)$, we define the truncated mean of μ by

$$M(R) = \int_{|x|<R} x\mu(dx).$$

The truncated mean of the distribution of a random variable X is denoted by $M_X(R)$. Let $\mathbf{C}$ denote the class of distributions on $[0, \infty)$ with s.v. truncated means.

THEOREM 2.7. *Suppose that $\nu(-\infty, 0] = 0$. Then ν is relatively stable if and only if $\nu \in \mathbf{C}$.*

Now we turn to comparison of the largest term L_n and the sum S_n. For simplicity, we restrict our attention to the case of ν concentrated on $(0, \infty)$; then $S_n > 0$, and we can consider L_n/S_n. The following facts are known.

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THEOREM 2.8. *L_n/S_n converges to* 0 *in probability as $n \to \infty$ if and only if $\nu \in \mathbf{C}$.*

THEOREM 2.9. *L_n/S_n converges to* 1 *in probability as $n \to \infty$ if and only if $\nu \in \mathbf{D}(0)$.*

THEOREM 2.10. *The following are equivalent*:
(1) *L_n/S_n has a non-trivial limit distribution,*
(2) *ν is attracted to a stable law of index $\alpha \in (0, 1)$,*
(3) *$\mathbf{E}(S_n/L_n - 1)$ tends to a positive finite limit.*

THEOREM 2.11. *If ν has finite mean m, then the following are equvalent*:
(1) *$(S_n - nm)/L_n$ has a non-degenerate limit distribution,*
(2) *ν is attracted to a stable law of index $\alpha \in (1, 2)$,*
(3) *$\mathbf{E}\{(S_n - nm)/L_n\}$ tends to a positive finite limit c.*
The α and c are related as $\alpha = (1 + c)/c$.

3. Decomposition of monotone regularly varying functions

In this section, general results on decomposition of monotone r.v. functions into the sums of monotone functions are given. In the first subsection, we complement our discussion in [7] of non-decreasing s.v. functions. Then we investigate non-increasing s.v. functions in the second subsection. The third subsection deals with monotone r.v. functions with non-zero indices. For each case, we consider three types of decomposition of a monotone function f into two components f_1, f_2: type I: $f_1 \in \mathbf{RV}$ and $f_2 \in \mathbf{RV}$; type II: $f_1 \in \mathbf{RV}$ and $f_2 \notin \mathbf{RV}$; type III: $f_1 \notin \mathbf{RV}$ and $f_2 \notin \mathbf{RV}$. Keep in mind that, when we consider type II decomposition, the numbering of f_1 and f_2 is made as above. We exclude the trivial decomposition where $f_1 = 0$ or $f_2 = 0$.

3.1. Non-decreasing slowly varying functions

The decomposition of non-decreasing s.v. functions is dealt with in [7]. The following definition is given there and Theorem 3.1 is the main result in [7].

DEFINITION. We say that a non-negative non-decreasing function f is dominatedly non-decreasing (resp. undominatedly non-decreasing) if $\limsup_{x\to\infty} (f(2x) - f(x)) < \infty$ (resp. $= \infty$).

THEOREM 3.1. *Let* $f \in \mathbf{SV}(\uparrow)$.

(1) *If* f *is dominatedly non-decreasing, then every decompoition of* f *is of type* I.

(2) *If* f *is undominatedly non-decreasing, then* f *has a type* III *decomposition.*

We show some facts on the decomposition, which are extensions of Proposition 3.7 and Theorem 3.8 of [7].

THEOREM 3.2 *Let* f *be an undominatedly non-decreasing s.v. function.*

(1) *Suppose that* f *is decomposed into* f_1 *and* f_2. *If* $f_2 \notin \mathbf{SV}$ *and* $\liminf_{j\to\infty} f_2(2x_j)/f_2(x_j) > 1$ *for some sequence* $x_j \to \infty$, *then*

$$\lim_{j\to\infty} f_2(x_j)/f(x_j) = 0.$$

(2) *For any constant* r *such that* $0 \leq r \leq \infty$, *there exists a type* II *decomposition of* f *into* f_1 *and* f_2 *satisfying*

$$\limsup_{x\to\infty} f_2(x)/f_1(x) = r. \tag{3.1}$$

Proof. (1) Notice that

$$(f(2x_j) - f(x_j))/f(x_j) \geq \{(f_2(2x_j) - f_2(x_j))/f_2(x_j)\}\{f_2(x_j)/f(x_j)\}.$$

Since the left-hand side converges to 0 as $j \to \infty$ by the slow variation of f and $(f_2(2x_j) - f_2(x_j))/f_2(x_j)$ has a positive lower bound, we get $\lim_{j\to\infty} f_2(x_j)/f(x_j) = 0$.

(2) If $r < \infty$, then the assertion is proved in [7]. Let $r = \infty$. By Theorem 3.1, f has a type III decomposition: $f = \tilde{f}_1 + \tilde{f}_2$. Set

$$\theta(x) = \sup_{t\geq x}(f(2t) - f(t))/f(t).$$

Obviously, θ is in $(\downarrow)_0$. Define ϕ by $\phi = l \circ \tilde{f}_2$, where l is in $\mathbf{NS} \cap (\downarrow)_0$ satisfying $\lim_{x\to\infty} \theta(x)/l(x) = 0$. Then it is easy to show that

$$\lim_{x\to\infty} \theta(x)/\phi(x) = 0. \tag{3.2}$$

Now define f_1 and f_2 by $f_1 = \tilde{f}_1 + \phi\tilde{f}_2$, $f_2 = (1-\phi)\tilde{f}_2$. Then, $f = f_1 + f_2$. Further, it is obvious that $f_2 \in (\uparrow)$ and $f_2 \notin \mathbf{SV}$. We will prove that f_1 and f_2 satisfy (3.1) with $r = \infty$ and $f_1 \in \mathbf{SV}(\uparrow)$. Since $\phi\tilde{f}_2 = \tilde{f}_2(l \circ \tilde{f}_2)$ and l is in $\mathbf{NS}$, $\phi\tilde{f}_2 \in (\uparrow)$ by Theorem 2.2. Hence $f_1 \in (\uparrow)$. Now let us prove that $f_1 \in \mathbf{SV}$. We have

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$$\begin{aligned}\frac{f_1(2x)}{f_1(x)} &= \frac{\tilde{f}_1(2x) + \phi(2x)\tilde{f}_2(2x)}{\tilde{f}_1(x) + \phi(x)\tilde{f}_2(x)} \\ &\leq \frac{\theta(x)f(x) + \tilde{f}_1(x) + \phi(2x)\{\theta(x)f(x) + \tilde{f}_2(x)\}}{\tilde{f}_1(x) + \phi(x)\tilde{f}_2(x)} \\ &\leq 1 + \frac{\theta(x)}{\phi(x)}(1 + \phi(x)).\end{aligned}$$

Here we have used that

$$(\tilde{f}_i(2x) - \tilde{f}_i(x))/f(x) \leq \theta(x) \quad (i = 1,2) \quad \text{and} \quad \phi(2x) \leq \phi(x).$$

By (3.2), the last term converges to 1, which shows that $f_1 \in \mathbf{SV}$. Noticing that $\limsup_{x\to\infty} \tilde{f}_2(x)/\tilde{f}_1(x) = \infty$ ([7], Proposition 3.7 (2)), we can prove that f_1 and f_2 satisfy (3.1) with $r = \infty$. The proof is complete. □

We add a theorem concerning decomposition of type I.

THEOREM 3.3. *Let* $f \in \mathbf{SV}(\uparrow)$ *and* p *and* q *be constants each that* $0 \leq p \leq q \leq \infty$. *If* $\lim_{x\to\infty} f(x) = \infty$, *then* f *has a type* I *decomposition into* f_1, f_2 *satisfying*

$$\liminf_{x\to\infty} f_2(x)/f_1(x) = p \quad \text{and} \quad \limsup_{x\to\infty} f_2(x)/f_1(x) = q. \tag{3.3}$$

Proof. If f is dominatedly non-decreasing, then the increase on the intervals $(2^j, 2^{j+1}]$ $(j = 1,2,\ldots)$ is uniformly bounded. We can construct f_1 and f_2 having the desired properties in such a way that only one of f_1 and f_2 increases on each interval.

Consider the case of undominatedly non-decreasing f. By Theorem 3.1, f has a type III decomposition: $f = \tilde{f}_1 + \tilde{f}_2$. If $0 < p \leq q < \infty$, define f_1 and f_2 as

$$f_1 = a\tilde{f}_1 + c\tilde{f}_2, \quad f_2 = b\tilde{f}_1 + d\tilde{f}_2,$$

where $p = b/a$, $q = d/c$ and $a + b = c + d = 1$. If $p = q = 0$ or if $p = q = \infty$, define f_1 and f_2 as

$$f_1 = f - \sqrt{f}, \quad f_2 = \sqrt{f}.$$

If $p = 0$, $0 < q < \infty$ or if $0 < p$, $q = \infty$, define f_1 and f_2 as

$$f_1 = c\tilde{f}_1 + (1 - \phi)\tilde{f}_2, \quad f_2 = d\tilde{f}_1 + \phi\tilde{f}_2,$$

where $q = d/c$, $c + d = 1$ and $\phi \in (\downarrow)$ as in the proof of the previous theorem.

If $p = 0$, $q = \infty$, then we define

$$f_1 = \phi_1 \tilde{f}_1 + (1 - \phi_2)\tilde{f}_2, \quad f_2 = (1 - \phi_1)\tilde{f}_1 + \phi_2 \tilde{f}_2,$$

where ϕ_i $(i = 1, 2)$ is given by $\phi_i = l \circ \tilde{f}_i$. It is not difficult to prove that f_1 and f_2 have the desired properties. □

3.2. Non-increasing slowly varying functions

We consider the decomposition of non-increasing s.v. functions. Theorem 3.4 and Proposition 3.5 show that the decomposition in this case is simpler than that of non-decreasing s.v. functions. Proposition 3.6, Theorems 3.7 and 3.8 show that the components have properties similar to the components of non-decreasing s.v. functions.

THEOREM 3.4. *Let f be in* $\mathbf{SV}(\downarrow)$. *If* $\lim_{x\to\infty} f(x) = 0$, *then f has a type* III *decomposition.*

Proof. Let

$$g(x) = \sup_{t \geq x}(f(t) - f(2t)) \quad \text{and} \quad h(x) = f(x) - g(x).$$

It follows from non-increasingness of f that both g and h are in $(\downarrow)$. Further,

$$\lim_{x\to\infty} g(x)/h(x) = 0, \tag{3.4}$$

since $f \in \mathbf{SV}$, $h \in \mathbf{SV}$ and $h \sim f$. We construct f_1 and f_2 oscillating between h and g. Using (3.4), choose $x_0 > 0$ such that

$$g(x) \leq h(3x) \quad \text{for all} \quad x \geq x_0. \tag{3.5}$$

For each ε $(0 < \varepsilon < 1)$, define

$$C(\varepsilon) = \{x : (1 - \varepsilon)g(x) \leq f(x) - f(2x)\}.$$

Notice that $C(\varepsilon)$ is an unbounded set. Assume that f_1 and f_2 are defined on $[x_0, x_j]$ and that

$$f_1(x_j) = g(x_j), \quad f_2(x_j) = h(x_j), \quad x_j \in C(\varepsilon).$$

For $x_j < x \leq 2x_j$, we define f_1 and f_2 as

$$f_1(x) = g(x_j) + s(f(x) - f(x_j)), f_2(x) = h(x_j) + t(f(x) - f(x_j)),$$

where $s, t > 0$ and $s + t = 1$. Define $x_{j+1} = \inf\{t : h(t) < f_1(2x_j)\}$. Then, by (3.5),

$$x_{j+1} > 2x_j.$$

If $f_1(2x_j) \leq$ (resp. $>$) $h(x_{j+1})$, then, define, for $2x_j < x \leq$ (resp. $<$) x_{j+1},

$$f_1(x) = f_1(2x_j), \quad f_2(x) = f(x) - f_1(2x_j).$$

Choose x_{j+2} such that $x_{j+2} > x_{j+1}$ and $x_{j+2} \in C(\varepsilon)$. For $x_{j+1} <$ (resp. $\leq$) $x \leq x_{j+2}$, define

$$f_1 = h, f_2 = g.$$

Exchanging the roles of f_1 and f_2 we define f_1 and f_2 on $(x_{j+2}, x_{j+4}]$. Thus, by induction, we can define f_1 and f_2 on $[x_0, \infty)$. Obviously $f = f_1 + f_2$. It is not difficult to show that $f_1, f_2 \in (\downarrow)$. Let us prove that $f_1, f_2, \notin \mathbf{SV}$. Since

$$\frac{f_1(x_j) - f_1(2x_j)}{f_1(x_j)} = s\frac{f(x_j) - f(2x_j)}{g(x_j)} \geq s(1 - \varepsilon).$$

we get

$$\limsup_{x\to\infty} \frac{f_1(x) - f_1(2x)}{f_1(x)} \geq s(1 - \varepsilon) > 0.$$

This implies that $f_1 \notin \mathbf{SV}$. Similarly, $f_2 \notin \mathbf{SV}$. □

Remark. A similar method gives a new proof of Theorem 3.1 (2). In this case, we make f_1 and f_2 oscillate between two unbounded non-decreasing functions

$$g(x) = \sup_{x\to\infty} (f(t) - f(t/2)) \quad \text{and} \quad h(x) = f(x) - g(x).$$

We can give a stronger assertion on decomposition of a non-increasing s.v. function as follows. Proof is essentially the same as that of Theorem 3.4 and omitted.

PROPOSITION 3.5. *Let f be in $\mathbf{SV}(\downarrow)$. If $\lim_{x\to\infty} f(x) = 0$, then, for each n, f can be represented as $f = \sum_{i=1}^{n} f_i$, where each f_i is in $(\downarrow)$ and the sum of an arbitrary proper sebset of the set $\{f_i : i = 1,2,\ldots,n\}$ is not in $\mathbf{SV}$. Moreover, f has a representation $f = \sum_{i=1}^{\infty} f_i$ with the same properties.*

The following proposition gives properties of components in decomposition of a non-increasing s.v. function. This corresponds to Proposition 3.7 of [7] and Theorem 3.2 (1) on non-decreasing s.v. functions. Proof is similar and omitted.

PROPOSITION 3.6. *Let f be in* $\mathbf{SV}(\downarrow)$. *Suppose that f is decomposed into f_1 and f_2.*

(1) *Then at least one of f_1 and f_2 satisfies*

$$\limsup_{x\to\infty} f_i(kx)/f_i(x) = 1 \quad \text{for every} \quad k > 1. \tag{3.6}$$

(2) *If $f_2 \notin \mathbf{SV}$ and $\limsup_{j\to\infty} f_2(2x_j)/f_2(x_j) < 1$ for some sequence $\{x_j\}$, then $\lim_{j\to\infty} f_2(x_j)/f(x_j) = 0$. Especially, if the decomposition is of type* III, *then*

$$\liminf_{x\to\infty} f_2(x)/f_1(x) = 0 \quad \text{and} \quad \limsup_{x\to\infty} f_2(x)/f_1(x) = \infty. \tag{3.7}$$

Remark. In (1), one of f_1 and f_2 can fail to satisfy (3.6). For example, consider $f(x) = (\log x)^{-1} + x^{-1}$.

We show a theorem on decomposition of type II.

THEOREM 3.7. *Let f be in* $\mathbf{SV}(\downarrow)$ *and let* $0 \le r \le \infty$. *If* $\lim_{x\to\infty} f(x) = 0$, *then f has a type* II *decomposition satisfying* (3.1).

Proof. By Theorem 3.4, f has a type III decomposition: $f = \tilde{f}_1 + \tilde{f}_2$. Assume $0 < r < \infty$. Define u by $r = u/(1-u)$. Then, $0 < u < 1$. Set $f_1 = \tilde{f}_1 + (1-u)\tilde{f}_2$ and $f_2 = u\tilde{f}_2$. Then, f_1 and f_2 satisfy the desired conditions. We can get (3.1) by (3.7). This finishes the proof in the case $0 < r < \infty$. If $r = 0$, then define $f_1 = f - \tilde{f}_2^2$ and $f_2 = \tilde{f}_2^2$ for large x. Let us consider the case of $r = \infty$. Set

$$\theta(x) = \sup_{t \ge x} (f(t) - f(2t))/f(t).$$

Obviously, $\theta \in (\downarrow)_0$. Set $\omega = 1/\tilde{f}_2$ and choose a continuous non-decreasing function η satisfying $\omega \le \eta$. Then $\eta(x)x$ is an increasing function. Let ψ be its inverse function. Define l_1 by $l_1 = l_0 \circ \psi$, where l_0 is an unbounded s.v. function satisfying $\lim_{x\to\infty} \theta(x) l_0(x) = 0$. It is easy to see that $l_1 \in \mathbf{SV}(\uparrow)$ is unbounded. Let l be in $\mathbf{NS} \cap (\downarrow)_0$ such that $\lim_{x\to\infty} l(x) l_1(x) = 1$. Define $\phi = l \circ \omega$. Then, since $\theta(x)/\phi(x) \le \theta(x)/l(\eta(x)x)$, we have

$$\lim_{x\to\infty} \theta(x)/\phi(x) = 0. \tag{3.8}$$

Now define f_1 and f_2 by $f_1 = \tilde{f}_1 + \phi \tilde{f}_2$, $f_2 = (1-\phi)\tilde{f}_2$. Then, $f_1 \in (\downarrow)$, $f_2 \notin \mathbf{SV}$ and $f = f_1 + f_2$. We have to prove that f_1 and f_2 satisfy (3.1), $f_2 \in (\downarrow)$ and $f_1 \in \mathbf{SV}$. Since $(1-\phi)\tilde{f}_2 = (1 - l(1/\tilde{f}_2))\tilde{f}_2$ and l is in $\mathbf{NS}$, we see that $f_2 \in (\downarrow)$. Now let us prove that $f_1 \in \mathbf{SV}$. Using $(\tilde{f}_i(x) - \tilde{f}_i(2x))/f(x) \leq \theta(x)$, we have

$$\begin{aligned}\frac{f_1(2x)}{f_1(x)} &= \frac{\tilde{f}_1(2x) + \phi(2x)\tilde{f}_2(2x)}{\tilde{f}_1(x) + \phi(x)\tilde{f}_2(x)} \\ &\geq 1 + \frac{-\theta(x)(1+\phi(2x))f(x) + (\phi(2x) - \phi(x))\tilde{f}_2(x)}{\tilde{f}_1(x) + \phi(x)\tilde{f}_2(x)} \\ &\geq 1 - (1+\phi(2x))\frac{\theta(x)}{\phi(x)} - \frac{(\phi(x) - \phi(2x))\tilde{f}_2(x)}{\tilde{f}_1(x) + \phi(x)\tilde{f}_2(x)}.\end{aligned}$$

By (3.8), the second term in the last line converges to 0 as $x \to \infty$. Let us denote by δ the third term with the minus sign deleted. If $\tilde{f}_1(x)/\tilde{f}_2(x) \geq 1$, then we have

$$\delta(x) \leq \frac{\phi(x) - \phi(2x)}{1 + \phi(x)} \leq \phi(x) \to 0 \quad \text{as} \quad \to \infty.$$

If $\tilde{f}_1(x)/\tilde{f}_2(x) \leq 1$, then $(\tilde{f}_2(x) - \tilde{f}_2(2x))/2\tilde{f}_2(x) \leq (f(x) - f(2x))/f(x) \leq \theta(x)$ and we get

$$\omega(2x) \leq (1 - 2\theta(x))^{-1}\omega(x),$$

which implies

$$\delta(x) \leq \frac{\phi(x) - \phi(2x)}{\phi(x)} \leq \frac{l(\omega(x)) - l((1-2\theta(x))^{-1}\omega(x))}{l(\omega(x))}.$$

The last expression tends to 0 by the slow variation of l. This concludes the proof that $f_1 \in \mathbf{SV}$. It is easy to show that f_1 and f_2 satisfy (3.1). □

We give a statement on decomposition of type I. The proof is similar to that of Theorem 3.3.

THEOREM 3.8. *Let f be in* $\mathbf{SV}(\downarrow)$ *and let* $0 \leq p \leq q \leq \infty$. *If* $\lim_{x\to\infty} f(x) = 0$, *then f has a type* I *decomposition satisfying* (3.3).

3.3. Monotone regularly varying functions with non-zero index

Now we study decomposition of monotone r.v. functions with non-zero index.

In the three types of the decomposition, type III is especially interesting because there is a remarkable difference between zero index case and non-zero index case. Since the results on non-decreasing r.v. functions are similar to those on non-increasing ones, we state them together.

LEMMA 3.9 *Let f be in* $\mathbf{RV}_\rho(\uparrow)$ *with* $\rho > 0$ (*resp.* $\mathbf{RV}_\rho(\downarrow)$ *with* $\rho < 0$) *and* l *be in* $\mathbf{SV}(\uparrow)$ (*resp.* $\mathbf{SV}(\downarrow)$). *Then there exist* $f_0 \in \mathbf{RV}(\uparrow)$ (*resp.* $\mathbf{RV}(\downarrow)$) *and* $l_0 \in \mathbf{SV}(\uparrow)$ (*resp.* $\mathbf{SV}(\downarrow)$) *satisfying*

$$f = l_0 f_0 \quad \text{and} \quad l_0 \sim l$$

Proof. It is sufficient to prove in the non-decreasing case. Assume that $f \in \mathbf{RV}_\rho(\uparrow)$ with $\rho > 0$ on $[1, \infty)$. Let $F = \log f$ and $\varepsilon_j = \log(l(2^{j+1})/l(2^j))$. Then, since

$$\lim_{x\to\infty} (F(2x) - F(x)) = \rho \log 2 \quad \text{and} \quad \lim_{j\to\infty} \varepsilon_j = 0,$$

we can assume that $\varepsilon_j \leq F(2x) - F(x)$ for all j and x without loss of generality. Define G and H inductively as follows: For $x = 1$, let $G(1) = \log l(1)$, $H(1) = F(1) - \log l(1)$. For $2^j < x \leq 2^{j+1}$, let

$$G(x) = G(2^j) + \frac{\varepsilon_j}{F(2^{j+1}) - F(2^j)}(F(x) - F(2^j)),$$

$$H(x) = H(2^j) + \frac{F(2^{j+1}) - F(2^j) - \varepsilon_j}{F(2^{j+1}) - F(2^j)}(F(x) - F(2^j)).$$

Define $l_0 = \exp G$ and $f_0 = \exp H$. It is easy to prove that l_0 and f_0 satisfy the above conditions. □

The following theorem is proved by this lemma and Theorem 3.3 or 3.8.

THEOREM 3.10. *Let f be in* $\mathbf{RV}_\rho(\uparrow)$ *with* $\rho > 0$ (*resp.* $\mathbf{RV}_\rho(\downarrow)$ *with* $\rho < 0$) *and let* $0 \leq p \leq q \leq \infty$. *Then f has a type* I *deomposition satisfying* (3.3).

Proof. Choose an unbounded non-decreasing s.v. function l and apply the above lemma to f and l. Then l_0 is also unbounded. Write l_0 as the sum of two funcitons l_1 and l_2 having the properties in Theorem 3.3. Then we define $f_1 = l_1 f_0$ and $f_2 = l_2 f_0$, which satisfy our conditions. In the non-increasing case, choose l in $\mathbf{SV} \cap (\downarrow)_0$ and use the above lemma and Theorem 3.8. □

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We show a theorem on decomposition of type II.

THEOREM 3.11. *Let f be in* $\mathbf{RV}(\uparrow)$ (*resp.* $\mathbf{RV}(\downarrow)$).
(1) *If a decomposition is of type* II, *then* $\liminf_{x\to\infty} f_2(x)/f_1(x) = 0$.
(2) *For any constant r such that $0 \le r \le \infty$, there exists a type* II *decomposition of f satisfying* $\limsup_{x\to\infty} f_2(x)/f_1(x) = r$.

Proof. (1) It is easy to see that the index of f_1 is equal to the index ρ of f. Assume that $f_2/f_1 > \varepsilon$ for some positive constant ε. By the regular variation of f,

$$\lim_{x\to\infty} \frac{f_1(kx) - k^\rho f_1(x)}{f_1(x) + f_2(x)} + \frac{f_2(kx) - k^\rho f_2(x)}{f_1(x) + f_2(x)} = 0$$

for every $k > 0$. Since

$$\limsup_{x\to\infty} \left| \frac{f_1(kx)/f_1(x) - k^\rho}{1 + f_2(x)/f_1(x)} \right| \le \limsup_{x\to\infty} | f_1(kx)/f_1(x) - k^\rho | = 0,$$

we get

$$\lim_{x\to\infty} \frac{f_2(kx) - k^\rho f_2(x)}{f_1(x) + f_2(x)} = 0.$$

Using $f \le (1 + \varepsilon^{-1}) f_2$, we have

$$\limsup_{x\to\infty} \left| \frac{f_2(kx)}{f_2(x)} - k^\rho \right| \le (1 + \varepsilon^{-1}) \limsup_{x\to\infty} \left| \frac{f_2(kx) - k^\rho f_2(x)}{f_1(x) + f_2(x)} \right| = 0.$$

This is contrary to that $f_2 \notin \mathbf{RV}$.

Proof of (2) is given in a similar way to the proof of the previous theorem by using Theorem 3.2 (resp. 3.7) instead of Theorem 3.3 (resp. 3.8). □

Remark. Notice that the proof of (1) does not use the assumption of monotonicity.

Now we proceed to the decomposition of type III, which is widely different from the case of slow variation in Proposition 3.6.

THEOREM 3.12. *Let f be in* $\mathbf{RV}_\rho(\uparrow)$ *with* $\rho > 0$ (*resp.* $\mathbf{RV}_\rho(\downarrow)$ *with* $\rho < 0$) *and let* $0 \le p < q \le \infty$. *Then f has a type* III *decomposition satisfying* (3.3).

Proof. We give outline of our proof in non-decreasing case. Non-increasing case can be treated in a similar way. Let ρ be the index of f. First we choose suitable h_i, $g_i \in (\uparrow)$ $(i = 1,2)$ satisfying

$$f = h_1 + h_2 = g_1 + g_2.$$

We will define f_i oscillating between h_i and g_i. We will choose a sequence $\{x_j\}$ in a suiatable manner and make f_1 (resp. f_2) increasing (resp. flat) in $[x_{2j-1}, x_{2j})$ and flat (resp. increasing) in $[x_{2j}, x_{2j+1})$. We divide our consideration into three cases, namely, *Case* 1: $0 < p < q < \infty$; *Case* 2: either $p = 0 < q < \infty$ or $0 < p < q = \infty$; *Case* 3: $p = 0$, $q = \infty$. In Case 2, we assume $p = 0 < q < \infty$, as the discussion for $0 < p < q = \infty$ is similar. Define h_i and g_i in each case as follows.

Case 1: $\quad h_1 = \dfrac{1}{q+1} f,\ h_2 = \dfrac{q}{q+1} f,\ g_1 = \dfrac{1}{p+1} f,\ g_2 = \dfrac{p}{p+1} f.$

Case 2: $\quad h_1 = \dfrac{1}{q+1} f,\ h_2 = \dfrac{q}{q+1} f,\ g_1 = f - \sqrt{f},\ g_2 = \sqrt{f}.$

Case 3: $\quad h_2 = g_1 = f - \sqrt{f},\ h_1 = g_2 = \sqrt{f}.$

Define the sequence $\{x_j\}$ inductively as

$$x_{2j+1} = \inf\{t : h_1(t) > g_1(x_{2j})\}, \quad x_{2j} = \inf\{t : g_2(t) > h_2(x_{2j-1})\}.$$

This sequence is increasing and satisfies

$$\lim_{j\to\infty} \frac{h_1(x_{2j+1})}{g_1(x_{2j})} = \lim_{j\to\infty} \frac{h_2(x_{2j-1})}{g_2(x_{2j})} = 1.$$

In each case, this equality means the following.

Case 1:

$$\lim_{j\to\infty} \frac{f(x_{2j+1})}{f(x_{2j})} = \lim_{j\to\infty} \left(\frac{x_{2j+1}}{x_{2j}}\right)^{\rho} = \frac{q+1}{p+1}, \tag{3.9}$$

$$\lim_{j\to\infty} \frac{f(x_{2j})}{f(x_{2j-1})} = \lim_{j\to\infty} \left(\frac{x_{2j}}{x_{2j-1}}\right)^{\rho} = \frac{q(p+1)}{p(q+1)}.$$

Case 2:

$$\lim_{j\to\infty} \frac{f(x_{2j+1})}{f(x_{2j})} = \lim_{j\to\infty} \left(\frac{x_{2j+1}}{x_{2j}}\right)^{\rho} = q+1,$$

$$\lim_{j\to\infty} \frac{\sqrt{f(x_{2j})}}{f(x_{2j-1})} = \frac{q}{q+1},\ \lim_{j\to\infty} \frac{x_{2j}}{x_{2j-1}} = \infty.$$

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Case 3:

$$\lim_{j\to\infty}\frac{f(x_{j+1})}{f(x_j)^2}=1,\quad \lim_{j\to\infty}\frac{x_{j+1}}{x_j}=\infty.$$

Define f_1 and f_2 in the following way. Let $f_1(x_0)=g_1(x_0)$ and $f_2(x_0)=g_2(x_0)$. For $x_{2j-1}<x<x_{2j}$, define

$$f_1(x)=f(x)-h_2(x_{2j-1}),\quad f_2(x)=h_2(x_{2j-1}).$$

For $x_{2j}<x<x_{2j+1}$, define

$$f_1(x)=g_1(x_{2j}),\quad f_2(x)=f(x)-g_1(x_{2j}).$$

At x_{2j} and x_{2j+1} define f_1 and f_2 as follows: If $g_2(x_{2j})\geq$ (resp. $<$) $h_2(x_{2j-1})$, then

$$f_1(x_{2j})=g_1(x_{2j}),\quad f_2(x_{2j})=g_2(x_{2j}).$$
$$(\text{resp. } f_1(x_{2j})=f(x_{2j})-h_2(x_{2j-1}),\quad f_2(x_{2j})=h_2(x_{2j-1})).$$

If $h_1(x_{2j+1})\geq$ (resp. $<$) $g_1(x_{2j})$, then

$$f_1(x_{2j+1})=h_1(x_{2j+1}),\quad f_2(x_{2j+1})=h_2(x_{2j+1}).$$
$$(\text{resp. } f_1(x_{2j+1})=g_1(x_{2j}),\quad f_2(x_{2j+1})=f(x_{2j+1})=g_2(x_{2j})).$$

Then

$$\lim_{j\to\infty}\frac{f_1(x_{2j+1})}{f_1(x_{2j})}=1\quad\text{and}\quad\lim_{j\to\infty}\frac{f_2(x_{2j})}{f_2(x_{2j-1})}=1.\tag{3.10}$$

Let us prove that f_1 and f_2 have the desired properties. Obviously, $f=f_1+f_2$. Non-decrease is easy to see. (3.9) and (3.10) show that $f_1,f_2\notin\mathbf{RV}$ because $\liminf_{x\to\infty}f_i(kx)/f_i(x)=1$, $i=1,2$, for some $k>1$. For the proof of (3.3), check that

$$\lim_{j\to\infty}\frac{f_2(x_{2j})}{f_1(x_{2j})}=\lim_{j\to\infty}\frac{g_2(x_{2j})}{g_1(x_{2j})}=p,\ \lim_{j\to\infty}\frac{f_2(x_{2j+1})}{f_1(x_{2j+1})}=\lim_{j\to\infty}\frac{h_2(x_{2j+1})}{h_1(x_{2j+1})}=q.$$

This completes the proof. □

In the above proof, we showed that f_1 (resp. f_2) $\notin\mathbf{RV}$ by its insufficient increase on some intervals, but, on the same intervals, f_2 (resp, f_1) increases rapidly. Actually, in Case 1,

$$\lim_{j\to\infty}\frac{f_2(x_{2j+1})}{f_2(x_{2j})}=\frac{q}{p}>\lim_{j\to\infty}\left(\frac{x_{2j+1}}{x_{2k}}\right)^{\rho}=\frac{q+1}{p+1}.$$

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This shows that $f_2 \notin \mathbf{RV}$, since it increases rapidly on $[x_{2k}, x_{2k+1})$. In general, when a component that is not occasionally small loses regular variation, the complementary component loses it simultaneously, but the manners how to lose it are opposite in the two components. This fact is shown below.

PROPOSITION 3.13. *Let f be in $\mathbf{RV}_\rho(\uparrow)$ with $\rho > 0$ (or $\mathbf{RV}_\rho(\downarrow)$ with $\rho < 0$). Assume that f is decomposed into f_1 and f_2. If a sequence $\{x_j\}$ satisfies $\liminf_{j\to\infty} f_1(kx_j)/f_1(x_j) > k^\rho$ (resp. $\limsup_{j\to\infty} f_1(kx_j)/f_1(x_j) < k^\rho$) for some $k > 0$ and $\liminf_{j\to\infty} f_1(x_j)/f_2(x_j) > 0$, then $\limsup_{j\to\infty} f_2(kx_j)/f_2(x_j) < k^\rho$ (resp. $\liminf_{j\to\infty} f_2(kx_j)/f_2(x_j) > k^\rho$).*

Proof. Choose ε, $r > 0$ such that

$$f_1(kx_j)/f_1(x_j) > k^\rho + \varepsilon, \tag{3.11}$$

$$f_1(x_j)/f_2(x_j) > r \tag{3.12}$$

for all large j. Select $\delta > 0$ such that $r\varepsilon - \delta(r+1) > 0$. We can assume that

$$|\, f(kx_j)/f(x_j) - k^\rho\,| < \delta. \tag{3.13}$$

Using (3.11) and (3.13), we have

$$f_2(kx_j) < -(k^\rho + \varepsilon) f_1(x_j) + f(kx_j) < -(k^\rho + \varepsilon) f_1(x_j) + k^\rho f(x_j) + \delta f(x_j).$$

Hence, by (3.12), we get

$$f_2(kx_j)/f_2(x_j) < k^\rho - (\varepsilon - \delta) f_1(x_j)/f_2(x_j) + \delta < k^\rho - r(\varepsilon - \delta) + \delta.$$

Thus we get $\limsup_{j\to\infty} f_2(kx_j)/f_2(x_j) \le k^\rho - \lambda$, where $\lambda = r\varepsilon - \delta(r+1) > 0$. The other case can be proved, similarly. □

4. Decomposition problem of distributions in the class characterized by regular variation

In this section, we apply the results in the preceeding section to the decomposition problem of probability measures. We investigate the classes $\mathbf{D}(\alpha)$ and $\mathbf{C}$. Those classes appear in connection with limit theorems for i.i.d. sequences (see Section 2). As we show, they are closed under convolution. But, we are mainly interested in properties of factors of distributions in these classes, as we studied $\mathbf{D}_2$ in [7].

The following theorem gives a relation between a distribution with r.v.

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right-tail and its factors. Proof does not need any result in Section 3.

THEOREM 4.1. *Let μ_i $(i = 1,2,\ldots,n)$ be distributions on $\mathbf{R}^1$ and let $\mu = \mu_1 * \cdots * \mu_n$. Then $\mu(x, \infty) \in \mathbf{RV}$ if and only if $\sum_{i=1}^n \mu_i(x, \infty) \in \mathbf{RV}$. In this case,*

$$\lim_{x\to\infty} \sum_{i=1}^n \mu_i(x, \infty)/\mu(x, \infty) = 1.$$

Proof. Let X_i be independent random variables with distribution μ_i and let $0 < \varepsilon < (n-1)^{-1}$. We claim that, for any $R > 0$,

$$\begin{aligned} &\prod_{i=1}^n \mathbf{P}(|X_i| < \varepsilon R) \sum_{i=1}^n \mathbf{P}(X_i > (1 + (n-1)\varepsilon)R) \\ &\quad \le \mathbf{P}\Big(\sum_{i=1}^n X_i > R\Big) \\ &\quad \le \sum_{i=1}^n \mathbf{P}(X_i > (1 - (n-1)\varepsilon)R) + \frac{1}{2}\Big(\sum_{i=1}^n \mathbf{P}(X_i > \varepsilon R)\Big)^2. \end{aligned} \tag{4.1}$$

In fact, the first inequality comes from the estimate

$$\begin{aligned} \mathbf{P}\Big(\sum_{i=1}^n X_i > R\Big) &\ge \sum_{i=1}^n \mathbf{P}(X_i > (1 + (n-1)\varepsilon)R) \prod_{j\ne i} \mathbf{P}(|X_j| < \varepsilon R) \\ &\ge \sum_{i=1}^n \mathbf{P}(X_i > (1 + (n-1)\varepsilon)R) \prod_{i=1}^n \mathbf{P}(|X_i| < \varepsilon R), \end{aligned}$$

and the second inequality is obtained from

$$\begin{aligned} \mathbf{P}\Big(\sum_{i=1}^n X_i > R\Big) &\le \sum_{i=1}^n \mathbf{P}(X_i > (1 - (n-1)\varepsilon)R) \\ &\quad + \mathbf{P}\Big(\max_{1\le i\le n} X_i \le (1 - (n-1)\varepsilon)R, \sum_{i=1}^n X_i > R\Big), \end{aligned}$$

since

$$\begin{aligned} &\mathbf{P}\Big(\max_{1\le i\le n} X_i \le (1 - (n-1)\varepsilon)R, \sum_{i=1}^n X_i > R\Big) \\ &\quad \le \mathbf{P}\Big(\min_{1\le i\le n} \sum_{j\ne i} X_j > (n-1)\varepsilon R\Big) \le \mathbf{P}(\cup_{i<j}\{\min\{X_i, X_j\} > \varepsilon R\}) \\ &\quad \le \sum_{i<j} \mathbf{P}(X_i > \varepsilon R)\mathbf{P}(X_j > \varepsilon R) \le \frac{1}{2}\Big(\Big(\sum_{i=1}^n \mathbf{P}(X_i > \varepsilon R)\Big)^2. \end{aligned}$$

Assume that $\sum_{i=1}^n \mathbf{P}(X_i > R) \in \mathbf{RV}_{-\alpha}$. Then, by (4.1), we get

$$(1+(n-1)\varepsilon)^{-\alpha} \le \liminf_{R\to\infty} \frac{\mathbf{P}(\sum_{i=1}^n X_i > R)}{\sum_{i=1}^n \mathbf{P}(X_i > R)} \tag{4.2}$$
$$\le \limsup_{R\to\infty} \frac{\mathbf{P}(\sum_{i=1}^n X_i > R)}{\sum_{i=1}^n \mathbf{P}(X_i > R)} \le (1-(n-1)\varepsilon)^{-\alpha}.$$

Thus we get

$$\lim_{R\to\infty} \frac{\mathbf{P}(\sum_{i=1}^n X_i > R)}{\sum_{i=1}^n \mathbf{P}(X_i > R)} = 1. \tag{4.3}$$

Conversely, assume that $\mathbf{P}(\sum_{i=1}^n X_i > R) \in \mathbf{RV}_{-\alpha}$. Then, by the first inequality in (4.1), it is easy to see that

$$\lim_{R\to\infty} \frac{(\sum_{i=1}^n \mathbf{P}(X_i > R))^2}{\mathbf{P}(\sum_{i=1}^n X_i > R)} = 0. \tag{4.4}$$

It follows from (4.1) that

$$\mathbf{P}\Big(\sum_{i=1}^n X_i > \frac{R}{1-(n-1)\varepsilon}\Big) - \frac{1}{2}\Big\{\sum_{i=1}^n \mathbf{P}\Big(X_i > \frac{\varepsilon R}{1-(n-1)\varepsilon}\Big)\Big\}^2$$
$$\le \sum_{i=1}^n \mathbf{P}(X_i > R)$$
$$\le \Big\{\prod_{i=1}^n \mathbf{P}(|X_i| < \frac{\varepsilon R}{1+(n-1)\varepsilon}\Big\}^{-1} \mathbf{P}(\sum_{i=1}^n X_i > \frac{R}{1+(n-1)\varepsilon}\Big).$$

Hence, using (4.4) and the regular variation of $\mathbf{P}(\sum_{i=1}^n X_i > R)$, we have (4.2). Letting $\varepsilon \downarrow 0$, we get (4.3). □

Remark. Noticing that $f_1 + f_2 \in \mathbf{RV}_\rho$ for $f_1 \in \mathbf{RV}_\rho$ and $f_2 \in \mathbf{RV}_{\rho'}$ with $\rho' \le \rho$, we see that $\mathbf{D}(\alpha)$ is closed under convolution.

Now, combining the above theorem with the results in Section 3, we can obtain many results on the class $\mathbf{D}(\alpha)$. Among them we give the following theorem.

THEOREM 4.2. *There exist distributions μ_1 and μ_2 such that neither of them belongs to $\bigcup_{0\le\beta<\infty}\mathbf{D}(\beta)$ but $\mu = \mu_1 * \mu_2$ belongs to $\mathbf{D}(\alpha)$. In general, for each n, there exist distributions $\mu_1, \ldots, \mu_n$ such that $\mu = \mu_1 * \cdots * \mu_n$ belongs to $\mathbf{D}(\alpha)$ but, for every proper subset S of $\{1,\ldots,n\}$, the convolution of $\{\mu_i : i \in S\}$ does not belong to $\bigcup_{0\le\beta<\infty}\mathbf{D}(\beta)$.*

Proof. Choose an arbitrary distribution ν in $\mathbf{D}(\alpha)$. Then,

$$\nu(x, \infty) = x^{-\alpha} l(x) \quad \text{with some} \quad l \in \mathbf{SV}.$$

By Theorems 3.4 and 3.12, there exist probability measures μ_1 and μ_2 on $[0, \infty)$ such that

$$\mu_1(x, \infty) + \mu_2(x, \infty) = x^{-\alpha} l(x) \text{ for sufficiently large } x,$$

where neither $\mu_1(x, \infty)$ nor $\mu_2(x, \infty)$ belongs to $\mathbf{RV}$. Then, by Theorem 4.1, we get

$$\nu(x, \infty) = \mu_1(x, \infty) + \mu_2(x, \infty) \sim \mu_1 * \mu_2(x, \infty) = \mu(x, \infty).$$

This means that μ belongs to $\mathbf{D}(\alpha)$.

Similarly, Proposition 3.5 and Lemma 3.9 yield the latter half of the theorem. $\square$

Remark. By using Theorems 3.7, 3.11 and 4.1 as in the proof of the above theorem, we can construct two distributions μ_1 and μ_2 such that μ_1 and the convolution $\mu = \mu_1 * \mu_2$ belong to $\mathbf{D}(\alpha)$ and μ_2 does not belong to $\cup_{0 \le \beta < \infty} \mathbf{D}(\beta)$.

Remark. Concerning the domain of attraction of a general stable distribution μ with characteristic function (2.1), we can show the existence of two distributions such that neither of them belongs to the domain of attraction of μ but their convolution belongs to it. In fact, choose a distribution μ_1 such that both right and left tails are r.v. with index $-\alpha$ but the ratio of them does not converge. Choose a distribution μ_2 satisfying

$$\mu_2(x, \infty) \sim (c_2/c_1)\mu_1(-\infty, -x], \quad \mu_2(-\infty, -x] \sim (c_1/c_2)\mu_1(x, \infty).$$

Then, by Theorem 4.1, $\mu_1 * \mu_2$ belongs to the domain of attraction of μ.

Remark. In spite of Theorems 3.4 and 3.12, it is not true that every distribution in $\mathbf{D}(\alpha)$ is decomposed into two distributions neither of which belongs to $\mathbf{D}(\alpha)$. For example, there exists an indecomposable distribution in $\mathbf{D}(\alpha)$ constructed as follows: Choose a distribution ν in $\mathbf{D}(\alpha)$ and a set $A = \{x_j\}$ such that the distances between the points in A are all distinct and $j < x_j < j + 1$. Define a distribution μ as $\mu(x, \infty) = \nu(x_j, \infty)$ for $x_j < x \le x_{j+1}$. Then, A is an indecomposable set (in the sense that if $A = A_1 + A_2$ then A_1 or A_2 is a one-point set) and $\mu(x, \infty) \sim \nu(x, \infty)$. Hence μ is an indecomposable disribution in $\mathbf{D}(\alpha)$.

Now we investigate properties of factors of distributions in $\mathbf{C}$. First we will give a sufficient condition for a distribution to have the property that all factors of it belong to $\mathbf{C}$. Obviously, any factor of a distribution with finite mean has finite mean, and hence belongs to $\mathbf{C}$. We extend this fact; if μ has a dominatedly non-decreasing truncated mean, then every factor of μ belongs to $\mathbf{C}$. Second, we give μ_1 and μ_2 such that neither μ_1 nor μ_2 belongs to $\mathbf{C}$, but the convolution $\mu_1 * \mu_2$ belongs to $\mathbf{C}$.

We prepare two propositions.

PROPOSITION 4.3. *Let X be a non-negative random variable with truncated mean $M(R)$. Then the following are equivalent:*

$$\limsup_{R\to\infty} (M(2R) - M(R)) < \infty. \tag{4.7}$$

$$\limsup_{R\to\infty} R\mathbf{P}(X > R) < \infty. \tag{4.8}$$

Proof. (4.8) implies (4.7) because

$$\begin{aligned} M(2R) - M(R) &= \mathbf{E}X1(R \leq X < 2R) \\ &\leq 2R\mathbf{P}(R \leq X < 2R) \leq 2R\mathbf{P}(X \geq R). \end{aligned}$$

Conversely, assume (4.7). Then, since $M(2R) - M(R) \geq R\mathbf{P}(R \leq X\ 2R)$, there exists a positive constant c such that $R\mathbf{P}(R \leq X < 2R) < c$ for every $R > 0$. Therefore we get $2^n R\mathbf{P}(2^n R \leq X < 2^{n+1} R) < c$ for every $n \in \mathbf{N}$. Summing up for all n, we have

$$R\mathbf{P}(X \geq R) = \sum_{n=0}^{\infty} R\mathbf{P}(2^n R \leq X < 2^{n+1} R) < \sum_{n=0}^{\infty} 2^{-n} c < \infty. \qquad \square$$

PROPOSITION 4.4. *For an arbitrary non-negative left-continuous non-decreasing s.v. function f on $[0, \infty)$, there exists a distribution μ on $[0, \infty)$ and a constant B such that*

$$f(x) = \int_{[0,x)} t\mu(dt) \quad \textit{for all} \quad x \geq B.$$

Proof is straightforward.

Let us denote by $\mathbf{C}_0$ the class of distributions on $[0, \infty)$ with dominatedly non-decreasing truncated mean. Note that $\mathbf{C}_0$ is a subclass of $\mathbf{C}$.

THEOREM 4.5. *Let X and Y be non-negative random variables and $Z = X + Y$. Then the distribution of Z belongs to $\mathbf{C}_0$ if and only if both of the distributions of X and Y belong to $\mathbf{C}_0$.*

Proof. Assume that the distributions of X and Y belong to $\mathbf{C}_0$. Then, since

$$R\mathbf{P}(X + Y > R) \leq R\mathbf{P}(X > R/2) + R\mathbf{P}(Y > R/2)$$

and

$$\limsup_{R\to\infty} R\mathbf{P}(X + Y > R) \leq 2(\limsup_{R\to\infty} R\mathbf{P}(X > R) + \limsup_{R\to\infty} R\mathbf{P}(Y > R)),$$

the distribution of Z is in $\mathbf{C}_0$ by Proposition 4.3.

Conversely, assume that the distribution of Z is in $\mathbf{C}_0$. Note that

$$R\mathbf{P}(X > R) \leq R\mathbf{P}(X + Y > R)$$

and consider the upper limits of both sides. Then, by Proposition 4.3, the distribution of X belongs to $\mathbf{C}_0$, and similarly that of Y. □

The class $\mathbf{C}_0$ is strictly bigger than the class of distribution of which all factors belong to $\mathbf{C}$. The following example shows this fact. Define μ by $\mu(\{2^j\}) = cj2^{-j}$ for $j = 1,2\ldots,$ where $c = (\sum_{j=1}^{\infty} j2^{-j})^{-1}$. Then μ is indecomposable, since the support of μ is an indecomposable set. We can prove that both μ and $\mu * \mu$ are in $\mathbf{C} \setminus \mathbf{C}_0$. Also we can prove that if $\mu * \mu = \mu_1 * \mu_2$ with non-trivial μ_1 and μ_2, then both μ_1 and μ_2 are idenitcal with μ up to convolution with trivial distributions. Proof is essentially the same as the discussion of Examples 1 and 2 for $\mathbf{D}_2$ in [7].

The following theorem gives a relation between a distribution with s.v. truncated mean and its factors.

THEOREM 4.6. *Let X_i $(i = 1,\ldots,n)$ be non-negative random variables with truncated mean $M_i(R)$. The distribution of the sum $S = \sum_{i=1}^n X_i$ belongs to $\mathbf{C}$ if and only if $\sum_{i=1}^n M_i(R) \in \mathbf{SV}$. In this case*

$$\lim_{R\to\infty} \sum_{i=1}^n M_i(R)/M_S(R) = 1. \tag{4.9}$$

Proof. Let $W(R) = \mathbf{E}S1\,(\max_{1\leq i\leq n} X_i < R)$. Then

$$W(R/n) \leq M_S(R) \leq W(R).$$

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Hence, $M_S(R) \in \mathbf{SV}$ if and only if $W(R) \in \mathbf{SV}$. If $M_S(R)$ and $W(R)$ are in $\mathbf{SV}$ then they are asymptotically equal and (4.9) is equivalent to

$$\lim_{R\to\infty} W(R)/\sum_{i=1}^{n} M_i(R) = 1. \tag{4.10}$$

Let $I(R) = \sum_{i=1}^{n} \mathbf{E}X_i 1(X_i < R,\ \max_{j\neq i} X_j \geq R)/\sum_{i=1}^{n} M_i(R)$. Then,

$$W(R)/\sum_{i=1}^{n} M_i(R) = 1 - I(R).$$

Assume that $M_S(R) \in \mathbf{SV}$. We have

$$I(R) \leq R\sum_{i=1}^{n} \mathbf{P}(\max_{j\neq i} X_j \geq R)/\sum_{i=1}^{n} M_i(R) \leq nR\mathbf{P}(S \geq R)/M_S(R).$$

The slow variation of $M_S(R)$ implies that $\lim_{R\to\infty} R\mathbf{P}(S \geq R)/M_S(R) = 0$. Hence we get (4.10).

Conversely, assume that $\sum_{i=1}^{n} M_i(R) \in \mathbf{SV}$. We use the estimate

$$\begin{aligned} I(R) &\leq R\sum_{i=1}^{n} \mathbf{P}(\max_{j\neq i} X_j \geq R)/\sum_{i=1}^{n} M_i(R) \\ &\leq (n-1)R\sum_{i=1}^{n} \mathbf{P}(X_i \geq R)/\sum_{i=1}^{n} M_i(R). \end{aligned}$$

Let $X_1', \ldots, X_n'$ be independent random variables such that, for each i, X_i' has the same distribution as X_i. Let $S' = \sum_{i=1}^{n} X_i'$ and $W'(R) = \mathbf{E}S'1(\max_{1\leq i\leq n} X_i' < R)$. Then

$$W'(R) = \sum_{i=1}^{n} \mathbf{E}X_i'1(X_i' < R)\prod_{j\neq i} \mathbf{P}(X_j' < R) = \sum_{i=1}^{n} M_i(R)\prod_{j\neq i} \mathbf{P}(X_j < R).$$

Therefore $W'(R)$ is asymptotically equal to $\sum_{i=1}^{n} M_i(R)$. Hence $W'(R)$ and $M_{S'}(R)$ are in $\mathbf{SV}$. Now use

$$R\sum_{i=1}^{n} \mathbf{P}(X_i \geq R)/\sum_{i=1}^{n} M_i(R) \leq nR\mathbf{P}(S' \geq R)/M_{S'}(R).$$

The right-hand side tends to 0 as $R \to \infty$ because of the slow variation of $M_{S'}(R)$. Hence $I(R) \to 0$. Therefore we get (4.10) and see that $W(R) \in \mathbf{SV}$. This finishes the proof. $\square$

Remark. The class $\mathbf{C}$ is also closed under convolution. This fact is shown by the above theorem in a similar way to $\mathbf{D}(\alpha)$.

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Now we can prove the following theorem from Theorems 3.1 and 4.6 combined with Proposition 3.3 of [7].

THEOREM 4.7. *There exist distributions μ_1 and μ_2 such that neither of them belongs to* $\mathbf{C}$ *but* $\mu = \mu_1 * \mu_2$ *belongs to* $\mathbf{C}$. *In general, for each n, there exist distributions $\mu_1, \ldots, \mu_n$ such that $\mu = \mu_1 * \cdots * \mu_n$ belongs to* $\mathbf{C}$ *but, for every proper subset S of $\{1, \ldots, n\}$, the convolution of $\{\mu_i : i \in S\}$ does not belong to* $\mathbf{C}$.

Proof. By Proposition 4.4, we can choose a distribution μ in $\mathbf{C}$ such that $\limsup_{R\to\infty}(M(2R) - M(R)) = \infty$. By Theorem 3.1, there exist measures μ_1^0 and μ_2^0 on $(0, \infty)$ such that $M(R) = M_1(R) + M_2(R)$, where $M_i(R) \notin \mathbf{SV}$ and $M_i(R) = \int_{(0,R)} x\mu_i^0(dx)$ for $i = 1,2$ and $\mu(0, \infty) = \sum_{i=1}^2 \mu_i^0(0, \infty)$. We define probability measures μ_i $(i = 1, 2)$ on $[0, \infty)$ by $\mu_i = \mu_i^0 + \delta_i$, where δ_i is a measure concentrated at $\{0\}$ with mass $1 - \mu_i^0(0, \infty)$. Then the truncated mean of μ_i is equal to $M_i(R)$. We define a probability measure $\tilde{\mu}$ by $\tilde{\mu} = \mu_1 * \mu_2$. Then, by Theorem 4.6, $\tilde{\mu}$ belongs to $\mathbf{C}$, but neither μ_1 nor μ_2 belongs to $\mathbf{C}$. Similarly, Proposition 3.3 of [7] yields the latter half of the theorem. □

Remark. By using Theorems 3.2 and 4.6, we can construct two distributions μ_1 and μ_2 such that μ_1 and the convolution $\mu = \mu_1 * \mu_2$ belong to $\mathbf{C}$ and μ_2 does not belong to $\mathbf{C}$.

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REFERENCES

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, "Regular Variation," Encyclopedia of Math. and Its Appl., Cambridge University Press, Cambridge, 1987.

[2] D.A. Darling, The influence of the maximum term in the addition of independent random variables, Trans. Amer. Math. Soc., **73** (1952), 95–107.

[3] W. Feller, "An introduction to Probability Theory and Its Applications," Vol. II 2nd ed., Wiley, New York, 1971.

[4] B. V. Gnedenko and and A. N. Kolmogorov, "Limit Distributions for Sums of Independent Random Variables," (English translation) 2nd ed., Addison Wesley, Cambridge Mass., 1968.

[5] Ju. V. Linnik and I. V. Ostrovskiĭ, "Decomposition of Random Variables and Vectors," (English translation) Trans. Math. Monographs, **48**, Amer. Math. Soc., Providence, Rhode Island, 1977.

[6] E. Seneta, "Regularly Varying Functions," Lecture Notes in Mathematics. **508**, Springer-Verlag, Berlin, 1976.

[7] T. Shimura, Decomposition of non-decreasing slowly varying functions and the domain of attraction of Gaussian distributions, J. Math. Soc. Japan, **43, No. 4** (1991), 775–793.

[8] H. G. Tucker, Convolutions of distributions attracted to stable laws, Ann. Math. Statist., **39** (1968), 1381–1390.

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