A NOTE ON THE KAWADA-INTO THEOREM

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If μ is a bounded regular Borel measure on a locally compact group G, and $L_1(G)$ denotes the class of complex-valued functions which are integrable with respect to the left Haar measure m of G, then, for each $f \in L_1(G)$,

$$\mu * f(y) = \int_G f(x^{-1}y)d\mu(x)$$

defines almost everywhere (a.e.) with respect to m a function $\mu * f$ which is again in $L_1(G)$. The measure μ will be called isotone on G if the mapping $f \to \mu * f$ is isotone, i.e. $f \geqq 0$ a.e. (m) if and only if $\mu * f \geqq 0$ a.e. (m).

J. H. Williamson called the hypothesis that every isotone measure has one-point support the Kawada-into theorem. (The Kawada-onto theorem, in which the additional assumption that the mapping $f \to \mu * f$ is onto $L_1(G)$ is made, was proved by Y. Kawada in (**2**).) He then produced a difficult counter-example using a non-unimodular matrix group (**5**). This note shows that his tentative conjecture that the theorem holds if the group is unimodular is false; precisely, we use his work to construct a unimodular group on which there is an isotone measure whose support is not one-point.

Proposition 1. *Let H be a closed subgroup of a locally compact group G, and let μ be a measure with support in H. Then if μ is isotone on H, it is isotone on G.*

(I am grateful to Dr S. Swierczkowski for pointing out this stronger form of my original result.)

Proof. We will denote by K the locally compact space of cosets

$$\bar{x} = Hx \ (x \in G),$$

and n will be the left Haar measure on H. Then there exists on K a Borel measure ν with the property that $m(E) = 0$ if and only if, for almost all $\bar{x} \in K$ and any choice of $x \in \bar{x}$, $n(\{t : t \in H;\ tx \in E\}) = 0$. (This is proved for Baire sets E and right Haar measures in Theorems 1 and 3 of (**4**); it follows for left Haar measures since the sets of measure zero in the left and right Haar measures are the same, and we deduce the result for Borel sets because Haar measures are completion regular (**1**, page 288).)

Suppose $\mu * f \geqq 0$ a.e. (m). Then for almost all $\bar{y} \in K$, $\mu * f(ty) \geqq 0$ for almost all $t \in H$, where y is chosen in $\bar{y}$. Now

$$\mu * f(ty) = \int_G f(x^{-1}ty)d\mu(x) = \int_H f(x^{-1}ty)d\mu(x),$$

and so, since μ is isotone on H and the function $t \rightarrow f(ty)$ is in $L_1(H)$ for almost all $\bar{y}$, for almost all $\bar{y}$, $f(ty) \geqq 0$ for almost all t, i.e. $f \geqq 0$ a.e. (m).

Since it is obvious that if μ is positive on H it is positive on G, the proof is finished.

Proposition 2. *There is a unimodular group on which the Kawada-into theorem does not hold.*

Proof. Let H be the group Williamson considered in (5), and take G to be the unimodular semi-direct product of the real line by H (3, page 120). Then Williamson's isotone measure on H is again isotone on G, and does not have one-point support.

We close with another equally tentative conjecture made by Williamson. His construction in (5) is apparently valid in any metrisable non-unimodular group, and so it follows from Proposition 1 that for the Kawada-into theorem to hold on a metrisable group G, it is necessary that every closed subgroup of G should be unimodular. Is this condition necessary and sufficient in general?

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