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# The log log law for multidimensional stochastic integrals and diffusion processes

Ludwig Arnold

Let for $t \in [a, b] \subset [0, \infty)$

$$X_t = c + \int_a^t f(s)ds + \int_a^t G(s)dW_s ,$$

where $W_s$ is an $n$-dimensional Wiener process, $f(s)$ an $n$-vector process and $G(s)$ an $n \times m$ matrix process. $f$ and $G$ are nonanticipating and sample continuous. Then the set of limit points of the net

$$\left\{\frac{X_{t+h}-X_t}{(2h \log \log 1/h)^{1/2}}\right\}_{e^{-1}>h\downarrow 0}$$

in $R^n$ is equal, almost surely, to the random ellipsoid $E_t = G(t)S_m$ , $S_m = \{x \in R^m : |x| \le 1\}$ . The analogue of Lévy's law is also given. The results apply to $n$-dimensional diffusion processes which are solutions of stochastic differential equations, thus extending the versions of Hinčin's and Lévy's laws proved by H.P. McKean, Jr, and W.J. Anderson.

## 1. Introduction

The local behavior of the paths of an $m$-dimensional standard Wiener

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process $W_t$ is described by Hinčin's local log log law and Lévy's law concerning the modulus of continuity. The log log law states that for fixed $t \geq 0$ the set of limit points of the net

$$\left\{\frac{W_{t+h}-W_t}{(2h \log \log 1/h)^{1/2}}\right\}_{e^{-1}>h\downarrow 0}$$

is equal, almost surely, to the $m$-dimensional closed unit sphere $S_m$, while Lévy's law states that the limit points of

$$\left\{\frac{W_{t+h}-W_t}{(2h \log 1/h)^{1/2}}\right\}_{1>h\downarrow 0,\ a\leq t\leq b-h}, \quad [a, b] \subset [0, \infty),$$

coincide, almost surely, with $S_m$. In the usual formulation of these laws, only the limit point of maximum modulus 1 is mentioned.

A random function on $[a, b] \subset [0, \infty)$ is called *nonanticipating* if its value at $t$ is measurable with respect to a σ-algebra $F_t$, where $F_t$ contains all the events generated by $\{W_s, s \leq t\}$, is independent of $\{W_{s+t}-W_t, s \geq 0\}$, and satisfies $F_{t_1} \subset F_{t_2}$ $(t_1 < t_2)$. Let $f(t)$ be an $n$-vector process and $G(t)$ an $n \times m$ matrix process, $f$ and $G$ being nonanticipating and sample continuous. If $c$ is an $n$-vector random variable measurable with respect to $F_a$, then the process

$$(1) \qquad X_t = c + \int_a^t f(s)ds + \int_a^t G(s)dW_s, \quad t \in [a, b],$$

(where the first integral is an ordinary Riemann integral and the second one is a stochastic (Itô) integral) is well-defined, nonanticipating, and sample continuous. We are looking for the analogues of Hinčin's and Lévy's laws for the paths of $X_t$.

## 2. The main result

THEOREM. *Let* $X_t$ *be the* $n$*-vector process defined by* (1). *Then* *(i) for fixed* $t \in [a, b)$ *, the set of limit points of the net*

$$\left\{\frac{X_{t+h}-X_t}{(2h \log \log 1/h)^{1/2}}\right\}_{e^{-1}>h\downarrow 0}$$

*is equal, almost surely, to the random closed ellipsoid*

$$E_t = G(t)S_m = \left\{x \in R^n : x = G(t)y,\ y \in S_m\right\} ;$$

*(ii) the set of limit points of*

$$\left\{\frac{X_{t+h}-X_t}{(2h \log 1/h)^{1/2}}\right\}_{1>h\downarrow 0,\ a\le t\le b-h}$$

*is equal, almost surely, to the random closed set* $\bigcup_{a\le t\le b} E_t$ .

Proof. *(i)* Since $c + \int_a^t f(s)ds$ is continuously differentiable, it is enough to prove the result for the stochastic integral

$$Y_t = \int_a^t G(s)dW_s .$$

Write for fixed $t \in [a, b)$

$$Y_{t+h} - Y_t = G(t)\left(W_{t+h}-W_t\right) + \int_t^{t+h} \left(G(s)-G(t)\right)dW_s . \tag{2}$$

The limit points of

$$\left\{\frac{G(t)\left(W_{t+h}-W_t\right)}{(2h \log \log 1/h)^{1/2}}\right\}_{h\downarrow 0}$$

are clearly, almost surely, equal to $G(t)S_m$ . The proof will be completed once we know that

$$\lim_{h\downarrow 0} \frac{\int_t^{t+h} \left(G_{ij}(s)-G_{ij}(t)\right)dW_s^j}{(2h \log \log 1/h)^{1/2}} = 0 \text{ almost surely, } 1 \le i \le n,\ 1 \le j \le m .$$

There exists a new scalar Wiener process $\overline{W}_t^{ij}$ so that

$$\int_t^{t+h} \big(G_{ij}(s)-G_{ij}(t)\big)dW_s^j = \overline{W}^{ij}_{\tau(h)}$$

with

$$\tau(h) = \int_t^{t+h} \big(G_{ij}(s)-G_{ij}(t)\big)^2 ds ,$$

(see McKean [3], p. 29). Since by the scalar log log law

$$(3) \qquad \overline{\lim_{h\downarrow 0}} \frac{|\overline{W}^{ij}_{\tau(h)}|}{\big(2\tau(h)\log\log 1/\tau(h)\big)^{1/2}} = 1 \text{ almost surely,}$$

and

$$\lim_{h\downarrow 0} \left\{\frac{2\tau(h)\log\log 1/\tau(h)}{2h\log\log 1/h}\right\}^{1/2} = \sqrt{\tau'(0)} = 0 \text{ almost surely,}$$

(see Anderson [1], Lemma 3.1.1), the result follows.

*(ii)* We proceed as in *(i)*. Lévy's law for $W_t$ holds in any small interval, thus giving us as possible limit points of

$$\left\{\frac{G(t)\big(W_{t+h}-W_t\big)}{(2h\log 1/h)^{1/2}}\right\}_{h\downarrow 0,\ a\le t\le b-n}$$

the set $\bigcup_{a\le t\le b} E_t$ . The remainder in (2) is treated as in *(i)* using Lévy's law for scalar processes instead of (3). Q.E.D.

We remark that the ellipsoid $E_t$ can also be written as

$$E_t = U(t)\sqrt{\Lambda(t)}\, S_n ,$$

where $GG^T = U\Lambda U^T$ , $U$ being the $n\times n$ eigenvector matrix and $\Lambda$ the $n\times n$ diagonal matrix of the eigenvalues $\lambda_i$ of $GG^T$ . Thus, $E_t$ is the ellipsoid whose main axes have length $\sqrt{\lambda_i}$ and point into the direction of the eigenvectors of $GG^T$ .

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The theorem contains numerous special cases, in particular the statements

$$\overline{\lim_{h\downarrow 0}} \frac{|X_{t+h}-X_t|}{(2h \log \log 1/h)^{1/2}} = \max_i \sqrt{\lambda_i(t)}$$

and

$$\overline{\lim_{\substack{h\downarrow 0\\ a\le t\le b-h}}} \frac{|X_{t+h}-X_t|}{(2h \log 1/h)^{1/2}} = \max_{a\le t\le b} \max_i \sqrt{\lambda_i(t)}$$

proved for a particular case by McKean ([3], p. 96), and a result of Anderson [1] for the components $X_t^i$,

$$\overline{\lim_{h\downarrow 0}} \frac{X_{t+h}^i-X_t^i}{(2h \log \log 1/h)^{1/2}} = \sqrt{\left(G(t)G(t)^T\right)_{ii}} \ .$$

## 3. Application to multidimensional diffusions

If the vector stochastic differential equation

$$dX_t = f(t, X_t)dt + G(t, X_t)dW_t \ , \quad X_a = c \ ,$$

has continuous coefficients satisfying the usual Lipschitz condition and growth restriction, the solution is unique and constitutes a vector diffusion process $X_t$ (see Gikhman-Skorokhod [2], p. 402). Our theorem applies to $X_t$, where the random ellipsoid $E_t = G(t, X_t)S_m$ now depends only on $t$ and the state $X_t$.

## References

[1] William James Anderson, "Local behaviour of solutions of stochastic integral equations", (Ph.D. thesis, McGill University, Montreal, 1969).

[2] I.I. Gikhman, A.V. Skorokhod, *Introduction to the theory of random processes* (Sounders, Philadelphia, London, Toronto, 1969).

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Centre de recherches mathématiques,
Université de Montréal,
Montréal,
Canada.