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The degree one Laguerre–Pólya class and the shuffle-word-embedding conjecture

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Abstract. We discuss the class of functions, which are well approximated on compacta by the geometric mean of the eigenvalues of a unital (completely) positive map into a matrix algebra or more generally a type II_1 factor, using the notion of a Fuglede–Kadison determinant. In two variables, the two classes are the same, but in three or more noncommuting variables, there are generally functions arising from type II_1 von Neumann algebras, due to the recently established failure of the Connes embedding conjecture. The question of whether or not approximability holds for scalar inputs is shown to be equivalent to a restricted form of the Connes embedding conjecture, the so-called shuffle-word-embedding conjecture.

1 Introduction

The *Laguerre–Pólya class* is the space of analytic functions, which are the uniform limit of real-rooted polynomials on compacta in $\mathbb{C}$. Such functions have a Hadamard factorization of the form

$$e^{a+bz-cz^2} z^k \prod \left(1 - \frac{z}{\rho_n}\right) e^{z/\rho_n},$$

where b is real, c is nonnegative, and the at most countably many nonzero roots ρ_n satisfy $\sum \frac{1}{|\rho_n|^2} < \infty$ (see, e.g., [14]).

Note that any real-rooted polynomial with positive leading coefficient can be written in the form $p(x) = c\det(A + xI)$, $c > 0$, where A is a self-adjoint matrix. For convenience, we normalize so that $c = 1$. If A is positive semi-definite, we would know that the zeros are on the negative real axis. Note that

$$p(x) = \det(A + xI) = e^{\operatorname{tr}\log(A+xI)}.$$

It is natural to consider the homogenized version of the class as the corresponding approximation theory provides more tools to work with. A natural homogeneous version of a determinantal representation is given by

$$p(x, y) = \det(Ax + By) = e^{\operatorname{tr}\log(Ax+By)},$$

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where A, B are positive semi-definite and sum to the identity. Note that such a function is naturally evaluated at matrix inputs by replacing the product with appropriate tensor products.

In [11], the author gave a natural log-convexification of the Laguerre–Pólya class by taking the radical of the Laguerre–Pólya class, by allowing limits of nth roots of functions in the Laguerre–Pólya class. Indeed, by taking radicals, the author in [11] showed that logarithms of the functions in this class are exactly (up to a minus sign) the ones with a so-called trace minimax property. These functions are interesting as they form a class of functions satisfying a trace inequality and they are automatically analytic. Taking the analogy with the theory of tracial von Neumann algebras, one naturally wants to consider the normalized trace instead of the trace, which via the tracial phrasing of our determinantal representation, we see corresponds to taking the geometric mean of the eigenvalues. This leads us to define the following class of functions. Define the *degree one Laguerre–Pólya class in two variables* to be the space of analytic functions which are uniform limits on compacta (i.e., limits in the compact-open topology) in $\{(x, y) \in \mathbb{C}^2 : \operatorname{Re} x > 0, \operatorname{Re} y > 0\}$ of functions of the form $e^{\frac{1}{n}\operatorname{tr}\log(Ax+By)}$, where A and B are positive semidefinite matrices summing to the identity of size n for some n. Note that if f is in this class, then $f(rx, ry) = rf(x, y)$ for $r > 0$, which inspired the qualification "degree one." Any limit in the compact-open topology of degree one Laguerre–Pólya class functions are of the form $e^{\tau\log(Ax+By)}$, where A, B are positive semidefinite elements of some type II_1 von Neumann algebra satisfying $A + B = I$, and τ is a normalized trace. We call this potentially larger class the *apparent degree one Laguerre–Pólya class in two variables*.

We have the following proposition.

Proposition 1.1 *The apparent degree one Laguerre–Pólya class in two variables equals the degree one Laguerre–Pólya class in two variables.*

Proof Let A and $I - A$ be positive semidefinite elements of a II_1 von Neumann algebra, and let τ be a normalized trace. The W$*$-algebra (or von Neumann algebra) generated by $A = A^*$ can be represented as $L^\infty([0,1], \mu)$, where μ is a σ-finite probability measure on $[0,1]$ by the spectral theorem (see [3, Chapter 7]). The space $L^\infty([0,1], \mu)$ forms an algebra of operators on the Hilbert space $L^2([0,1], \mu)$, where each $f \in L^\infty([0,1], \mu)$ is identified with the multiplication operator $\psi \mapsto f\psi$ with trace $\tau(f) = \int_{[0,1]} f\,d\mu$. Taking M_n, the finite-dimensional subspace of $L^2([0,1], \mu)$ spanned by $1, x, \ldots, x^n$, the eigenvalues of the restriction of multiplication by x (which represents A) to the subspace M_n has eigenvalues equal to the zeros of the nth orthogonal polynomial with respect to the measure μ. These spectral measures μ_n representing the action of multiplication by x restricted to M_n converge weakly to the measure μ (see [15, Section 6.2]). This yields that the degree one Laguerre–Pólya class exhausts the apparent degree one Laguerre–Pólya class in two variables. ∎

Our goal is to establish the failure of such a correspondence in several variables for matrix inputs and in the scalar case give an equivalence with a relaxation of the Connes embedding problem to finite tracial approximation of the BMV polynomials, which we call the *shuffle-word-embedding conjecture*.

2 The Laguerre–Pólya class in several noncommuting variables

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and consider the nc domain

$$\mathcal{D} = \cup_{k\geq 1}\mathcal{D}_k,\ \mathcal{D}_k = \{(X_1, \dots, X_n) \in (\mathbb{K}^{k\times k})^n : \sum_{j=1}^{n} \|X_j\| < 1\}.$$

We let $\mathcal{F}$ be a real or complex von Neumann algebra with a faithful tracial state τ. On $d \times d$ matrices over $\mathbb{K}$ we let Tr denote the normalized trace; i.e., $\mathrm{Tr}[(m_{i,j})_{i,j=1}^{d}] = \frac{1}{d}\sum_{i=1}^{d} m_{ii}$. On the (spatial) tensor product $\mathcal{F}\bar{\otimes}\mathbb{K}^{d\times d}$, we introduce the product trace, which with a slight abuse of notation, will also be denoted by τ. In particular, $\tau(A \otimes M) = \tau(A)\mathrm{Tr}(M)$. Let now $A_1, \dots, A_n \in \mathcal{F}$ be Hermitian (complex case) or symmetric (real case) contractions, and define the function $f_{(A_1,\dots,A_n)} = f_A : \mathcal{D} \to \mathbb{K}$ via

$$f_A(X) = f_A(X_1, \dots, X_n) = e^{\tau(\log(I-\sum_{j=1}^{n} A_j\otimes X_j))},$$

where the logarithm may be defined via $\log(I - M) = -\sum_{m=1}^{\infty} \frac{M^m}{m}$, $\|M\| < 1$.

It should be noted that the class introduced here is an affinized version of the Laguerre–Pólya class introduced in the introduction.

We have the following basic properties, that are also satisfied by the Fuglede–Kadison determinant.

Lemma 2.1 (i) *For $X \in \mathcal{D}_k$ and $Y \in \mathcal{D}_l$, we have*

$$f_A(X)^k f_A(Y)^l = f_A(X \oplus Y)^{k+l}.$$

(ii) *For $U \in \mathbb{K}^{k\times k}$ unitary and $X \in \mathcal{D}_k$, we have $f_A(UXU^*) = f_A(X)$.*

Proof Let tr denote the usual (not normalized) trace of a matrix. Note that for a $k \times k$ matrix M, a $l \times l$ matrix N and $B \in \mathcal{F}$, we have that

$$(k + l)\tau(B \otimes (M \oplus N)) = (k + l)\tau(B)\mathrm{Tr}(M \oplus N) = \tau(B)\mathrm{tr}(M \oplus N) =$$

$$\tau(B)\mathrm{tr}(M) + \tau(B)\mathrm{tr}(N) = k\tau(B)\mathrm{Tr}(M) + l\tau(B)\mathrm{Tr}(N).$$

The lemma follows now easily. ∎

2.1 Functions in this class encode tracial moments

Two words w and $\hat{w}$ are called *cyclically equivalent* if there exist words w_1 and w_2 so that $w = w_1w_2$ and $\hat{w} = w_2w_1$. We have the following lemma, which can also be deduced from [10, Theorem 3.1: Spurnullstellensatz].

Lemma 2.2 *Consider a word $w(X_1, \dots, X_n)$ of length m in n letters $X_1, \dots, X_n$. then there exist $m \times m$ matrices $T_1, \dots, T_n$ so that*

$$\mathrm{Tr}\, w(T_1, \dots, T_n) > 0$$

and

$$\operatorname{Tr} \tilde{w}(T_1, \ldots, T_n) = 0$$

for any word $\tilde{w}$ of length at most m that is not cyclically equivalent to w.

Proof Let the word w equal $w(X_1, \ldots, X_n) = X_{i_m} \ldots X_{i_1}$, where $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Put $a_0 = \varnothing$, $a_k = X_{i_k} \ldots X_{i_1}$, $k = 1, \ldots, m$. Let now $T_j = (t_{rs}^{(j)})_{r,s=0}^{m-1}$, $j = 1, \ldots, n$, be defined via

$$t_{rs}^{(j)} = \begin{cases} 1, & \text{if} \quad X_j a_s = a_r, \\ 1, & \text{if} \quad r = 0, s = m-1, X_j a_{m-1} = a_m, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that T_j only has nonzero entries in the diagonal $\{(i,j) : j - i = m - 1 (\text{mod } m)\}$, so that the product of k matrices T_j only has nonzero entries in the diagonal $\{(i,j) : j - i = m - k (\text{mod } m)\}$. Thus for any word $\tilde{w}$ of length less than m, we have that $\tilde{w}(T_1, \ldots, T_n)$ has zeros on the main diagonal, and thus its trace equals 0. When $\tilde{w}$ has length m it is not hard to see that $\tilde{w}(T_1, \ldots, T_n)$ has 1's on the diagonal if and only if $\tilde{w}$ is cyclically equivalent to w, and otherwise there will only be zeros on the diagonal. It may be helpful to think of the operators $T_1, \ldots, T_n$ also in the following way.

Let us denote the words that are cyclically equivalent to w as follows: $w_0 = w = a_0 w, w_1 = X_{i_1} X_{i_m} \ldots X_{i_2} =: a_1 b_1, \ldots, w_{m-1} = X_{i_{m-1}} \ldots X_{i_1} X_{i_m} =: a_{m-1} b_{m-1}$. In this setting, we have that $t_{rs}^{(j)} = 1$ if and only if $r - s = 1 (\text{mod } m)$, X_j is the last letter of w_s and w_r is obtained from w_s by moving X_j to the front. In this interpretation, the product $T_j T_k$ has a 1 in position (r,s) if and only if $r - s = 2 (\text{mod } m)$, and w_r is obtained from w_s by moving X_k to the front and subsequently X_j to the front. Similarly, any product of T_j's can be interpreted. ■

As an example, consider the word $w = X_1 X_2 X_1 X_2$. Then the matrices T_1 and T_2 in the proof of Lemma 2.2 are

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now

$$T_1 T_2 T_1 T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We now describe our fundamental lemma which relates the equality of tracial moments to the equality of our associated functions.

Lemma 2.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be complex von Neumann algebras with faithful tracial states τ and $\hat{\tau}$, respectively. Let $A_1, \ldots, A_n \in \mathcal{F}$ and $B_1, \ldots, B_n \in \mathcal{G}$ be Hermitian contractions.*

Then $f_A(X) = f_B(X)$, *for all* $X \in \mathcal{D}$, *if and only if* $\tau(p(A)) = \hat{\tau}(p(B))$ *for every noncommutative polynomial* p.

The same result also holds when $\mathcal{F}$ *and* $\mathcal{G}$ *are real von Neumann algebras and* A_i *and* B_i *are real symmetric contractions.*

Proof Let us write $M(X) = \sum_{i=1}^n A_i \otimes X_i$, where $X \in \mathcal{D}$. We now have that

$$f_A(X) = e^{-\tau(\sum_{m=1}^\infty \frac{M(X)^m}{m})} = 1 - \tau(\sum_{m=1}^\infty \frac{M(X)^m}{m}) + \frac{1}{2!}(\tau(\sum_{m=1}^\infty \frac{M(X)^m}{m}))^2 - \dots.$$

Extracting the degree k term, we obtain

$$-\tau(\frac{M(X)^k}{k}) + \frac{1}{2!}\sum_{j=1}^k \binom{k}{j}\tau(\frac{M(X)^j}{j})\tau(\frac{M(X)^{k-j}}{k-j}) -$$
$$\frac{1}{3!}\sum_{j_1+j_2+j_3=k, j_i\geq 1}\binom{k}{j_1, j_2, j_3}\tau(\frac{M(X)^{j_1}}{j_1})\tau(\frac{M(X)^{j_2}}{j_2})\tau(\frac{M(X)^{j_3}}{j_3}) + \dots.$$

Let now $U = (u_{ij})_{i,j=1}^k$ be the $k \times k$ unitary circulant with $u_{ij} = 1$ when $j = i + 1 \bmod k$, and 0 elsewhere. Then Tr U^j equals 1 when k divides j, and 0 otherwise. If we now let $x_i = U \otimes X_i$, $i = 1, \dots n$, we have that $x = (x_1, \dots, x_n) \in \mathcal{D}$, and the degree k term in $f_A(x)$ now equals $-\tau(\frac{M(X)^k}{k})$. Indeed, whenever $1 \leq j < k$, we have that $\tau(\frac{M(X)^j}{j}) = 0$ since Tr $U^j = 0$. Next, notice that $M(X)^k = \sum_w w(A_1, \dots, A_n) \otimes w(X_1, \dots, X_n)$, where the sum is taken over all words of length k. Since the monomials are linearly independent by Lemma 2.2, the result follows. ∎

2.2 Due to the Connes embedding problem, the apparent class is bigger

The Klep–Schweighofer formulation of the now refuted *Connes embedding conjecture* states that for any tuple of contractive self-adjoint operators $A_1, \dots, A_n$ in a type II_1 von Neumann algebra with trace τ and any matrix of polynomials m, for any $\varepsilon > 0$, there are matrices $B_1, \dots, B_n$ such that

$$|\tau(m_{ij}(A_1, \dots, A_n)) - \mathrm{Tr}(m_{ij}(B_1, \dots, B_n))| < \varepsilon.$$

Here, Tr denotes the normalized trace. The Connes embedding conjecture fails [8]. We use the above formulation in showing next that the apparent class is larger.

Proposition 2.1 *There exists an* $\delta > 0$ *and a choice of* f_A *so that for any* f_B, *where* B *is a tuple of matrices, we have that* $\|\mathrm{coeff}(f_A - f_B)\|_\infty \geq \delta$. *That is, there exists an* n *so that the apparent degree one Laguerre–Pólya class in* n *variables is strictly larger than the degree one Laguerre–Pólya class in* n *variables.*

Proof Due to the failure of the Connes embedding conjecture [8], it follows from [9, Proposition 3.17] (complex case) and [2, Proposition 3.2] (real case) that there exists a II_1 factor $\mathcal{F}$ with a faithful tracial state τ, an $\varepsilon > 0$, $n, k \in \mathbb{N}$, and self-adjoint/symmetric contractions $A_1, \dots, A_n$ so that for all $s \in \mathbb{N}$ and self-adjoint/symmetric contractions $B_1, \dots, B_n \in \mathbb{K}^{s\times s}$ there exists a word w_0 in n letters of length k so that

$$|\tau(w_0(A_1, \dots, A_n)) - \mathrm{Tr}(w_0(B_1, \dots, B_n))| \geq \varepsilon.$$

So by Lemma 2.3, we are done. ∎

3 The scalar/commutative case

3.1 BMV polynomials, and the shuffle-word-embedding conjecture

The BMV polynomials [1] are the coefficients of t^ω in $(t_1A_1 + \cdots + t_nA_n)^k$, where A_i are our noncommuting indeterminants and t_i are commuting with everything. For example, for $k = n = 2$, we see $A^2, AB + BA, B^2$. Stahl [13] showed that such polynomials in two variables always have positive trace on tuples of positive semidefinite matrices. We introduce now the following conjecture.

The shuffle-word-embedding conjecture: For any tuple of contractive self-adjoint operators $A_1, \dots, A_k$ in a type II_1 von Neumann algebra with trace τ and any matrix of BMV polynomials m, for any $\varepsilon > 0$ there are matrices $B_1, \dots, B_n$ such that

$$|\tau(m_{ij}(A_1, \dots, A_n)) - \mathrm{Tr}(m_{ij}(B_1, \dots, B_n))| < \varepsilon.$$

Note that the shuffle-word-embedding conjecture significantly strengthens Stahl's result in these sense that it would imply that it held for inputs in tracial von Neumann algebras. (Eremenko's version [4] of Stahl's proof [13] kind of relies on finitary aspects of complex analysis, and generalization via a proof along those lines appears intractable.)

The shuffle-word-embedding conjecture would imply that Stahl's theorem holds for every II_1 factor, that is, for every BMV polynomial p, $\tau(p(A)) \geq 0$ if $A_1, \dots, A_n$ is a tuple of positive semidefinite operators in a II_1 factor.

Proposition 3.1 *The apparent degree one Laguerre–Pólya is the closure of the degree one Laguerre–Pólya class in the compact-open topology if and only if the shuffle-word-embedding conjecture is true.*

Proof We will show that $f_A(x_1, \dots, x_n) = f_B(x_1, \dots, x_n)$, for all scalars x_j with $\sum_{i=1}^n |x_j| < 1$, if and only if $\tau(p(A)) = \tau(p(B))$ for every BMV polynomial p, and thus we would be done. Note that instead of scalars x_j, we may also use simultaneously diagonalizable x_j's.

Proceeding in a similar way as in the proof of Lemma 2.3, let $U = (u_{ij})_{i,j=1}^k$ be the $k \times k$ unitary circulant with $u_{ij} = 1$ when $j = i + 1 \bmod k$, and 0 elsewhere. Then $\mathrm{Tr}\ U^j$ equals 1 when k divides j, and 0 otherwise. If we now let $X_i = x_iU$, $i = 1, \dots n$, we have that $X = (X_1, \dots, X_n) \in \mathcal{D}$ whenever $\sum_{i=1}^n |x_j| < 1$, and the degree k term in $f_A(X)$ now equals $-\tau(\frac{M(X)^k}{k})$. Indeed, whenever $1 \leq j < k$ we have that $\tau(\frac{M(X)^j}{j}) = 0$ since $\mathrm{Tr}\ U^j = 0$. Next, notice that $M(X)^k = \sum_w w(A_1, \dots, A_n) \otimes w(X_1, \dots, X_n)$, where the sum is taken over all words of length k. On scalar inputs, this reduces to $(t_1A_1 + \cdots + t_nA_n)^k$, so we are done by linear independence of monomials. This linear independence can be established in a similar way as in Lemma 2.2, where now we choose C_k the $k \times k$ circular shift, and we take operators T of the form $I \otimes \cdots \otimes I \otimes C_k \otimes I \otimes \cdots \otimes I$. ■

4 Summary and the homogenized version

A homogeneous real-stable polynomial with no definite roots in three variables is of the form

$$p(x, y, z) = \det(Ax + By + Cz) = e^{\operatorname{tr}\log(Ax+By+Cz)},$$

where A, B, C are positive semi-definite and $A + B + C$ is the identity. Thus, in the one or two variable degree one Laguerre–Pólya class, we can view the Laguerre–Pólya class as the product of the eigenvalues of the image of some matrix valued unital positive map on $\mathbb{C}^{d\times d}$. (See Hanselka's treatment [6] of the Helton–Vinnikov theorem [7] for the most optimal version and [12] and [5] for elementary approaches of determinantal representations of homogeneous hyperbolic polynomials in three variables/real zero polynomials in two variables.) Replacing the product with the geometric mean corresponds to replacing the trace with a normalized trace, which we will generally do, as it will convexify certain problems.

Let Ω be a class of tuples of positive definite operators in various tracial von Neumann algebras. We define the *degree one Laguerre–Pólya class on* Ω to be the closure of maps of the form $e^{\tau(\log L(x))}$, where L is some matrix-valued completely positive map and τ denotes the normalized trace with respect to the topology on compact subsets (in the norm topology) of Ω. Call the *apparent degree one Laguerre–Pólya class on* Ω the set of maps of the form $e^{\tau(\log L(x))}$, where L is a unital completely positive map into some II_1 factor. Weak compactness of unital completely positive maps gives that such a class is a normal family.

Let $\mathbb{P}^d$ denote the class of all d-tuples of positive operators from a tracial von Neumann algebra, running through all von Neumann algebras. Approximation theory fails there, essentially due to the failure of Connes embedding conjecture.

Theorem 4.1 *The degree one Laguerre–Pólya class on $\mathbb{P}^d$ is strictly smaller than the apparent one when d is large enough.*

Approximation theory will be related to our shuffle-word-embedding conjecture.

Theorem 4.2 *The degree one Laguerre–Pólya class on $(\mathbb{R}^+)^d$ equals apparent one if and only if the shuffle-word-embedding conjecture is true.*

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