Proceedings of the Edinburgh Mathematical Society (2000) **43**, 243–260 ©

ON δ-LIE SUPERTRIPLE SYSTEMS ASSOCIATED WITH (ε, δ)-FREUDENTHAL–KANTOR SUPERTRIPLE SYSTEMS

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(Received 11 November 1997)

Abstract We will present an investigation of (ε, δ)-Freudenthal–Kantor supertriple systems that are intimately related to Lie supertriple systems and Lie superalgebras. We can also introduce a super analogue of Nijenhuis tensor and almost-complex structure in differential geometry.

Keywords: Freudenthal–Kantor supertriple system; Lie supertriple system; Lie superalgebra

AMS 1991 *Mathematics subject classification:* Primary 17A40; 17B60

1. Introduction

It is well known that triple systems are intimately related to Lie algebras and Jordan algebras, for example, by

$$[x, y, z] := [[x, y], z] \quad \text{and} \quad \langle xyz \rangle = x(yz) - y(xz) + (xy)z,$$

respectively, for Lie product $[x, y]$ and Jordan product xy. These systems are actually special cases of more general δ-Lie triple systems and Freudenthal–Kantor triple systems defined by certain identities among triple products (cf. [**8, 12, 13, 21**]).

Conversely, simple Freudenthal–Kantor triple systems have been used to construct all simple Lie algebras together with the concept of root systems and Cartan matrix (cf. [**8, 11, 12, 14**]). Also, some classes of Jordan algebra can be similarly obtained from another type of triple system called Jordan–Lie triple systems [**21**].

Furthermore, these triple-product systems have been successfully applied, in many branches of mathematical physics, to solutions of the Yang–Baxter equation, para-statistics and nonlinear Schrödinger equations (cf. [**18–20, 24**]).

In this note, we discuss super-generalization of these triple systems in some detail. Especially, we will show in Theorem 4.2 and Proposition 4.5 a δ-Lie supertriple system can always be constructed from any (ε, δ)-Freudenthal–Kantor supertriple system ($\varepsilon, \delta = 1$ or -1). Further, the construction in Proposition 4.5 is intimately related to the existence of a Nijenhuis tensor in an algebraic context (see works of Leites and Rothstein and co-workers [**1**, **22**]).

We recall the fact that each bounded symmetric domain in a differential manifold can be obtained from a certain Jordan triple system and vice versa. Moreover, it is also well known that the tangent space of a symmetric homogeneous manifold corresponds to a Lie triple system in algebraic notation [**6**]. Since the Nijenhuis tensor is related to complex structure in differential geometry, these facts suggest that supertriple systems may play some role in differentiable super manifolds, although the possible connection has yet to be studied in detail.

We organize the present note as follows.

Let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal–Kantor supertriple system and $T(\delta)$ be the δ-Lie supertriple system associated with $U(\varepsilon,\delta)$. Then the contents in this article are described as follows.

(i) Several examples of (ε,δ)-Freudenthal–Kantor supertriple systems.

(ii) The correspondence of $U(\varepsilon,\delta)$ and $T(\delta)$.

(iii) Nijenhuis tensor condition for complex superstructure of $T(\delta)$.

(iv) Examples of simple Lie superalgebras associated with our triple systems.

We shall mainly employ the notation and terminology in [**8**, **19**, **21**].

2. Examples of (ε,δ)-Freudenthal–Kantor supertriple systems

To make this article as self-contained as possible, we shall first introduce a notion of (ε,δ)-Freudenthal–Kantor supertriple systems [**8**–**12**] that generalizes many earlier works [**2**,**3**,**5**,**14**,**25**] on the subject. Hereafter, we are concerned with algebras and triple systems that are finite dimensional over a field Φ of characteristic $\neq 2$.

We shall denote the degree (or grade or signature) for Z_2-graded vector space $V = V_0 \oplus V_1$ by

$$\deg x = \begin{cases} 0, & \text{if } x \in V_0, \\ 1, & \text{if } x \in V_1. \end{cases}$$

From now on, we often write, for simplicity,

$$(-1)^{xy} = (-1)^{\deg x \deg y},$$
$$(-1)^{x_i x_j} = (-1)^{ij}, \quad \text{if } x_i \in V_i,\ x_j \in V_j.$$

Definition 2.1. For $\varepsilon = \pm 1$, $\delta = \pm 1$, a vector space $U(\varepsilon,\delta) = \Sigma_i \oplus U_i(\varepsilon,\delta)$ over a field Φ with a triple product $\langle -,-,-\rangle$ is said to be a (ε,δ)-Freudenthal–Kantor supertriple system if

$$\langle a_i b_j c_k \rangle \in U_{i+j+k}(\varepsilon,\delta) \tag{U0}$$

$$[L(a_i,b_j), L(c_k,d_l)] = L(\langle a_i b_j c_k\rangle, d_l) + \varepsilon(-1)^{(i+j)k+ij} L(c_k, \langle b_j a_i d_l\rangle), \tag{U1}$$

$$K(\langle a_i b_j c_k\rangle, d_l) + (-1)^{(i+j)k} K(c_k, \langle a_i b_j d_l\rangle) + \delta(-1)^{j(k+l)} K(a_i, K(c_k,d_l) b_j) = 0, \tag{K1}$$

https://doi.org/10.1017/S0013091500020903 Published online by Cambridge University Press

where $L(a_i, b_j)c_k = \langle a_i b_j c_k\rangle$ and

$$K(a_i, b_j)c_k = (-1)^{jk}\langle a_i c_k b_j\rangle - \delta(-1)^{i(j+k)}\langle b_j c_k a_i\rangle,$$

for $a_i \in U_i(\varepsilon,\delta)$, $b_j \in U_j(\varepsilon,\delta)$, $c_k \in U_k(\varepsilon,\delta)$, $[E,F] := EF - (-1)^{\deg E \deg F} FE$.

That is, a (ε,δ)-Freudenthal–Kantor supertriple system is a Z_2-graded vector space together with bilinear maps $L(a_i, b_j)$ and $K(a_i, b_j) : U(\varepsilon,\delta) \times U(\varepsilon,\delta) \to \operatorname{End} U(\varepsilon,\delta)$, such that (U0), (U1) and (K1) are satisfied.

Remark 2.2 (see [10]). We note that the following identities are equivalent:

(U1) and (K1) $\Longleftrightarrow$ (U1) and

$K(K(a_i, b_j)c_k, d_l) - (-1)^{kl+(i+j)(k+l)} L(d_l, c_k)K(a_i, b_j) + \varepsilon K(a_i, b_j)L(c_k, d_l) = 0.$

Definition 2.3 (see [8, 10]). If an endomorphism D of the supertriple system U satisfies the identity

$$[D, L(a_i, b_j)] = L(Da_i, b_j) + (-1)^{\deg D \deg a_i} L(a_i, Db_j),$$

then it is said to be a superderivation of U.

For degree 0, this identity is equivalent to

$$D\langle abc\rangle = \langle (Da)bc\rangle + \langle a(Db)c\rangle + \langle ab(Dc)\rangle.$$

Remark 2.4. When U be a (ε,δ)-Freudenthal–Kantor supertriple system, then the endomorphism

$$S(a_i, b_j) := L(a_i, b_j) + \varepsilon(-1)^{ij} L(b_j, a_i)$$

is a superderivation of $U(\varepsilon,\delta)$, and we denote by $\operatorname{Inn}\operatorname{Der} U(\varepsilon,\delta)$ the set of these superderivations [8, 10].

If we have $K(x, y) = 0$ identically, $U(\varepsilon,\delta)$ is called a (ε,δ)-Jordan supertriple system. Alternately, suppose that there exists a bilinear form $\langle -,-\rangle$ such that

$$\langle a_i, b_j\rangle = -\varepsilon(-1)^{ij}\delta_{ij}\langle b_j, a_i\rangle, \tag{2.1}$$

where $a_i \in U_i, b_j \in U_j$ and δ_{ij} is the Kronecker delta. Then, if $K(x, y)$ satisfies

$$K(a_i, b_j) = \langle a_i, b_j\rangle \operatorname{Id}, \tag{2.2}$$

we call $U(\varepsilon,\delta)$ balanced. Here Id stands for the identity map. Assuming $\langle x, y\rangle$ to be not identically zero, the balanced (ε,δ)-Freudenthal–Kantor supertriple system implies the validity of the following relations in addition to $(U1)$:

(i) $\varepsilon = \delta$;

(ii) $\langle a_i b_j c_k\rangle - \delta(-1)^{ij+jk+ki}\langle c_k b_j a_i\rangle = (-1)^{jk}\langle a_i, c_k\rangle b_j$;

(iii) $\langle a_i b_j c_k\rangle - \varepsilon(-1)^{ij}\langle b_j a_i c_k\rangle = -\varepsilon\langle a_i, b_j\rangle c_k$;

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(iv) $\langle\langle a_i b_j c_k\rangle, d_l\rangle + (-1)^{(i+j)k}\langle c_k, \langle a_i b_j d_l\rangle\rangle = -\delta\langle a_i, b_j\rangle\langle c_k, d_l\rangle$; and

(v) $\langle Da_i, b_j\rangle + (-1)^{\deg a_i \deg D}\langle a_i, Db_j\rangle = 0$, where D is a superderivation of $U(\varepsilon,\delta)$.

However, since its derivation is standard, we need not go into detail.

Also, for the case of degree 0 with $\varepsilon = -1$, $\delta = 1$ and $K(a,b) \equiv 0$ (identically zero), (U0), (U1) and (K1) reproduce the definition of the well-known Jordan triple system as follows:

$$\langle abc\rangle = \langle cba\rangle,$$
$$\langle ab\langle cde\rangle\rangle = \langle\langle abc\rangle de\rangle - \langle c\langle bad\rangle e\rangle + \langle cd\langle abe\rangle\rangle.$$

We will now present the following examples.

Example 2.5. Let W be a vector space over Φ equipped with a symmetric bilinear form $\langle x,y\rangle$. Then W is a Jordan triple system with respect to the product

$$\langle xyz\rangle := \langle x,y\rangle z + \langle y,z\rangle x - \langle z,x\rangle y.$$

Example 2.6. Let W be as above. Then $(W,[xyz])$ is a Lie triple system with respect to the new product

$$[xyz] := \langle xyz\rangle - \langle yxz\rangle$$

(for the definition of a Lie triple system, see § 3 below or [**7**, **9**, **13**]).

Remark 2.7. We note that the connection between totally geodesic submanifolds of a Riemannian globally symmetric space and Lie triple systems is presented in Helgason's book [**6**].

Proposition 2.8. *Let $U(\varepsilon,\delta) = U(\varepsilon,\delta)_0 \oplus U(\varepsilon,\delta)_1$ be a Z_2-graded vector space with a bilinear form $\langle -,-\rangle$ satisfying equation (2.1). Then, we find that*

(i) *Both*

$$\langle a_i b_j c_k\rangle = \alpha\langle a_i, b_j\rangle c_k + \beta\langle b_j, c_k\rangle a_i$$

and

$$\langle a_i b_j c_k\rangle = \alpha\langle a_i, b_j\rangle c_k + \beta\{\langle b_j, c_k\rangle a_i + \varepsilon(-1)^{jk}\langle a_i, c_k\rangle b_j\}$$

satisfy the condition (U1) for any constants α and $\beta \in \Phi$.

(ii)

$$\langle a_i b_j c_k\rangle = \langle b_j, c_k\rangle a_i$$

is a (ε,δ)-Freudenthal–Kantor supertriple system, i.e. it satisfies (U1) and (K1).

(iii) *Both*

$$\langle a_i b_j c_k\rangle = \langle a_i, b_j\rangle c_k - \varepsilon\delta\langle b_j, c_k\rangle a_i,$$

and

$$\langle a_i b_j c_k\rangle = \langle a_i, b_j\rangle c_k + \langle b_j, c_k\rangle a_i + \varepsilon(-1)^{jk}\langle a_i, c_k\rangle b_j,$$

with $\varepsilon = -\delta$ (only for the second equation), are (ε,δ)-Jordan supertriple systems.

(iv)

$$\langle a_i, b_j, c_k\rangle = \tfrac{1}{2}\{-\varepsilon\langle a_i, b_j\rangle c_k + \varepsilon\langle b_j, c_k\rangle a_i + (-1)^{jk}\langle a_i, c_k\rangle b_j\}$$

gives a balanced (ε, δ)-Freudenthal–Kantor supertriple system.

Proof. Since the calculations are somewhat lengthy, we will give the detailed proof only for statement (ii) here. From the definition of the triple product, it follows that

$$\begin{aligned}
\langle a_i b_j \langle c_k d_l e_m\rangle\rangle &= \langle d_l, e_m\rangle\langle b_j, c_k\rangle a_i,\\
\langle\langle a_i b_j c_k\rangle d_l e_m\rangle &= \langle b_j, c_k\rangle\langle d_l, e_m\rangle a_i,\\
\varepsilon(-1)^{k(i+j)+ij}\langle c_k\langle b_j a_i d_l\rangle e_m\rangle &= \varepsilon(-1)^{k(i+j)+ij}\langle a_i, d_l\rangle\langle b_j, e_m\rangle c_k,\\
(-1)^{(k+l)(i+j)}\langle c_k d_l\langle a_i b_j e_m\rangle\rangle &= (-1)^{(k+l)(i+j)}\langle b_j, e_m\rangle\langle d_l, a_i\rangle c_k.
\end{aligned}$$

On the other hand, $\langle a_i, d_l\rangle = -(-1)^{il}\varepsilon\langle d_l, a_i\rangle\delta_{i,l}$ by straightforward calculation, and we obtain

$$[L(a_i, b_j), L(c_k, d_l)] = L(\langle a_i b_j c_k\rangle, d_l) + \varepsilon(-1)^{k(i+j)+ij}L(c_k, \langle b_j c_i d_l\rangle).$$

By the same method, we have

$$\begin{aligned}
K(\langle a_i b_j c_k\rangle, d_l)e_m &= (-1)^{lm}\langle\langle a_i b_j c_k\rangle e_m d_l\rangle - \delta(-1)^{(i+j+k)(m+l)}\langle d_l e_m\langle a_i b_j c_k\rangle\rangle\\
&= (-1)^{lm}\langle b_j, c_k\rangle\langle e_m, d_l\rangle a_i - \delta(-1)^{(i+j+k)(m+l)}\langle b_j, c_k\rangle\langle e_m, a_i\rangle d_l,
\end{aligned}$$

$$\begin{aligned}
&(-1)^{k(j+i)}K(c_k, \langle a_i b_j d_l\rangle)e_m\\
&\quad= (-1)^{(i+j)k}(-1)^{m(i+j+l)}\langle c_k e_m\langle a_i b_j d_l\rangle\rangle - \delta(-1)^{(i+j)k}(-1)^{k(i+j+l+m)}\langle\langle a_i b_j d_l\rangle e_m c_k\rangle\\
&\quad= (-1)^{(i+j)k}(-1)^{m(i+j+l)}\langle b_j, d_l\rangle\langle e_m, a_i\rangle c_k - \delta(-1)^{k(l+m)}\langle b_j, d_l\rangle\langle e_m, c_k\rangle a_i,
\end{aligned}$$

$$\begin{aligned}
&\delta(-1)^{j(k+l)}K(a_i, K(c_k, d_l)b_j)e_m\\
&\qquad= \delta(-1)^{j(k+l)}K(a_i, (-1)^{lj}\langle b_j, d_l\rangle c_k - \delta(-1)^{k(l+j)}\langle b_j, c_k\rangle d_l)e_m\\
&\qquad= \delta(-1)^{jk}\langle b_j, d_l\rangle((-1)^{km}\langle a_i e_m c_k\rangle - \delta(-1)^{i(k+m)}\langle c_k e_m a_i\rangle)\\
&\qquad\quad - (-1)^{j(k+l)+k(l+j)}\langle b_j, c_k\rangle((-1)^{lm}\langle a_i e_m d_l\rangle - \delta(-1)^{i(l+m)}\langle d_l e_m a_i\rangle)\\
&\qquad= \delta(-1)^{jk}\langle b_j, d_l\rangle((-1)^{km}\langle e_m, c_k\rangle a_i - \delta(-1)^{i(k+m)}\langle e_m, a_i\rangle c_k)\\
&\qquad\quad - (-1)^{l(j+k)}\langle b_j, c_k\rangle((-1)^{lm}\langle e_m, d_l\rangle a_i - \delta(-1)^{i(l+m)}\langle e_m, a_i\rangle d_l).
\end{aligned}$$

Adding these identities, we get

$$K(\langle a_i b_j c_k\rangle, d_l) + (-1)^{(i+j)k}K(c_k, \langle a_i b_j d_l\rangle) + \delta(-1)^{j(k+l)}K(a_i, K(c_k, d_l)b_j) = 0.$$

This completes the proof for (ii). □

For the case of degree 0 (V_0 refers to bosonic space), it follows that $U(\varepsilon, \delta)$ is a (ε, δ)-Freudenthal–Kantor triple system with respect to the product

$$\langle abc\rangle := \langle b, c\rangle a.$$

For the case of degree 1 (V_1 refers to fermionic space) or degree 0, we have the following proposition.

Proposition 2.9. *Let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal–Kantor supertriple system of degree 1 (that is, $U(\varepsilon,\delta)=V_1$) equipped with the triple product $\langle abc\rangle$, $a,b,c,\in V_1$. Then $U(\varepsilon,\delta)$ is a $(-\varepsilon,-\delta)$-Freudenthal–Kantor triple system (that is, $U(\varepsilon,\delta)=V_0$ may be regarded now as degree 0) equipped with the new triple product $\langle abc\rangle$, $a,b,c\in V_0$.*

3. Examples of (ε,δ)-Jordan supertriple systems and balanced systems

In this section, we shall first consider the special case $K(a,b)=0$ for all $a,b\in U(\varepsilon,\delta)$, which defines a (ε,δ)-Jordan supertriple system. We can redefine this alternatively by

$$\langle a_ib_jc_k\rangle=\delta(-1)^{k(i+j)+ij}\langle c_kb_ja_i\rangle, \tag{J1}$$

$$\begin{aligned}\langle a_ib_j\langle c_kd_le_m\rangle\rangle=\langle\langle a_ib_jc_k\rangle d_le_m\rangle+\varepsilon(-1)^{k(i+j)+ij}\langle c_k\langle b_ja_id_l\rangle e_m\rangle\\ +(-1)^{(k+l)(i+j)}\langle c_kd_l\langle a_ib_je_m\rangle\rangle,\end{aligned} \tag{J2}$$

which is often more convenient.

Remark 3.1. We note that

$$\text{(J1)}\iff K(a,b)=0,\quad \text{for all } a,b\in U(\varepsilon,\delta),$$

and

$$\text{(J2)}\iff \text{(U1)}.$$

For the case of degree 0, and if $(\varepsilon,\delta)=(-1,1)$, then this notion reduces to a Jordan triple system (cf. § 1), but if $(\varepsilon,\delta)=(1,-1)$, then this defines an anti-Jordan triple system (cf. [**9**]).

We have already given examples of (ε,δ)-Jordan triple systems in Proposition 2.8. For example, we have the following example.

Example 3.2. Let $W=\Sigma_i\oplus W_i$ be a vector space equipped with a super symmetric bilinear form $\langle x_i,y_j\rangle=(-1)^{ij}\langle y_j,x_i\rangle\delta_{ij}$ for $x_i\in W_i, y_j\in W_j$. Then

$$\langle x_iy_jz_k\rangle=\langle x_i,y_j\rangle z_k+\langle y_j,z_k\rangle x_i-(-1)^{(i+j)k}\langle z_k,x_i\rangle y_j$$

defines a $(-1,1)$-Jordan supertriple system on W, while

$$\langle x_iy_jz_k\rangle=\langle x_i,y_j\rangle z_k-\langle y_j,z_k\rangle x_i$$

gives a $(-1,-1)$-Jordan supertriple system.

Example 3.3. Let $W=\Sigma_i\oplus W_i$ be a vector space equipped with a super antisymmetric bilinear form $\langle x_i,y_j\rangle=-(-1)^{ij}\langle y_j,x_i\rangle\delta_{ij}$. Then

$$\langle x_i,y_j,z_k\rangle=\langle x_i,y_j\rangle z_k+\langle y_j,z_k\rangle x_i-(-1)^{(i+j)k}\langle z_k,x_i\rangle y_j$$

defines a $(1,-1)$-Jordan supertriple system on W, while

$$\langle x_i,y_j,z_k\rangle=\langle x_i,y_j\rangle z_k-\langle y_j,z_k\rangle x_i$$

gives a $(1,1)$-Jordan supertriple system on W.

We also have other types of Jordan supertriple systems as follows.

Example 3.4. Let xy be an associative algebra, i.e.

$$(xy)z = x(yz) := xyz.$$

Then

$$\langle a_i b_j c_k \rangle = a_i b_j c_k + \delta(-1)^{ij+jk+ki} c_k b_j a_i$$

defines a $(-\delta, \delta)$-Jordan supertriple system (the case of $\varepsilon = -\delta$).

Example 3.5. Let $x \cdot y = (-1)^{xy} y \cdot x$ be a Jordan superalgebra (cf. [**21**]). Then

$$\langle a_i b_j c_k \rangle = a_i \cdot (b_j \cdot c_k) + (a_i \cdot b_j) \cdot c_k - (-1)^{jk} (a_i \cdot c_k) \cdot b_j$$

defines a $(-1, 1)$-Jordan supertriple system.

Next let us consider examples of balanced Freudenthal–Kantor supertriple systems other than that discussed in Proposition 2.8. Suppose that we have $\varepsilon = \delta$, and introduce the second triple product by

$$a_i \cdot b_j \cdot c_k := \langle a_i, b_j, c_k \rangle + \tfrac{1}{2}\varepsilon \langle a_i, b_j \rangle c_k.$$

Then, equations (K1) and (U1) can be rewritten as

(i) $a_i \cdot b_j \cdot c_k - \varepsilon(-1)^{ij} b_j \cdot a_i \cdot c_k = 0;$

(ii) $a_j \cdot b_j \cdot c_k - \varepsilon(-1)^{jk} a_i \cdot c_k \cdot b_j = \tfrac{1}{2}\varepsilon\{2\langle b_j, c_k \rangle a_i - \langle a_i, b_j \rangle c_k - (-1)^{(i+j)k} \langle c_k, a_i \rangle b_j\};$

(iii) $a_i \cdot b_j \cdot (x_k \cdot y_l \cdot z_m) = (a_i \cdot b_j \cdot x_k) \cdot y_l \cdot z_m + (-1)^{(i+j)k} x_k \cdot (a_i \cdot b_j \cdot y_l) \cdot z_m + (-1)^{(i+j)(k+l)} x_k \cdot y_l \cdot (a_i \cdot b_j \cdot z_m);$

(iv) $\langle a_i \cdot b_j \cdot c_k, d_l \rangle + (-1)^{(i+j)k} \langle c_k, a_i \cdot b_j \cdot d_l \rangle = 0.$

If we define the left multiplication operation $\hat{L}(a_i, b_j)$ by

$$\hat{L}(a_i, b_j) c_k := a_i \cdot b_j \cdot c_k,$$

then (iii) implies that $\hat{L}(a_i, b_j)$ is a superderivation of the supertriple system.

Such a system has been called symplectic ($\varepsilon = 1$) and orthogonal ($\varepsilon = -1$) and has been used to construct solutions of the Yang–Baxter equation [**18**]. Especially, the octonionic triple system [**19**] offers an example of the case of degree 0 ($\varepsilon = -1$). For other examples, see also [**11, 12**].

4. δ-Lie supertriple systems $T(\delta)$ associated with $U(\varepsilon, \delta)$

In order to make this paper as self-contained as possible, we shall briefly recall the basic concept for δ-Lie supertriple systems from the previous papers [**20, 21**].

For $\delta = \pm 1$, a vector space $T(\delta) = \Sigma_i \oplus T_i$ over Φ with a triple product $[-,-,-]$ is called a δ-Lie supertriple system if

$$[x_i y_j z_k] \in T_{i+j+k}, \tag{T0}$$

$$[x_i y_j z_k] = -\delta(-1)^{ij}[y_j x_i z_k], \tag{T1}$$

$$(-1)^{ik}[x_i y_j z_k] + (-1)^{ji}[y_j z_k x_i] + (-1)^{kj}[z_k x_i y_j] = 0, \tag{T2}$$

$$[x_i y_j [z_k u_l v_m]] = [[x_i y_j z_k] u_l v_m] + (-1)^{(i+j)k}[z_k [x_i y_j u_l] v_m] + (-1)^{(k+l)(i+j)}[z_k u_l [x_i y_j v_m]], \tag{T3}$$

for all $x_i \in T_i$, $y_j \in T_j$, $z_k \in T_k$, $u_l \in T_l$, $v_m \in T_m$.

In the case of $\delta = 1$ (respectively, $\delta = -1$), $T(\delta)$ is said to be a Lie supertriple system (respectively, anti-Lie supertriple system).

Furthermore, for the case of degree 0 and $\delta = 1$ it is said to be a Lie triple system, while for the case of degree 0 and $\delta = -1$ it is called an anti-Lie triple system (cf. [**9**,**13**]).

Remark 4.1. The above identity (T3) implies that the mapping $L(x_i, x_j) : x_k \to [x_i x_j x_k]$ is a superderivation of the supertriple system $T(\delta)$. We denote it by $L(T(\delta), T(\delta))$ and it is said to be an inner superderivation of $T(\delta)$.

We can now give an example as follows.

Let $\langle x_j, y_k \rangle = \delta(-1)^{jk}\langle y_k, x_j \rangle$. Then, the triple product

$$[x_i y_j z_k] = \langle y_j, z_k \rangle x_i - \delta(-1)^{ij}\langle x_i, z_k \rangle y_j$$

defines a δ-Lie supertriple system.

Next we shall investigate the correspondence of (ε, δ)-Freudenthal–Kantor supertriple systems and δ-Lie supertriple systems.

Theorem 4.2. *Let $U(\varepsilon, \delta)$ be a (ε, δ)-Freudenthal–Kantor supertriple system. If P is a grade-preserving linear transformation of $U(\varepsilon, \delta)$ such that $P\langle x_i y_j z_k \rangle = \langle P x_i, P y_j, P z_k \rangle$ and $P^2 = -\varepsilon\delta \,\mathrm{Id}$, then $(U(\varepsilon, \delta), [-,-,-])$ is a Lie supertriple system (the case of $\delta = 1$) or an anti-Lie supertriple system (the case of $\delta = -1$) with respect to the triple product*

$$[x_i y_j z_k] := \langle x_i P y_j z_k \rangle - \delta(-1)^{ij}\langle y_j P x_i z_k \rangle + \delta(-1)^{kj}\langle x_i P z_k y_j \rangle - (-1)^{i(j+k)}\langle y_j P z_k x_i \rangle. \tag{4.1}$$

For the proof of Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let $(U(\varepsilon, \delta), \langle xyz \rangle)$ be a (ε, δ)-Freudenthal–Kantor supertriple system equipped with $L(x, y)$ and $K(x, y)$ given by*

$$L(a_i, b_j)c_k := \langle a_i b_j c_k \rangle \tag{4.2a}$$

$$K(a_i b_j)c_k := (-1)^{jk}\langle a_i c_k b_j \rangle - \delta(-1)^{i(j+k)}\langle b_j c_k a_i \rangle. \tag{4.2b}$$

Setting

$$M(a_i, b_j) := L(a_i, P b_j) - \delta(-1)^{ij} L(b_j, P a_i), \tag{4.3a}$$

$$Q(a_i, b_j) := \delta K(a_i, b_j) P, \tag{4.3b}$$

they satisfy a Lie super equation

$$\text{(i) } [M(a_i,b_j),M(c_k,d_e)] = M(M(a_i,b_j)c_k,d_e) + (-1)^{(i+j)k}M(c_k,M(a_i,b_j)d_e); \quad (4.4\,a)$$

$$\text{(ii) } [M(a_i,b_j),Q(c_k,d_e)] = Q(M(a_i,b_j)c_k,d_e) + (-1)^{(i+j)k}Q(c_k,M(a_i,b_j)d_e); \quad (4.4\,b)$$

$$\text{(iii) } [Q(a_i,b_j),Q(c_k,d_e)] = M(Q(a_i,b_j)c_k,d_e) + (-1)^{(i+j)k}M(c_k,Q(a_i,b_j)d_e); \quad (4.4\,c)$$

where $[X,Y]$ *stands for the super commutator*

$$[X,Y] = XY - (-1)^{XY}YX. \tag{4.5}$$

Moreover, $Q(x,y)$ *obeys constraint equations*

$$\begin{aligned} Q(a_i,b_j)Q(c_k,d_e) &= L(Q(a_i,b_j)c_k,Pd_e) - \delta(-1)^{ke}L(Q(a_i,b_j)d_e,Pc_k) \\ &= -(-1)^{j(k+e)}L(a_i,PQ(c_k,d_e)b_j) + \delta(-1)^{i(j+k+e)}L(b_j,PQ(c_k,d_e)a_i). \end{aligned} \tag{4.6}$$

Proof. Equation $(4.4\,a)$ can be shown straightforwardly, while equation $(4.4\,b)$ follows readily from the validity of

$$\begin{aligned} L(a_i,Pb_j)Q(c_k,d_e) + \delta(-1)^{(i+j)(k+e)+ij}Q(c_k,d_e)L(b_j,Pa_i) \\ = Q(L(a_i,Pb_j)c_k,d_e) + (-1)^{k(i+j)}Q(c_k,L(a_i,Pb_j)d_e). \end{aligned}$$

In order to prove equation (4.6), we omit unnecessary complications due to the presence of sign factors $(-1)^{ij}$, etc., by omitting them in the following proof. We can easily supply them if necessary. Changing notations with this understanding, we calculate

$$\begin{aligned} Q(u,v)Q(x,y)z &= K(u,v)P\{\langle x,Pz,y\rangle - \delta\langle y,Pz,x\rangle\} \\ &= -\varepsilon\delta K(u,v)\{\langle Px,z,Py\rangle - \delta\langle Py,z,Px\rangle\} \\ &= -\varepsilon\delta\langle u,\langle Px,z,Py\rangle,v\rangle + \varepsilon\langle v,\langle Px,z,Py\rangle,u\rangle \\ &\quad + \varepsilon\langle u,\langle Py,z,Px\rangle,v\rangle - \varepsilon\delta\langle v,\langle Py,z,Px\rangle,u\rangle. \end{aligned}$$

We note next the validity of

$$\begin{aligned} -\varepsilon\delta\langle u,\langle Px,z,Py\rangle,v\rangle &= -\delta\langle z,Px,\langle u,Py,v\rangle\rangle \\ &\quad + \delta\langle\langle z,Px,u\rangle,Py,v\rangle + \delta\langle u,Py,\langle z,Px,v\rangle\rangle, \end{aligned}$$

and similarly for other expressions from (U1). We then find

$$\begin{aligned} Q(u,v)Q(x,y)z = &-\delta K(z,K(u,v)Py)Px - \langle K(u,v)Py,Px,z\rangle \\ &+ \delta K(u,\langle z,Px,v\rangle)Py - K(v,\langle z,Px,u\rangle)Py + K(z,K(u,v)Px)Py \\ &+ \delta\langle K(u,v)Px,Py,z\rangle - K(u,\langle z,Py,v\rangle)Px + \delta K(v,\langle z,Py,u\rangle)Px. \end{aligned}$$

Using now the relation (K1), which gives

$$\delta K(z,K(u,v)Py) + K(\langle z,Py,u\rangle,v) + K(u,\langle z,Py,v\rangle) = 0,$$

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we then obtain

$$Q(u,v)Q(x,y)z = -\langle K(u,v)Py, Px, z\rangle + \delta\langle K(u,v)Px, Py, z\rangle,$$

which is equivalent to the first relation in equation (4.6). Next, we note the identity

$$L(\langle u,v,x\rangle, y) + \varepsilon L(x, \langle v,u,y\rangle) = -L(\langle x,y,u\rangle, v) - \varepsilon L(u, \langle y,x,v\rangle),$$

which can be derived from (U1) by letting $a_i \leftrightarrow c_k$ $b_j \leftrightarrow d_e$. Letting $x \leftrightarrow u$, this leads to

$$L(K(u,x)v,y) + L(K(x,u)y,u) + \varepsilon L(u, K(y,v),x) + \varepsilon L(x, K(v,y)u) = 0,$$

which leads to the second relation in equation (4.6). Finally, equation (4.4 c) then follows from equation (4.6). This completes the proof of Lemma 3.2. □

The proof of Theorem 4.2 is now simple. If we introduce the left multiplication operator $L_0(a_i, b_j)$ by

$$L_0(a_i,b_j)c_k = [a_i, b_j, c_k],$$

we then find

$$L_0(a_i,b_j) = M(a_i,b_j) + Q(a_i,b_j),$$

so that Lemma 4.3 gives the desired result

$$[L_0(a_i,b_j), L_0(c_k,d_e)] = L_0([a_i,b_j,c_k],d_e) + (-1)^{(i+j)k} L_0(c_k, [a_i,b_j,d_e]),$$

after some calculation. This proves (T3). the validity of both (T1) and (T2) follows directly from equation (4.1).

In particular, if $\varepsilon = -1$, $\delta = 1$, $K(x,y) = 0$ for all $x,y \in U(\varepsilon,\delta)$, $P = \mathrm{Id}$ and the case of degree 0 (that is, $U(\varepsilon,\delta)$ is a Jordan triple system), then the resulting Lie triple product becomes

$$[xyz] := \langle xyz\rangle - \langle yxz\rangle, \tag{4.7}$$

as in Example 2.6.

This special construction implies the construction of Lie algebra or Lie triple systems from Jordan algebras or Jordan triple systems, respectively (see, for example, [**9**]).

Corollary 4.4 (see [9]). *Let $(U, \langle -,-,-\rangle)$ be an anti-Jordan triple system. Then $(U, [-,-,-])$ is an anti-Lie triple system with respect to the product*

$$[xyz] := \langle xyz\rangle + \langle yxz\rangle$$

(that is, the case of $P = \mathrm{Id}$, $\varepsilon = 1$, $\delta = -1$, $K(x,y) = 0$ for all $x,y \in U(\varepsilon,\delta)$).

Proposition 4.5. *If we let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal–Kantor supertriple system, then vector space $U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a Lie supertriple system (the case of $\delta = 1$)*

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or an anti-Lie supertriple system (the case of $\delta = -1$*) with respect to the triple product defined by*

$$\left[\begin{pmatrix} a_i \\ b_i \end{pmatrix}\begin{pmatrix} c_j \\ d_j \end{pmatrix}\begin{pmatrix} e_k \\ f_k \end{pmatrix}\right] = \begin{pmatrix} L(a_i,d_j) - \delta(-1)^{ij}L(c_j,b_i) & \delta K(a_i,c_j) \\ -\varepsilon K(b_i,d_j) & \varepsilon((-1)^{ij}L(d_j,a_i) - \delta L(b_i,c_j)) \end{pmatrix}\begin{pmatrix} e_k \\ f_k \end{pmatrix}. \tag{4.8}$$

Proof. Let $V(\varepsilon,\delta) := U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$, which is also a (ε,δ)-Freudenthal–Kantor supertriple system with triple product given by

$$\left\langle \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \begin{pmatrix} e_k \\ f_k \end{pmatrix} \right\rangle := \begin{pmatrix} \langle a_i c_j e_k \rangle \\ \langle b_i d_j f_k \rangle \end{pmatrix}.$$

Identifying

$$P = \begin{pmatrix} 0 & \mathrm{Id} \\ -\varepsilon\delta\,\mathrm{Id} & 0 \end{pmatrix}$$

in Theorem 4.2, we obtain

$$\begin{aligned}\left[\begin{pmatrix} a_i \\ b_i \end{pmatrix}\begin{pmatrix} c_j \\ d_j \end{pmatrix}\begin{pmatrix} e_k \\ f_k \end{pmatrix}\right] &= \left\langle \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ -\varepsilon\delta\,\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} c_j \\ d_j \end{pmatrix}, \begin{pmatrix} e_k \\ f_k \end{pmatrix} \right\rangle \\ &\quad - \delta(-1)^{ij}\left\langle \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ -\varepsilon\delta\,\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} e_k \\ f_k \end{pmatrix} \right\rangle \\ &\quad + \delta(-1)^{kj}\left\langle \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ -\varepsilon\delta\,\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} e_k \\ f_k \end{pmatrix}, \begin{pmatrix} c_j \\ d_j \end{pmatrix} \right\rangle \\ &\quad - (-1)^{i(j+k)}\left\langle \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ -\varepsilon\delta\,\mathrm{Id} & 0 \end{pmatrix}\begin{pmatrix} e_k \\ f_k \end{pmatrix}, \begin{pmatrix} a_i \\ b_i \end{pmatrix} \right\rangle,\end{aligned}$$

which implies the identity (4.4). This completes the proof. □

From this Proposition, we can obtain the δ-Lie supertriple system $U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ associated with $U(\varepsilon,\delta)$ and denote it by $T(\delta)$ (for special cases of degree 0 and $\delta = 1$, see [**8**] or [**11**]).

In ending this section, we may note the following for the case of degree 0 (cf. [**8, 9**]). It is well known that if $(W, [xyz])$ is a Lie triple system over Φ, then $L := L(W,W) \oplus W$ is a Lie algebra with respect to the product

$$[D + x, D' + y] = [D, D'] + L(x,y) + Dy - D'x,$$

where D, D', $L(x,y)$ are elements of inner derivations $L(W,W)$ of W, $x, y \in W$. This method is the canonical construction of a Lie algebra from a Lie triple system. We can readily generalize it for the case of super-Lie triple system ($\delta = 1$). For anti-Lie supertriple systems ($\delta = -1$), we can similarly construct a doubly graded Lie super algebra as in [**21**].

5. Almost-complex superstructure of $T(\delta)$

We recall that a (ε,δ)-Freudenthal–Kantor supertriple system over a field Φ is said to be balanced if there exists a bilinear supersymmetric form $\langle\cdot,\cdot\rangle$, such that

$$K(x_i,x_j) := \langle x_i,x_j\rangle\,\mathrm{Id}, \tag{5.1}$$

$$\langle x_i,x_j\rangle = (-1)^{ij}\delta_{ij}\langle x_j,x_i\rangle. \tag{5.2}$$

For a balanced (ε,δ)-Freudenthal–Kantor supertriple system $U(\varepsilon,\delta)$ and the associated δ-Lie supertriple system $T(\delta)$ as in Proposition 4.5, there exist two endomorphisms E and F of $T(\delta)$ as follows:

$$E := \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix}.$$

Then we have

$$H := [E,F] = EF - FE = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}, \quad [H,E] = 2E \quad \text{and} \quad [H,F] = -2F.$$

Thus, we can obtain a three-dimensional simple Lie algebra

$$\langle H,E,F\rangle_{\mathrm{gen}} \simeq sl(2,\Phi).$$

For the remainder of this section, we assume the case of $\varepsilon\delta = 1$. We set

$$E(X,Y) := E[EX,Y] + E[X,EY],$$

where we have introduced $[X,Y]$ for $X,Y \in U(\varepsilon,\delta)\oplus U(\varepsilon,\delta)$ by

$$\left[\begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} c_j \\ d_j \end{pmatrix}\right] = \begin{pmatrix} L(a_i,d_j) - \delta(-1)^{ij}L(c_j,b_i) & \delta K(a_i,c_j) \\ -\varepsilon K(b_i,d_j) & \varepsilon(-1)^{ij}L(d_j,a_i) - \varepsilon\delta L(b_i,c_j) \end{pmatrix},$$

or, equivalently,

$$\left[\begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} c_j \\ d_j \end{pmatrix}\right]\cdot\begin{pmatrix} e_k \\ f_k \end{pmatrix} := \left[\begin{pmatrix} a_i \\ b_i \end{pmatrix}, \begin{pmatrix} c_j \\ d_j \end{pmatrix}, \begin{pmatrix} e_k \\ f_k \end{pmatrix}\right].$$

We then have the following.

Theorem 5.1. *Let $U(\varepsilon,\delta)$ be a (ε,δ)-Freudenthal–Kantor supertriple system and $T(\delta)$ be a δ-Lie supertriple system associated with $U(\varepsilon,\delta)$, as in Proposition 4.5. Then the following are equivalent.*

(1) $E(X,Y) = 0$, *for* $X = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $Y = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$.

(2) $\delta(-1)^{ij}L(x_j,x_i) - L(x_i,x_j) = 0$, *for* $x_i \in U_i(\varepsilon,\delta)$, $x_j \in U_j(\varepsilon,\delta)$.

Proof. For

$$X = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} x_j \\ y_j \end{pmatrix},$$

we have

$$\begin{aligned}
E(X,Y) &:= \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} y_i \\ 0 \end{pmatrix}, \begin{pmatrix} x_j \\ y_j \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} x_i \\ y_i \end{pmatrix}, \begin{pmatrix} y_j \\ 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} L(y_i, y_j) & \delta K(y_i, x_j) \\ 0 & \varepsilon(-1)^{ij} L(y_j, y_i) \end{pmatrix} \\
&\qquad + \begin{pmatrix} 0 & \mathrm{Id} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\delta(-1)^{ij} L(y_j, y_i) & \delta K(x_i, y_j) \\ 0 & -\varepsilon\delta L(y_i, y_j) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \varepsilon(-1)^{ij} L(y_j, y_i) - \varepsilon\delta L(y_i, y_j) \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus we obtain

$$E(X,Y) = 0 \iff \delta(-1)^{ij} L(y_j, y_i) - L(y_i, y_j) = 0.$$

This completes the proof. □

Theorem 5.2. *Let $U(\varepsilon, \delta)$ and $T(\delta)$ be as in Theorem 5.1. Then the following are equivalent.*

(1) $(-1)^{ij} L(x_j, x_i) - \delta L(x_i, x_j) = K(x_i, x_j)$, *for* $x_i \in U_i(\varepsilon, \delta)$, $x_j \in U_j(\varepsilon, \delta)$.

(2) $F = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix}$ *is a derivation of* $T(\delta)$.

Proof. (1) $\iff$ (2): by straightforward calculations, we have

$$\begin{aligned}
&F \left[\begin{pmatrix} a_i \\ b_i \end{pmatrix} \begin{pmatrix} c_j \\ d_j \end{pmatrix} \begin{pmatrix} e_k \\ f_k \end{pmatrix} \right] \\
&\quad = F \begin{pmatrix} L(a_i, d_j) - \delta(-1)^{ij} L(c_j, b_i) & \delta K(a_i, c_j) \\ -\varepsilon K(b_i, d_j) & \varepsilon((-1)^{ij} L(d_j, a_i) - \delta L(b_i, c_j)) \end{pmatrix} \begin{pmatrix} e_k \\ f_k \end{pmatrix} \\
&\quad = \begin{pmatrix} 0 & 0 \\ L(a_i, d_j) - \delta(-1)^{ij} L(c_j, b_i) & \delta K(a_i, c_j) \end{pmatrix} \begin{pmatrix} e_k \\ f_k \end{pmatrix},
\end{aligned}$$

$$\left[F\begin{pmatrix}a_i\\b_i\end{pmatrix}\begin{pmatrix}c_j\\d_j\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix}\right]=\left[\begin{pmatrix}0\\a_i\end{pmatrix}\begin{pmatrix}c_j\\d_j\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix}\right]$$
$$=\begin{pmatrix}-\delta(-1)^{ij}L(c_j,a_i) & 0\\ -\varepsilon K(a_i,d_j) & \varepsilon(-\delta L(a_i,c_j))\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix},$$

$$\left[\begin{pmatrix}a_i\\b_i\end{pmatrix}F\begin{pmatrix}c_j\\d_j\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix}\right]=\left[\begin{pmatrix}a_i\\b_i\end{pmatrix}\begin{pmatrix}0\\c_j\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix}\right]$$
$$=\begin{pmatrix}L(a_i,c_j) & 0\\ -\varepsilon K(b_i,c_j) & \varepsilon(-1)^{ij}L(c_j,a_i)\end{pmatrix}\begin{pmatrix}e_k\\f_k\end{pmatrix},$$

$$\left[\begin{pmatrix}a_i\\b_i\end{pmatrix}\begin{pmatrix}c_j\\d_j\end{pmatrix}F\begin{pmatrix}e_k\\f_k\end{pmatrix}\right]$$
$$=\left[\begin{pmatrix}a_i\\b_i\end{pmatrix}\begin{pmatrix}c_j\\d_j\end{pmatrix}\begin{pmatrix}0\\e_k\end{pmatrix}\right]$$
$$=\begin{pmatrix}L(a_i,d_j)-\delta(-1)^{ij}L(c_j,b_i) & \delta K(a_i,c_j)\\ -\varepsilon K(b_i,d_j) & \varepsilon((-1)^{ij}L(d_j,a_i)-\delta L(b_i,c_j))\end{pmatrix}\begin{pmatrix}0\\e_k\end{pmatrix}.$$

Thus we find that F is a derivation of $T(\delta)$

$$\iff \begin{pmatrix}0\\(L(a_i,d_j)-\delta(-1)^{ij}L(c_j,b_i))e_k+\delta K(a_i,c_j)f_k\end{pmatrix}$$
$$=\begin{pmatrix}-\delta(-1)^{ij}L(c_j,a_i)e_k\\-\varepsilon K(a_i,d_j)e_k+\varepsilon(-\delta)L(a_i,c_j)f_k\end{pmatrix}$$
$$+\begin{pmatrix}L(a_i,c_j)e_k\\-\varepsilon K(b_i,c_j)e_k+\varepsilon(-1)^{ij}L(c_j,a_i)f_k\end{pmatrix}$$
$$+\begin{pmatrix}\delta K(a_i,c_j)c_k\\\varepsilon((-1)^{ij}L(d_j,a_i)-\delta L(b_i,c_j))e_k\end{pmatrix}$$
$$\iff \delta(-1)^{ij}L(c_j,a_i)+L(a_i,c_j)+\delta K(a_i,c_j)=0,$$

and

$$(L(a_i,d_j)-\delta(-1)^{ij}L(c_j,b_i))e_k+\delta K(a_i,c_j)f_k$$
$$=-\varepsilon K(a_i,d_j)e_k+\varepsilon(-\delta)L(a_i,c_j)f_k-\varepsilon K(b_i,c_j)e_k$$
$$+\varepsilon(-1)^{ij}L(c_j,a_i)f_k+\varepsilon((-1)^{ij}L(d_j,a_i)-\delta L(b_i,c_j))e_k.$$

On the other hand, from $\varepsilon\delta=1$, $\varepsilon=\pm1$ and $\delta=\pm1$, we have $\varepsilon=\delta$. Hence, we get

$$-\delta L(a_i,d_j)+(-1)^{ij}L(d_j,a_i)=K(a_i,d_j).$$

This completes the proof. □

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We next consider an operator J of $T(\delta)$ such that $J := \delta E - \varepsilon F$. Then we have $J^2 = -\varepsilon\delta\,\mathrm{Id}$. We call it a (ε, δ) almost-complex superstructure.

We put

$$N(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y].$$

We then obtain the following.

Theorem 5.3. *Let $U(\varepsilon,\delta)$, $T(\delta)$ and $N(X,Y)$ be as above. Then the following two relations are equivalent to each other.*

(1) $N(X,Y) = 0$, *for* $X = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$, $Y = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$.

(2) $K(x_i, x_j) = (-1)^{ij} L(x_j, x_i) - \delta L(x_i, x_j)$, *for* $x_i \in U_i(\varepsilon,\delta)$, $x_j \in U_j(\varepsilon,\delta)$.

Proof. We can verify by the same arguments as have been used for Theorem 5.1 and Theorem 5.2, and so we omit them. □

Remark 5.4. For the case of degree 0 and $\varepsilon\delta = 1$, the above condition $N(X,Y) = 0$ is an analogue of the Nijenhuis tensor condition for integrability in differential geometry (cf. [**4**,**6**,**15**]). It is well known that this condition, $N(X,Y) = 0$, is equivalent to the condition of complex structure. The same condition also appears in some class of integrable dynamical system [**17**].

Remark 5.5. From Theorems 5.3 and 5.2, it follows that for the endomorphism F of $T(\delta)$, F is a derivation $\Longleftrightarrow N(X,Y) = 0$.

6. Example of complex structure of degree 0

In this section, although we can generalize our result to the case of superspace, we will restrict ourselves to the case of degree 0 for simplicity.

Let $U(\varepsilon,\delta)$ be a (ε,δ)-balanced Freudenthal–Kantor triple system with degree 0 for the rest of the paper. For the case of $\delta = 1$, from $K(x,y)z = \langle xyz\rangle - \langle yzx\rangle$, it holds that $K(x,y) := \langle x,y\rangle = -\langle y,x\rangle$. For the case of $\delta = -1$, from $K(x,y)z = \langle xzy\rangle + \langle yzx\rangle$, it holds that $K(x,y) := \langle x,y\rangle = \langle y,x\rangle$. Furthermore, we recover the fact that the case of $\varepsilon\delta = -1$ does not occur in (ε,δ)-balanced Freudenthal–Kantor triple systems, because $K(x,y) = \langle x,y\rangle = -\delta\langle y,x\rangle = L(y,x) - \varepsilon L(x,y)$.

Theorem 6.1. *Let (ε,δ) be a (ε,δ)-balanced Freudenthal–Kantor triple system and $T(\delta)$ be the δ-Lie triple system associated with $U(\varepsilon,\delta)$. Then this triple system $T(\delta)$ has a complex structure.*

Proof. From the results of § 4, it follows that

$$T(\delta) \text{ has a complex structure,} \tag{6.1}$$

$$\Longleftrightarrow N(X,Y) = 0 \text{ (Nijenhuis operator} = 0), \tag{6.2}$$

$$\Longleftrightarrow K(a,b) = L(b,a) - \varepsilon L(a,b), \tag{6.3}$$

where $K(a,b)c = \langle acb\rangle - \delta\langle bca\rangle$.

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On the other hand, from the property of our balanced triple system, we obtain the relations:

(i) $\langle a,b\rangle c = K(a,b)c = L(a,c)b - L(b,c)a = L(b,a)c - L(a,b)c$ (if $\delta = 1$, $\varepsilon = 1$).

(ii) $\langle a,b\rangle c = K(a,b)c = L(a,c)b + L(b,c)a = L(b,a)c + L(a,b)c$ (if $\delta = -1$, $\varepsilon = -1$).

These relations satisfy the identity (6.3).

This completes the proof. □

For $\varepsilon = 1$ and $\delta = 1$, we have studied all simple Lie algebras associated with simple balanced Freudenthal–Kantor triple systems in [**11**, **12**].

For $\varepsilon = -1$ and $\delta = -1$, we give an example of $(-1,-1)$ balanced Freudenthal–Kantor triple and the simple Lie superalgebra associated with it as follows.

Example 6.2. Let U be a vector space over Φ equipped with a symmetric bilinear form $\langle\cdot,\cdot\rangle \in \Phi^*$. Then $(U, \langle -,-,-\rangle)$ is a $(-1-1)$ balanced Freudenthal–Kantor triple system over Φ with respect to the product

$$\langle xyz\rangle := \tfrac{1}{2}(\langle x,y\rangle z + \langle z,x\rangle y - \langle y,z\rangle x). \tag{6.4}$$

Thus, from the anti-Lie triple product, we have

$$\left[\begin{pmatrix} x \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right] = \begin{pmatrix} L(x,y) & 0 \\ 0 & -L(y,x) \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where $L(x,y)z = \langle xyz\rangle$.

Therefore, by the concept of the standard embedding Lie superalgebra $L(T)$ associated with $T := U \oplus U$ (cf. [**9**,**12**]), we obtain the simple Lie superalgebra $C(n)$, where $\dim_\Phi U = N = 2(n-1)$.

In fact, let

$$a_{ij} := L\left(\begin{pmatrix} e_i \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ e_j \end{pmatrix}\right),$$

then we obtain $a_{ij} = a_{ji}$ by (6.4). From straightforward calculations,

$$\dim_\Phi L(T) = \tfrac{1}{2}N(N+1) - 1 + 2N + 2 = 2n^2 + n - 2.$$

Next we give an example of a $(-1,-1)$-Jordan triple system and the simple Lie superalgebra associated with it.

Example 6.3. Let W be a set of matrices $\mathrm{Mat}(m,n;\Phi)$. Then $\langle xyz\rangle = x\,{}^{\mathrm{T}}yz - z\,{}^{\mathrm{T}}yx$ defines on W a $(-1,-1)$-Jordan triple system, where ${}^{\mathrm{T}}x$ denotes the transpose matrix of x, and $x,y,z \in W$. Thus, from an application of Proposition 4.5, we obtain an anti-Lie triple system T as follows

$$T = W \oplus W = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_i \in \mathrm{Mat}(m,n:\Phi)\right\}, \quad m \neq n,$$

https://doi.org/10.1017/S0013091500020903 Published online by Cambridge University Press

with respect to the product

$$\left[\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}\right] = \begin{pmatrix}L(x_1,y_2)+L(y_1,x_2) & 0\\ 0 & -L(y_2,x_1)-L(x_2,y_1)\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix},$$

where $L(x,y)z = \langle xyz \rangle$.

From the standard embedding Lie superalgebra, we have the Lie superalgebra $L(T)$ of type $A(n-1, m-1)$ characterized by

$$L(T,T) = A_{n-1} \oplus A_{m-1} \oplus \mathrm{Id} = \text{the inner derivations of } T,$$
$$L(T) = spl(m,n),$$
$$\dim L(T) = (n+m)^2 - 1, \quad \dim L(T,T) = n^2 + m^2 - 1,$$

where A_n denote the classical simple Lie algebra of type A_n. This triple system does not become a triple system equipped with the complex structure, because $0 = K(x,y) \neq L(y,x) + L(x,y)$.

References

1. F. A. BEREZIN AND D. A. LEITES, Supermanifolds, *Dokl. Akad. Nauk SSSR* **224** (1975), 1218–1222.
2. J. R. FAULKNER AND J. C. FERRAR, Simple anti-Jordan pairs, *Commun. Algebra* **8** (1980), 993–1013.
3. H. FREUDENTHAL, Beziehungen der F_7 and E_8 zur Oktavenbener, I, II, *Indag. Math.* **16** (1954), 218–230, 363–386.
4. A. FRÖLICHER, Zur Differentialgeometrie der Komplexen Structuren, *Math. Ann.* **129** (1955), 50–95.
5. W. HEIN, A construction of Lie algebras by triple systems, *Trans. Am. Math. Soc.* **205** (1975), 79–95.
6. S. HELGASON, *Differential geometry, Lie groups and symmetric spaces* (Academic, New York, 1978).
7. N. JACOBSON, Lie and Jordan triple systems, *Am. J. Math.* **71** (1949), 149–170.
8. N. KAMIYA, A structure theory of Freudenthal–Kantor triple systems, *J. Alg.* **110** (1987), 108–123.
9. N. KAMIYA, A construction of anti-Lie triple systems from a class of triple systems, *Mem. Fac. Sci. Shimane University* **22** (1988), 51–62.
10. N. KAMIYA, A structure theory of Freudenthal–Kantor triple systems, II, *Commun. Math., Univ. Sancti Pauli* **38** (1989), 41–60.
11. N. KAMIYA, The construction of all simple Lie algebras over C from balanced Freudenthal–Kantor triple system, in *Contributions to General Algebra*, vol. 7, pp. 205–213 (Hölder–Pichler–Tempsky, Wien, B. G. Teubner, Stuttgart, 1991).
12. N. KAMIYA, On Freudenthal–Kantor triple systems and generalized structurable algebras, Nonassociative algebras and its applications, In *Mathematics and its Applications*, vol. 303, pp. 198–203 (Kluwer, Dordrecht, 1994).
13. N. KAMIYA, The nil radical for Lie or anti-Lie triple systems, *J. Alg.* **156** (1993), 226–232.
14. I. L. KANTOR, Models of exceptional Lie algebras, *Sov. Math. Dokl.* **14** (1973), 254–258.
15. S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry, I and II* (Interscience, New York, 1963).

16. A. Newlander and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, *Ann. Math.* **65** (1957), 391–404.
17. S. Okubo, Nijenhuis tensor, BRST cohomology and related algebras, *Alg. Group Geom.* **6** (1989), 65–112.
18. S. Okubo, Triple products and Yang–Baxter equation, I and II, *J. Math. Phys.* **34** (1993), 3273–3291 (I), 3292–3315 (II).
19. S. Okubo, *Introduction to octonion and other non-associative algebras in physics* (Cambridge University press, 1995).
20. S. Okubo and N. Kamiya, Quasi-classical Lie-superalgebra and Lie supertriple systems, University of Rochester report UR–1462 (1996).
21. S. Okubo and N. Kamiya, Jordan–Lie superalgebras and Jordan–Lie triple systems, *J. Alg.* **198** (1997), 388–411.
22. M. Rothstein, Deformations of complex supermanifolds, *Proc. Am. Math. Soc.* **95** (1985), 255–266.
23. M. Scheunert, *The theory of Lie superalgebras* (Springer, Berlin, 1979).
24. S. I. Svinolupov, Generalized Schrödinger equations and Jordan pairs, *Commun. Math. Phys.* **143** (1992), 559–575.
25. K. Yamaguti and A. Ono, On representation of Freudenthal–Kantor triple systems $U(\varepsilon,\delta)$, *Bull. Fac. School, Ed. Hiroshima University, Part II* **7** (1984), 43–51.