Fractal dimension of potential singular points set in the Navier–Stokes equations under supercritical regularity

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The main objective of this paper is to answer the questions posed by Robinson and Sadowski [**22**, p. 505, Commun. Math. Phys., 2010] for the Navier–Stokes equations. Firstly, we prove that the upper box dimension of the potential singular points set $\mathcal{S}$ of suitable weak solution u belonging to $L^q(0, T; L^p(\mathbb{R}^3))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $2 \leqslant q < \infty$ and $2 < p < \infty$ is at most $\max\{p, q\}(\frac{2}{q} + \frac{3}{p} - 1)$ in this system. Secondly, it is shown that $1 - 2s$ dimension Hausdorff measure of potential singular points set of suitable weak solutions satisfying $u \in L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^3))$ for $0 \leqslant s \leqslant \frac{1}{2}$ is zero, whose proof relies on Caffarelli–Silvestre's extension. Inspired by Barker–Wang's recent work [**1**], this further allows us to discuss the Hausdorff dimension of potential singular points set of suitable weak solutions if the gradient of the velocity is under some supercritical regularity.

Keywords: Navier–Stokes equations; suitable weak solutions; box dimension; Hausdorff dimension

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1. Introduction

We consider the three-dimensional incompressible non-stationary Navier–Stokes equations

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla \Pi = 0, & \operatorname{div} u = 0 \text{ in } \mathbb{R}^3 \times (0, T), \\ u|_{t=0} = u_0(x) & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases} \tag{1.1}$$

Here u describes the velocity of the flow and the scalar function Π represents the pressure of the fluid. The initial data $u_0(x)$ satisfies the divergence-free condition.

The full regularity of solutions of the three-dimensional Navier–Stokes equations is not known, the partial regularity theory of suitable weak solutions of this system starting from Scheffer's work [**23**–**25**] is well-known. The famous Caffarelli–Kohn–Nirenberg theorem in [**4**] is that the one-dimensional parabolic Hausdorff measure of the potential space–time singular points set $\mathcal{S}$ of suitable weak solutions to the 3D Navier–Stokes equations is zero. The critical tool is the following so-called ϵ-regularity criterion: there is an absolute constant ϵ such that if

$$\limsup_{\varrho \to 0} \frac{1}{\varrho} \iint_{Q(\varrho)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant \epsilon, \tag{1.2}$$

then u is bounded in a neighbourhood of $(0,0)$, where $Q(\varrho) := B(\varrho) \times (-\varrho^2, 0)$ and $B(\varrho)$ denotes the ball of centre 0 and radius ϱ. To this end, Caffarelli–Kohn–Nirenberg [**4**] established an ϵ-regularity criterion at one scale

$$\|u\|_{L^3(Q(1))} + \|u\Pi\|_{L^1(Q(1))} + \|\Pi\|_{L^{1,5/4}(Q(1))} \leqslant \epsilon. \tag{1.3}$$

An alternative approach of Caffarelli–Kohn–Nirenberg theorem based on blow-up argument was due to Lin, Ladyzhenskaya and Seregin [**17**, **18**], where the corresponding ϵ-regularity criterion at one scale reads

$$\|u\|_{L^3(Q(1))} + \|\Pi\|_{L^{3/2}(Q(1))} \leqslant \epsilon. \tag{1.4}$$

In what follows, a point $z = (x,t)$ in (1.1) is said to be regular if u belongs to L^∞ at a neighbourhood of z. Otherwise, it is called singular. Estimates of the size of potential singular points set in the 3D Navier–Stokes equations can be found in [**1**, **9**, **14**, **15**, **20**–**22**, **30**, **31**].

On the other hand, the integral (Serrin) type conditions based on the velocity, the gradient of the velocity or the pressure lead to the full regularity of Leray–Hopf weak solutions of the 3D Navier–Stokes equations. Precisely, a weak solution u is smooth on $(0, T]$ if it satisfies one of the following three conditions

(1) Serrin [**27**], Struwe [**26**], Escauriaza, Seregin and Šverák [**7**]

$$u \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } 2/p + 3/q = 1, \quad q \geqslant 3. \tag{1.5}$$

(2) Beirao da Veiga [**2**]

$$\nabla u \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } 2/p + 3/q = 2, \quad q > 3/2. \tag{1.6}$$

(3) Berselli and Galdi [**3**], Zhou [**36**, **37**]

$$\Pi \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } 2/p + 3/q = 2, \quad q > 3/2. \tag{1.7}$$

The aforementioned integral (Serrin) type conditions can be seen as the critical regularity, which is scale invariant under the natural scaling of the Navier–Stokes equations (1.1). The full regularity means that the set of $\mathcal{S}$ is empty. The natural (supercritical) regularity $u \in L^q(0, T; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} = \frac{3}{2}$ in suitable weak solutions means that

$$\dim_H(\mathcal{S}) \leqslant 1 \text{ and } \dim_B(\mathcal{S}) \leqslant 5/3, \tag{1.8}$$

which can be found in [**4**, **20**] and $\dim_H(S)$ and $\dim_B(S)$ denote the Hausdorff dimension and box dimension of a set S, respectively. A natural question

is whether the suitable weak solutions satisfying supercritical regularity $u \in L^q(0, T; L^p(\mathbb{R}^3))$ with $1 < \frac{2}{q} + \frac{3}{p} < \frac{3}{2}$ lower the fractal dimension in (1.8). In this direction, Gustafson, Kang and Tsai [**9**] proved that the Hausdorff dimension of the potential singular points set $\mathcal{S}$ of a Leray–Hopf weak solution belonging to $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p}$ with $\frac{2}{p} + \frac{2}{q} < 1$ and $\frac{3}{p} + \frac{1}{q} < 1$ is at most $3 - q + \frac{2q}{p}, p > q$ or $2 - q + \frac{3q}{p}, p \leqslant q$. Robinson and Sadowski [**22**] showed that the upper box dimension of potential singular points set $\mathcal{S}$ of a suitable weak solution belonging to $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $3 < p, q < \infty$ is no greater than

$$\max\{p, q\} \left(\frac{2}{q} + \frac{3}{p} - 1 \right). \tag{1.9}$$

In addition, the Hausdorff dimension of potential singular points set $\mathcal{S}$ of a suitable weak solution belonging to $\nabla u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $2 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{5}{2}$ with $2 < p \leqslant q < \infty$ is less than or equal to

$$\max\{p, q\} \left(\frac{2}{q} + \frac{3}{p} - 2 \right). \tag{1.10}$$

In [**22**, Conclusion, Page 9], Robinson and Sadowski mentioned some natural questions from their results:

(1) It would be interesting to relax the assumption $q > 3$ in (1.9) and obtain the same bound for any $q \geqslant 2$;

(2) Similarly in (1.10) one would like to relax the condition $q \geqslant p$.

(3) In order to obtain (1.9) in a bounded domain we would require the analogue of Lemma 2 (estimates for the pressure when $u \in L^q(0, T; L^p(\Omega))$).

(4) An order of magnitude harder is to determine whether any of these partial regularity results can be proved for general weak solutions, and not only suitable weak solutions.

In this paper, our first result is the following theorem.

THEOREM 1.1. *Let u be a suitable weak solution belonging to $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $2 \leqslant q < \infty$ and $2 < p < \infty$. Then, the upper box dimension of its potential singular points set $\mathcal{S}$ is at most $\max\{p, q\} \left(\frac{2}{q} + \frac{3}{p} - 1 \right)$.*

REMARK 1.2. Theorem 1.1 answers Robinson and Sadowski's first question (1).

As observed in [**9**], the weak solutions in spaces $L^q(0, T; L^p(\mathbb{R}^3))$ with $\frac{2}{p} + \frac{2}{q} < 1$ and $\frac{3}{p} + \frac{1}{q} < 1$ are suitable weak solutions. Therefore, towards the Robinson and Sadowski's fourth question (4), we have

COROLLARY 1.3. *Let u be a Leray–Hopf weak solution belonging to $u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $\frac{2}{p} + \frac{2}{q} < 1$ and $\frac{3}{p} + \frac{1}{q} < 1$. Then, the*

upper box dimension of its potential singular points set $\mathcal{S}$ is at most $\max\{p, q\}\left(\frac{2}{q} + \frac{3}{p} - 1\right)$.

With a slight modification of the proof of theorem 1.1 and using the ϵ-regularity criterion at one scale without pressure in [**33**], we can obtain a parallel result of (1.9) in a bounded domain, which is corresponding to Robinson and Sadowski's third issue.

THEOREM 1.4. *Let u be a suitable weak solution belonging to $u \in L^q(0, T; L^p(\Omega))$ for $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $\frac{5}{2} < q, p < \infty$. Then, the upper box dimension of its interior potential singular points set $\mathcal{S}$ is at most $\max\{p, q\}\left(\frac{2}{q} + \frac{3}{p} - 1\right)$.*

Roughly, the following figures (Figures 1–4) summarize the known upper box dimension of its potential singular points set $\mathcal{S}$ of suitable weak solutions under supercritical regularity in the Navier–Stokes equations.

Next, we study the Robinson and Sadowski's second issue involving the gradient of the velocity with additional regularity. It seems that this problem is more complicated. Very recently, in the other direction, Barker and Wang [**1**] estimate the Hausdorff dimension of the singular set for the Navier–Stokes equations with supercritical assumptions on the pressure. There are two new ingredients in their proof. The first one is the higher integrability of the solutions with certain super-critical assumptions on pressure in the Navier–Stokes equations. The second one is the ϵ-regularity criterion in terms of quantity $|\nabla u|^2|u|^{q-2}$ with $2 < q < 3$, which usually arises in the L^p type energy estimates of the Navier–Stokes equations. In the spirit of [**1**], we consider the ϵ-regularity criterion via quantity $\Lambda^{s+1}u$ with $s > 0$, which usually appears in the $\dot{H}^{s+1}$ type energy estimates of the Navier–Stokes equations. One naturally invokes the Caffarelli–Silvestre extension used in [**6**, **19**, **29**] to overcome non-local derivatives. However, since $s > 0$, one requires higher-order Caffarelli–Silvestre (Yang) extensions [**35**]. To this end, we observe that the following identity due to [**6**], for $\alpha = s + 1 > 1$,

$$c_\alpha \int_{\mathbb{R}^4_+} y^{3-2\alpha}|\nabla^*(\nabla u)^*|^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^3}\left|(-\Delta)^{\frac{\alpha-1}{2}}\nabla u\right|^2(x, t)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^3}|(-\Delta)^{\frac{\alpha}{2}}u|^2(x, t)\,\mathrm{d}x,$$

that is,

$$\|u\|^2_{\dot{H}^{s+1}} = c_s \int_{\mathbb{R}^4_+} y^{1-2s}|\nabla^*(\nabla u)^*|^2(x, y, t)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.11}$$

which helps us to reduce the proof of theorem 1.5 to show theorem 1.6 just by Caffarelli–Silvestre extension rather than higher-order (Yang) extension. Theorem 1.5 can be viewed as the interpolation between the Caffarelli–Kohn–Nirenberg theorem and Kozono-Taniuchi regular class $L^2(0, T; BMO)$, which is of independent interest.

THEOREM 1.5. *Let u be a suitable weak solution belonging to $u \in L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^3))$ for $0 \leqslant s \leqslant \frac{1}{2}$. Then, $\mathcal{H}^{1-2s}(\mathcal{S}) = 0$.*
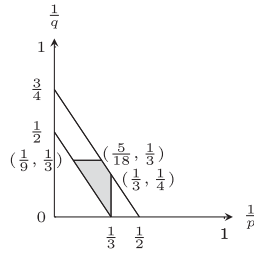
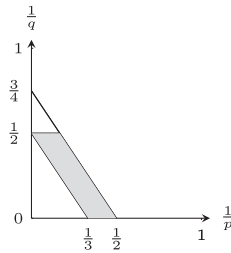Figure 1. Robinson–Sadowski results on $\mathbb{R}^3$.
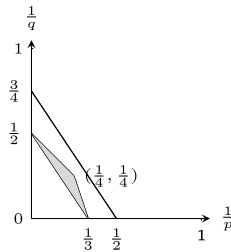

Figure 2. Theorem 1.1 on $\mathbb{R}^3$.

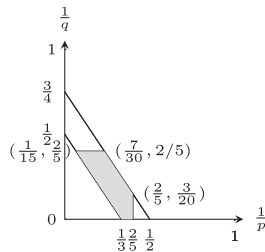Figure 3. Corollary 1.3 weak solutions.

Figure 4. Theorem 1.4 on bounded domain.

PROPOSITION 1.6. *Suppose that u is a suitable weak solution to* (1.1). *Then there exists an absolute positive constant ϵ_{01} such that $(0,0)$ is a regular point if*

$$\limsup_{\mu \to 0} \frac{1}{\mu^{1-2s}} \iint_{Q^*(\mu)} y^{1-2s} |\nabla^*(\nabla u)^*|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \leqslant \epsilon_{01}. \qquad (1.12)$$
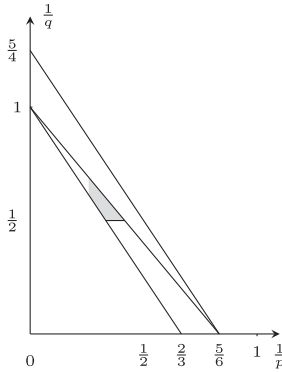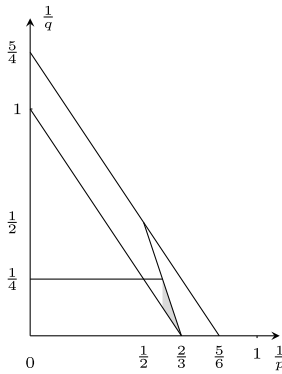
Figure 5. First part of corollary 1.7.

Figure 6. Second part of corollary 1.7.

As an application of theorem 1.5 and the energy estimate of the Navier–Stokes equations, we can partially answer the Robinson and Sadowski's second question.

COROLLARY 1.7. *Let u be a suitable weak solution belonging to* $\nabla u \in L^q(0,T; L^p(\mathbb{R}^3))$ *for* $2 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{5}{2}$.

(1) *If* $\frac{5}{2} - \frac{3}{p} - \frac{5}{2q} \geqslant 0, 2 < p < \frac{54 + 12\sqrt{14}}{25}, 1 < q \leqslant 2$, *then* $\mathcal{H}^{\frac{2\left(\frac{2}{q} + \frac{3}{p} - 2\right)}{1 - \frac{1}{q}}}(\mathcal{S}) = 0$.

(2) *If* $2 - \frac{3}{p} - \frac{1}{q} \geqslant 0, \frac{3}{2} < p < \frac{12}{7}, q \geqslant 4$, *then* $\mathcal{H}^{q\left(\frac{2}{q} + \frac{3}{p} - 2\right)}(\mathcal{S}) = 0$.

Currently, the Hausdorff dimension of suitable weak solutions with the gradient of the velocity under supercritical regularity are summarized in the following figures (Figures 5–7).

The remainder of this paper is organized as follows. In §2, we begin with the notations and the definition of fractal dimension including the Box dimension and Hausdorff dimension. Then we recall the Caffarelli and Silvestre's generalized extension for the fractional Laplacian operator and ϵ-regularity criterion at one scale.
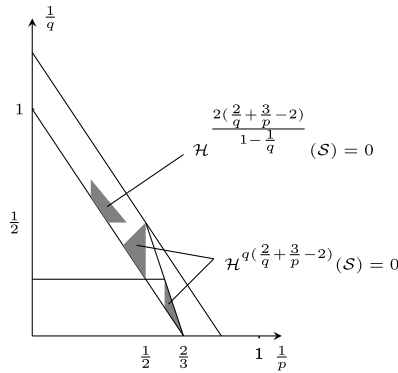
Figure 7. Known Hausdorff dimension of the gradient of the velocity.

Section 3 is devoted to the proof of theorem 1.1 concerning Box dimension. Partial regularity results involving Hausdorff dimension is proved in §4.

2. Preliminaries

First, we introduce some notations used in this paper. Throughout this paper, we denote

$$B(x, \mu) = \{y \in \mathbb{R}^3 \mid |x - y| < \mu\}, \quad B(\mu) := B(0, \mu),$$
$$Q(x, t, \mu) = B(x, \mu) \times (t - \mu^2, t), \quad Q(\mu) := Q(0, 0, \mu),$$
$$B^*(x, \mu) = B(x, \mu) \times (0, \mu), \quad B^*(\mu) := B^*(0, \mu),$$
$$Q^*(x, t, \mu) = B(x, \mu) \times (0, \mu) \times (t - \mu^2, t), \quad Q^*(\mu) := Q^*(0, 0, \mu).$$

For $p \in [1, \infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in X and $\|f(\cdot, t)\|_X$ belonging to $L^p(0, T)$. For simplicity, we write $\|f\|_{L^{p,q}(Q(\mu))} := \|f\|_{L^p(-\mu^2, 0; L^q(B(\mu)))}$ and $\|f\|_{L^p(Q(\mu))} := \|f\|_{L^p L^p(Q(\mu))}$. We shall denote by $\langle f, g \rangle$ the L^2 inner product of f and g. The classical Sobolev norm $\|\cdot\|_{H^s}$ is defined as $\|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 \, d\xi$, $s \in \mathbb{R}$. We denote by $\dot{H}^s$ homogenous Sobolev spaces with the norm $\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi$. Denote the average of f on the ball $B(\mu)$ by $\overline{f}_\mu$. Γ denotes the standard normalized fundamental solution of Laplace equation in $\mathbb{R}^3$. We denote by Div the divergence operator in $\mathbb{R}^{n+1}$ and ∇^* the gradient operator in $\mathbb{R}^{n+1}$. $|S|$ represents the Lebesgue measure of the set S. We will use the summation convention on repeated indices. C is an absolute constant which may be different from line to line unless otherwise stated in this paper.

DEFINITION 2.1. *The (upper) box-counting dimension of a set X is usually defined as*

$$\dim_B(X) = \limsup_{\epsilon \to 0} \frac{\log N(X, \epsilon)}{-\log \epsilon},$$

where $N(X, \epsilon)$ is the minimum number of balls of radius ϵ required to cover X.

Let $\beta > 0$, $\delta > 0$ and $\Omega \times I$ can be covered by the union of series of parabolic balls $Q(r)$ with radius r_j less than δ for $j \in \mathbb{N}$. Define

$$\mathcal{P}^\beta_\delta(\Omega \times I) = \inf \left\{ \sum r_j^\beta \ \Big| \ \Omega \times I \subseteq \cup Q(r_j), \ r_j < \delta, \ j \in \mathbb{N} \right\}$$

and $\mathcal{P}^\beta(\Omega \times I) = \lim_{\delta \to 0} \mathcal{P}^\beta_\delta(\Omega \times I)$. If there is β_0 such that $\mathcal{P}^\beta(\Omega \times I) = \infty$ if $\beta < \beta_0$ and $\mathcal{P}^\beta(\Omega \times I) = 0$ if $\beta > \beta_0$, then β_0 is called as the parabolic Hausdorff dimension and $\mathcal{P}^\beta(\Omega \times I)$ is the parabolic Hausdorff measure. The details of fractal dimension can be found in [8].

Next, we focus on Caffarelli and Silvestre's generalized extension for the fractional Laplacian operator $(-\Delta)^\alpha$ with $0 < \alpha < 1$ in [5]. The fractional power of Laplacian in $\mathbb{R}^3$ can be interpreted as

$$(-\Delta)^\alpha u = -C_\alpha \lim_{y \to 0_+} y^{1-2\alpha} \partial_y u^*,$$

where u^* satisfies

$$\begin{cases} \mathrm{Div}\,(y^{1-2\alpha} \nabla^* u^*) = 0 \, \text{in } \mathbb{R}^4_+, \\ u^*|_{y=0} = u, x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

As a by-product of the above equation, for any $v|_{y=0} = u$, it holds

$$\int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant \int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^* v|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.2}$$

Moreover, from § 3.2 in [5], the definition of the $\dot{H}^\alpha$ norm can be written as

$$\|u\|^2_{\dot{H}^\alpha} = \int_{\mathbb{R}^3} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \, \mathrm{d}\xi = \int_{\mathbb{R}^4_+} y^{1-2\alpha} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.3}$$

We recall the following observation due to [6], for $\alpha > 1$,

$$c_\alpha \int_{\mathbb{R}^4_+} y^{3-2\alpha} |\nabla^*(\nabla u)^*|^2 (x,y,t) \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^3} \left| (-\Delta)^{\frac{\alpha-1}{2}} \nabla u \right|^2 (x,t) \, \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 (x,t) \, \mathrm{d}x.$$

Hence,

$$\|u\|^2_{\dot{H}^{s+1}} = c_s \int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^*(\nabla u)^*|^2 (x,y,t) \, \mathrm{d}x \, \mathrm{d}y. \tag{2.4}$$

Based on the natural scaling of the Navier–Stokes equations, we set the following two dimensionless quantities:

$$E^*_*(\nabla^*(\nabla u)^*; \mu) = \frac{1}{\mu^{1-2s}} \iint_{Q^*(\mu)} y^{1-2s} |\nabla^*(\nabla u)^*|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t, \quad E_*(\nabla u; \mu)$$

$$= \frac{1}{\mu} \iint_{Q(\mu)} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

To make our paper more self-contained and more readable, we outline the proof of the Poincaré inequality concerning Caffarelli and Silvestre's generalized extension.

LEMMA 2.2. *Let u and u^* be defined in* (2.1). *There exist a constant C such that*

$$\|u - \overline{u}_\mu\|_{L^{\frac{6}{3-2s}}(B(\mu/2))} \leqslant C \left(\int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}, \qquad (2.5)$$

$$\|u - \overline{u}_\mu\|_{L^2(B(\mu/2))} \leqslant C\mu^s \left(\int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}. \qquad (2.6)$$

Proof. Consider the usual cut-off functions

$$\eta_1(x) = \begin{cases} 1, & x \in B(\hbar\mu), \ 0 < \hbar < 1, \\ 0, & x \in B^c(\mu), \end{cases}$$

and

$$\eta_2(y) = \begin{cases} 1, & 0 \leqslant y \leqslant \hbar\mu, \\ 0, & y > \mu, \end{cases}$$

satisfying

$$0 \leqslant \eta_1, \eta_2 \leqslant 1, \quad \text{and} \quad \mu|\partial_x \eta_1(x)| + \mu|\partial_y \eta_2(y)| \leqslant C.$$

By the Young inequalities, (2.3) and (2.2), we arrive at

$$\begin{aligned}
\|u\eta_1\|^2_{\dot{H}^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^* (u\eta_1)^*|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\leqslant C \int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^* (u^* \eta_2 \eta_1)|^2 \, \mathrm{d}x \, \mathrm{d}y \\
&\leqslant C\mu^{-2} \int_{B^*(\mu)} y^{1-2s} |u^*|^2 \, \mathrm{d}x \, \mathrm{d}t + C \int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

This guarantees that

$$\begin{aligned}
\|(u - &\overline{u^*}_{B^*(\mu)})\eta_1\|^2_{\dot{H}^s(\mathbb{R}^3)} \\
&\leqslant C\mu^{-2} \int_{B^*(\mu)} y^{1-2s} |u^* - \overline{u^*}_{B^*(\mu)}|^2 \, \mathrm{d}x \, \mathrm{d}t + C \int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y, \quad (2.7)
\end{aligned}$$

where $\overline{u^*}_{B^*(\mu)} = \frac{1}{|B^*(\mu)|} \int_{B^*(\mu)} y^{1-2s} u^* \, \mathrm{d}x \, \mathrm{d}y$ and $|B^*(\mu)| = \int_{B^*(\mu)} y^{1-2s} \, \mathrm{d}y \, \mathrm{d}x$.

Thanks to the classical weighted Poincaré inequality, we infer that

$$\int_{B^*(\mu)} y^{1-2s}|u^* - \overline{u^*}_{B^*(\mu)}|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant C\mu^2 \int_{B^*(\mu)} y^{1-2s}|\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.8}$$

Plugging (2.8) into (2.7), we observe that

$$\|(u - \overline{u^*}_{B^*(\mu)})\eta_1\|_{\dot{H}^s(\mathbb{R}^3)}^2 \leqslant C \int_{B^*(\mu)} y^{1-2s}|\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

This together with the Sobolev embedding yields that

$$\left(\int_{B(\hbar\mu)} |u - \overline{u^*}_{B^*(\mu)}|^{\frac{6}{3-2s}} \, \mathrm{d}x\right)^{\frac{3-2s}{3}} \leqslant \left(\int_{\mathbb{R}^3} |(u - \overline{u^*}_{B^*(\mu)})\eta_1|^{\frac{6}{3-2s}} \, \mathrm{d}x\right)^{\frac{3-2s}{3}}$$

$$\leqslant \|(u - \overline{u^*}_{B^*(\mu)})\eta_1\|_{\dot{H}^s(\mathbb{R}^3)}^2$$

$$\leqslant C \int_{B^*(\mu)} y^{1-2s}|\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y. \tag{2.9}$$

It follows from $u^* = u(x) + \int_0^y \partial_z u^* \, \mathrm{d}z$ and the Hölder inequality that

$$\left|\overline{u^*}_{B^*(\mu)} - \overline{u}_\mu\right| = \frac{1}{|B^*(\mu)|} \left|\int_{B^*(\mu)} y^{1-2s} \int_0^y \partial_z u^* \, \mathrm{d}z\right|$$

$$\leqslant C\frac{1}{|B^*(\mu)|} \int_{B(\mu)} \int_0^\mu y^{1-2s} \left(\int_0^y z^{1-2s}|\partial_z u^*|^2 \, \mathrm{d}z\right)^{1/2}$$

$$\times \left(\int_0^y z^{-(1-2s)} \, \mathrm{d}z\right)^{1/2} \, \mathrm{d}y \, \mathrm{d}x$$

$$\leqslant C\mu^{s-\frac{3}{2}} \left(\int_{B^*(\mu)} z^{1-2s}|\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}z\right)^{1/2}. \tag{2.10}$$

Combining (2.9) with the latter inequality, we deduct that

$$\left(\int_{B(\hbar\mu)} |u - \overline{u}_\mu|^{\frac{6}{3-2s}}\right)^{\frac{3-2s}{6}} \leqslant \left(\int_{B(\hbar\mu)} |u - \overline{u^*}_{B^*(\mu)}|^{\frac{6}{3-2s}}\right)^{\frac{3-2s}{6}}$$

$$+ \left(\int_{B(\hbar\mu)} |\overline{u^*}_{B^*(\mu)} - \overline{u}_\mu|^{\frac{6}{3-2s}}\right)^{\frac{3-2s}{6}}$$

$$\leqslant C \left(\int_{B^*(\mu)} y^{1-2s}|\nabla^* u^*|^2\right)^{\frac{1}{2}},$$

which means (2.5) and (2.6). $\qquad\square$

PROPOSITION 2.3. *([**11**]) Let the pair (u, Π) be a suitable weak solution to the 3D Navier–Stokes system* (1.1) *in $Q(1)$. There exists an absolute positive constant ϵ depending only on p and q such that if the pair (u, Π) satisfies*

$$\|u\|_{L^{q,p}(Q(1))} + \|\Pi\|_{L^1(Q(1))} \leqslant \epsilon, \tag{2.11}$$

for $1 \leqslant 2/q + 3/p < 2, 1 \leqslant p, q \leqslant \infty$, then $u \in L^\infty(Q(1/2))$.

PROPOSITION 2.4. *([**33**]) Let the pair (u, Π) be a suitable weak solution to the 3D Navier–Stokes system* (1.1) *in $Q(1)$. For any $\delta > 0$, there exists an absolute positive constant ϵ such that if u satisfies*

$$\iint_{Q(1)} |u|^{\frac{5}{2}+\delta} \, \mathrm{d}x \, \mathrm{d}t \leqslant \epsilon, \tag{2.12}$$

then $u \in L^\infty(Q(1/16))$.

For the generalization of the ϵ-regularity criterion (2.12) at one scale without pressure, the reader may refer to recent work [**16**] by Kwon. Next, we recall the Leibniz rule for fractional derivatives and product estimates for the fractional Laplacian.

LEMMA 2.5. *([**12**]) Let $\alpha > 0$, $p \in (1, \infty)$ and $p_i \in (1, \infty)$, $i = 1, 2, 3, 4$. Then there exists a positive constant C such that*

$$\|\Lambda^\alpha(fg) - f\Lambda^\alpha g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{\alpha-1} g\|_{L^{p_2}} + \|\Lambda^\alpha f\|_{L^{p_3}} \|g\|_{L^{p_4}}) \tag{2.13}$$

and

$$\|\Lambda^\alpha(fg)\|_{L^p} \leqslant C(\|\Lambda^\alpha f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|\Lambda^\alpha g\|_{L^{p_4}}), \tag{2.14}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

For the convenience of readers, we present the known fractional Gagliardo–Nirenberg inequality at the end of this section.

PROPOSITION 2.6. *([**10**, **32**, **34**]) Suppose that $u \in L^q(\mathbb{R}^n)$ and $\Lambda^s u \in L^r(\mathbb{R}^n)$. Let $0 \leqslant \sigma < s < \infty$ and $1 < q, r \leqslant \infty$. Then there exists a positive constant $C = C(n, q, p, r, s, \sigma)$ such that*

$$\|\Lambda^\sigma u\|_{L^p(\mathbb{R}^n)} \leqslant C\|u\|_{L^q(\mathbb{R}^n)}^\theta \|\Lambda^s u\|_{L^r(\mathbb{R}^n)}^{1-\theta} \tag{2.15}$$

with

$$\frac{n}{p} - \sigma = \theta\frac{n}{q} + (1 - \theta)\left(\frac{n}{r} - s\right), \tag{2.16}$$

where $0 < \theta < 1 - \frac{\sigma}{s}$ and $s - \frac{n}{r} \neq \sigma - \frac{n}{p}$.

3. Box dimension of possible singular points set of suitable weak solutions

This section contains the proof of theorem 1.1, corollary 1.3 and theorem 1.4. The key point is an application of the ϵ-regularity criteria (2.11) and (2.12) at one scale.

Proof of theorem 1.1. We present the proof by contradiction. We suppose that $\dim_B(\mathcal{S}) > \max\{p, q\}(\frac{2}{q} + \frac{3}{p} - 1)$. We pick a constant α such that $\alpha_0 = \max\{p, q\}\left(\frac{2}{q} + \frac{3}{p} - 1\right) < \alpha < \dim_B(\mathcal{S})$. Therefore, using the definition of the box dimension, we know that there exists a sequence $\delta_j \to 0$ such that $N(\mathcal{S}, \delta_j) > \delta_j^{-\alpha}$. We assume that $(x_i, t_i)_{i=1}^{N(\mathcal{S}, \delta_j)}$ be a collection of δ_j- separated points in $\mathcal{S}$. By the regularity criterion in proposition 2.3, for any $(x_i, t_i) \in \mathcal{S}$, we get

$$\int_{t_i - \delta_j^2}^{t_i} \left[\left(\int_{B_i(\delta_j)} |u|^p \, dx \right)^{\frac{q}{p}} + \left(\int_{B_i(\delta_j)} |\Pi|^{p/2} \, dx \right)^{\frac{q}{p}} \right] dt > \delta_j^{(-p+3)\frac{q}{p}+2} \varepsilon_1,$$

where $B_i(\mu) := B(x_i, \mu)$. Thus we have

$$\sum_{i=1}^{N(\mathcal{S}, \delta_j)} \int_{t_i - \delta_j^2}^{t_i} \left[\left(\int_{B_i(\delta_j)} |u|^p \, dx \right)^{\frac{q}{p}} + \left(\int_{B_i(\delta_j)} |\Pi|^{p/2} \, dx \right)^{\frac{q}{p}} \right] dt$$
$$> N(\mathcal{S}, \delta_j) \delta_j^{(-p+3)\frac{q}{p}+2} \varepsilon_1. \tag{3.1}$$

The pressure equation helps us to obtain, for $p > 2$, $q \geqslant 2$,

$$\|\Pi\|_{L^{q/2}(0,T;L^{p/2}(\mathbb{R}^3))}^{p/2} \leqslant C \|u\|_{L^q(0,T;L^p(\mathbb{R}^3))}^p. \tag{3.2}$$

For the case $\frac{q}{p} > 1$, we know that $\alpha_0 = q\left(\frac{2}{q} + \frac{3}{p} - 1\right)$.

Now, we can apply the inequality $\sum_{i=1}^{N(\mathcal{S}, \delta_j)} (a_i)^{\frac{q}{p}} \leqslant (\sum_{i=1}^{N(\mathcal{S}, \delta_j)} a_i)^{\frac{q}{p}}$ to control the left-hand side of (3.1) by $\|u\|_{L^q(0,T;L^p(\mathbb{R}^3))}^q + \|\Pi\|_{L^{q/2}(0,T;L^{p/2}(\mathbb{R}^3))}^{q/2}$. This together with (3.2) implies that

$$C \geqslant N(\mathcal{S}, \delta_j) \delta_j^{(-p+3)\frac{q}{p}+2} \varepsilon_1 \geqslant \delta_j^{(-p+3)\frac{q}{p}+2-\alpha} \varepsilon_1. \tag{3.3}$$

We immediately get a contradiction as $j \to \infty$.

For the rest case $\frac{q}{p} \leqslant 1$, we invoke the inequality $\sum_{i=1}^{N(\mathcal{S}, \delta_j)} (a_i)^{\frac{q}{p}} \leqslant N^{(1-\frac{q}{p})}(\mathcal{S}, \delta_j)(\sum_{i=1}^{N(\mathcal{S}, \delta_j)} a_i)^{\frac{q}{p}}$ in the proof. With a slight modification of the above proof, we see that $C \geqslant N^{\frac{q}{p}}(\mathcal{S}, \delta_j) \delta_j^{(-p+3)\frac{q}{p}+2} \varepsilon_1$. This means that

$$C \geqslant N(\mathcal{S}, \delta_j) \delta_j^{\frac{p}{q}[(-p+3)\frac{q}{p}+2]} \varepsilon_1 \geqslant \delta_j^{\frac{p}{q}[(-p+3)\frac{q}{p}+2]-\alpha} \varepsilon_1. \tag{3.4}$$

This led a contradiction as $j \to \infty$. The proof of this theorem is achieved. $\qquad \square$

Proof of corollary 1.3. It follows from $1 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{3}{2}$ with $\frac{2}{p} + \frac{2}{q} < 1$ and $\frac{3}{p} + \frac{1}{q} < 1$ that $u \in L^4(0, T; L^4(\mathbb{R}^n))$. Thanks to the work [**28**], we observe that u is a suitable weak solution. Following the path of the above proof, we complete the proof. $\square$

Proof of theorem 1.4. As the same manner of proof of theorem 1.1 and replacing the application of the regularity criterion (2.11) by (2.12), the proof of this theorem is completed. $\square$

4. Hausdorff dimension of possible singular points set of suitable weak solutions

First, we prove proposition 1.6. As an application of this proposition, we can achieve the proof of corollary 1.7. To this end, we prove the following lemma, which roughly indicates that the smallness of $\nabla^*(\nabla u)^*$ yields the smallness of ∇u.

LEMMA 4.1. *For $0 < \mu \leqslant \frac{1}{2}\rho$, there is an absolute constant C independent of μ and ρ, such that*

$$E_*(\nabla u; \mu) \leqslant C\left(\frac{\rho}{\mu}\right) E_*^*(\nabla^*(\nabla u)^*; \rho) + C\left(\frac{\mu}{\rho}\right)^2 E_*(\nabla u; \rho).$$

Proof. With the help of the triangle inequality, the Hölder inequality and (2.6), we see that

$$\begin{aligned}
\int_{B(\mu)} |u|^2 \, \mathrm{d}x &\leqslant C \int_{B(\mu)} |u - \bar{u}_\rho|^2 + C \int_{B(\mu)} |\bar{u}_\rho|^2 \\
&\leqslant C\left(\int_{B\left(\frac{\rho}{2}\right)} |u - \bar{u}_\rho|^2\right) + C\frac{\mu^3}{\rho^3}\left(\int_{B(\rho)} |u|^2\right) \\
&\leqslant C\rho^{2s}\left(\int_{B^*(\rho)} y^{1-2s}|\nabla^* u^*|^2 \, \mathrm{d}x \, \mathrm{d}y\right) + C\frac{\mu^3}{\rho^3}\left(\int_{B(\rho)} |u|^2\right), \quad (4.1)
\end{aligned}$$

that is,

$$\int_{B(\mu)} |\nabla u|^2 \, \mathrm{d}x \leqslant C\rho^{2s}\left(\int_{B^*(\rho)} y^{1-2s}|\nabla^*(\nabla u)^*|^2 \, \mathrm{d}x \, \mathrm{d}y\right) + C\frac{\mu^3}{\rho^3}\left(\int_{B(\rho)} |\nabla u|^2\right).$$

Integrating in time on $(-\mu^2, 0)$ this inequality, we obtain

$$\iint_{Q(\mu)} |\nabla u|^2 \, \mathrm{d}x \leqslant C\rho^{2s}\left(\iint_{Q^*(\rho)} y^{1-2s}|\nabla^*(\nabla u)^*|^2 \, \mathrm{d}x \, \mathrm{d}y\right) + C\frac{\mu^3}{\rho^3}\left(\iint_{Q(\rho)} |\nabla u|^2\right),$$

which leads to

$$E_*(\nabla u; \mu) \leqslant C\left(\frac{\rho}{\mu}\right) E_*^*(\nabla^*(\nabla u)^*; \rho) + C\left(\frac{\mu}{\rho}\right)^2 E_*(\nabla u; \rho).$$

This achieves the proof of this lemma. $\square$

Proof of proposition 1.6. From (1.12), we know that there exists a constant ρ_1 such that

$$E_*^*(\nabla^*(\nabla u)^*; \rho) \leqslant \epsilon_{01}, \text{for all } 0 < \rho \leqslant \rho_1.$$

Combining this with lemma 4.1, we conclude that, for $0 < \rho \leqslant \rho_1$,

$$E_*(\nabla u; \mu) \leqslant C\left(\frac{\rho}{\mu}\right) E_*^*(\nabla^*(\nabla u)^*; \rho) + C\left(\frac{\mu}{\rho}\right)^2 E_*(\nabla u; \rho)$$

$$\leqslant C_1\left(\frac{\rho}{\mu}\right) \epsilon_{01} + C_1\left(\frac{\mu}{\rho}\right)^2 E_*(\nabla u; \rho).$$

Before going further, we set $\lambda = \frac{\mu}{\rho} \leqslant \frac{1}{2}$. Hence, there holds

$$E_*(\nabla u; \lambda\rho) \leqslant C_1\lambda^{-1}\epsilon_{01} + C_1\lambda^2 E_*(\nabla u; \rho).$$

Choosing λ sufficiently small such that $q = C_1\lambda^2 < 1$ and taking ϵ_{01} such that $C_1\lambda^{-1}\epsilon_{01} \leqslant \frac{(1-q)\epsilon\lambda}{2}$, we obtain

$$E_*(\nabla u; \lambda\rho) \leqslant \frac{(1-q)\lambda}{2}\epsilon + qE_*(\nabla u; \rho).$$

Iterating this inequality, we infer that

$$E_*(\nabla u; \lambda^k\rho) \leqslant \frac{\lambda}{2}\epsilon + q^k E_*(\nabla u; \rho). \tag{4.2}$$

Based on the definition of $E_*(\nabla u; \rho)$, there is a positive sufficiently large number K_0 such that

$$q^{K_0} E_*(\nabla u; \rho_1) \leqslant q^{K_0}\frac{C\|\nabla u\|_{L_{t,x}^2}^2}{\rho_1} \leqslant \frac{\lambda\epsilon}{2}. \tag{4.3}$$

We write $\rho_2 = \lambda^{K_0}\rho_1$, then, for all $0 < \rho \leqslant \rho_2$, there is a positive constant $k \geqslant K_0$ such that $\lambda^{k+1}\rho_1 \leqslant \rho \leqslant \lambda^k\rho_1$ and

$$E_*(\nabla u; \rho) \leqslant \frac{1}{\lambda^{k+1}\rho_1}\iint_{Q(\lambda^k\rho_1)}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \qquad = \frac{1}{\lambda}E_*(\nabla u; \lambda^k\rho_1)$$

$$\leqslant \frac{1}{\lambda}\left[q^k E_*(\nabla u; \rho_1) + \frac{1}{2}\lambda\epsilon\right]$$

$$\leqslant \epsilon,$$

where (4.2) and (4.3) were used.

Finally, the famous ϵ-regularity criterion (1.2) helps us to finish the proof of this proposition.   □

Now we are in a position to complete the proof of theorem 1.5.

Proof of theorem 1.5. For the case $s = 0$, we complete the proof by the Caffarelli–Kohn–Nirenberg theorem in [**4**]. For the other borderline case $s = 1/2$, by

the fact $\dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow BMO$ and the Serrin class $L^2(0, T; BMO)$ due to Kozono and Taniuchi [**13**], we know there is no singular point in the weak solutions of the Navier–Stokes equations. Hence, we achieve the proof of two borderline cases. For the rest cases, from (2.4), we derive from $u \in L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^3))$ with $0 < s < \frac{1}{2}$ that

$$\int \int_{\mathbb{R}^4_+} y^{1-2s} |\nabla^*(\nabla u)^*|^2(x, y, t) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t < +\infty.$$

At this stage, the Vitali covering lemma used in [**4**] together with proposition 1.6 yields that $1 - 2s$ dimension of potential singular points set of suitable weak solutions satisfying $u \in L^2(0, T; \dot{H}^{s+1}(\mathbb{R}^3))$ for $0 < s < \frac{1}{2}$ is zero. The process is standard, hence, we omit the detail here.

In summary, the desired result is derived. □

The proof of corollary 1.7 is a consequence of the following two lemmas.

LEMMA 4.2. *Let $\nabla u \in L^q(0, T; L^p(\mathbb{R}^3))$ for $2 \leqslant \frac{2}{q} + \frac{3}{p} \leqslant \frac{5}{2}$ with $\frac{5}{2} - \frac{3}{p} - \frac{5}{2q} \geqslant 0, 2 < p < \frac{54+12\sqrt{14}}{25}, 1 < q \leqslant 2$. Then*

$$u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; \dot{H}^{1+s}(\mathbb{R}^3)),$$

where $0 \leqslant s = \frac{\frac{5}{2} - \frac{3}{p} - \frac{5}{2q}}{1 - \frac{1}{q}} \leqslant \frac{1}{2}$.

Proof. The incompressible condition allow us to get

$$\langle u \cdot \nabla \Lambda^s u, \Lambda^s u \rangle = 0. \tag{4.4}$$

Multiplying the Navier–Stokes equations with $\Lambda^{2s} u$, using the divergence-free condition and (4.4), we know that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^s u\|^2_{L^2(\mathbb{R}^3)} + \|\Lambda^{s+1} u\|^2_{L^2(\mathbb{R}^3)} = -\langle \Lambda^s(u \cdot \nabla u) - u \cdot \nabla(\Lambda^s u), \Lambda^s u \rangle.$$

The Hölder inequality guarantees that

$$|\langle \Lambda^s(u \cdot \nabla u) - u \cdot \nabla(\Lambda^s u), \Lambda^s u \rangle| \leqslant \|\Lambda^s(u \cdot \nabla u) - u \cdot \nabla(\Lambda^s u)\|_{L^2(\mathbb{R}^3)} \|\Lambda^s u\|_{L^2(\mathbb{R}^3)}.$$

By means of the Leibniz rule for fractional derivatives (2.13), we infer that

$$\|\Lambda^s(u \cdot \nabla u) - u \cdot \nabla(\Lambda^s u)\|_{L^2(\mathbb{R}^3)} \leqslant C \|\nabla u\|_{L^p(\mathbb{R}^3)} \|\Lambda^s u\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^3)}, p > 2.$$

Consequently, we arrive at

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^s u\|^2_{L^2(\mathbb{R}^3)} + \|\Lambda^{s+1} u\|^2_{L^2(\mathbb{R}^3)} \leqslant C \|\nabla u\|_{L^p(\mathbb{R}^3)} \|\Lambda^s u\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^3)} \|\Lambda^s u\|_{L^2(\mathbb{R}^3)}. \tag{4.5}$$

We conclude by the fractional Gagliardo–Nirenberg inequality (2.15) and the Sobolev inequality that,

$$\|\Lambda^s u\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^3)} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{\frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}}} \|u\|_{L^{\frac{3}{2}-s}(\mathbb{R}^3)}^{\frac{\frac{5}{2}-s-\frac{6}{p}}{\frac{5}{2}-s-\frac{3}{p}}} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{\frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}}} \|\Lambda^s u\|_{L^2(\mathbb{R}^3)}^{\frac{\frac{5}{2}-s-\frac{6}{p}}{\frac{5}{2}-s-\frac{3}{p}}}, \quad (4.6)$$

where we require

$$0 \leqslant \frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}} \leqslant 1, \quad \frac{5}{2}-s-\frac{3}{p} > 0 \quad \text{and} \quad s \leqslant \frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}}.$$

Indeed, in the light of the definition of s and $1 < q \leqslant 2$, we observe that $\frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}} \leqslant 1$. In addition, taking advantage of the definition of s again, we know that $\frac{5}{2}-s-\frac{3}{p} > 0$. Some straightforward computations yield that $\frac{9-\sqrt{56}}{6} < \frac{1}{p} < \frac{9+\sqrt{56}}{6}$ guarantees that $s \leqslant \frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}}$.

Inserting (4.6) into (4.5), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u\|_{L^2(\mathbb{R}^3)}^2 + \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}^2 \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{\frac{\frac{3}{p}}{\frac{5}{2}-s-\frac{3}{p}}+1} \|\Lambda^s u\|_{L^2(\mathbb{R}^3)}^{\frac{\frac{5}{2}-s-\frac{6}{p}}{\frac{5}{2}-s-\frac{3}{p}}+1}$$

$$\leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{q} \|\Lambda^s u\|_{L^2(\mathbb{R}^3)}^{\frac{\frac{5}{2}-s-\frac{6}{p}}{\frac{5}{2}-s-\frac{3}{p}}+1}. \quad (4.7)$$

Thanks to $\frac{\frac{5}{2}-s-\frac{6}{p}}{\frac{5}{2}-s-\frac{3}{p}} \leqslant 1$, we derive from (4.7) and $\nabla u \in L^q(0,T;L^p(\mathbb{R}^3))$ for $2 \leqslant \frac{2}{q}+\frac{3}{p} \leqslant \frac{5}{2}$ that $u \in L^2(0,T;\dot{H}^{1+s}(\mathbb{R}^3))$. □

LEMMA 4.3. *Let u be a suitable weak solution belonging to $\nabla u \in L^q(0,T;L^p(\mathbb{R}^3))$ for $2 \leqslant \frac{2}{q}+\frac{3}{p} \leqslant \frac{5}{2}$ with $2-\frac{3}{p}-\frac{1}{q} \geqslant 0, \frac{3}{2} < p < \frac{12}{7}, q \geqslant 4$. Then*

$$u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^{1+s}(\mathbb{R}^3)),$$

where $0 \leqslant s = \frac{2-\frac{3}{p}-\frac{1}{q}}{\frac{2}{q}} \leqslant \frac{1}{2}$.

Proof. In view of the standard energy estimate, the integration by parts and the incompressible condition, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u\|_{L^2(\mathbb{R}^3)}^2 + \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}^2 = \langle \Lambda^s(u_j u_i), \Lambda^{s+1}u_i \rangle.$$

It follows from the Hölder inequality that

$$|\langle \Lambda^s(u_j u_i), \Lambda^{s+1}u_i \rangle| \leqslant \|\Lambda^s(u_j u_i)\|_{L^2(\mathbb{R}^3)}\|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}.$$

We deduce from the product estimates for the fractional Laplacian (2.14) and the Sobolev embedding that

$$\|\Lambda^s(u_j u_i)\|_{L^2(\mathbb{R}^3)} \leqslant C\|\Lambda^s u\|_{L^{\frac{6p}{5p-6}}(\mathbb{R}^3)} \|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)}$$

$$\leqslant C\|\Lambda^s u\|_{L^{\frac{6p}{5p-6}}(\mathbb{R}^3)} \|\nabla u\|_{L^p(\mathbb{R}^3)}, \ \frac{6}{5} < p < 3.$$

Combining the above estimates together, we observe that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|^2_{L^2(\mathbb{R}^3)} + \|\Lambda^{s+1}u\|^2_{L^2(\mathbb{R}^3)} \leqslant C\|\Lambda^s u\|_{L^{\frac{6p}{5p-6}}(\mathbb{R}^3)} \|\nabla u\|_{L^p(\mathbb{R}^3)} \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}. \tag{4.8}$$

According to the fractional Gagliardo–Nirenberg inequality (2.15) and the Sobolev inequality, we discover that

$$\|\Lambda^s u\|_{L^{\frac{6p}{5p-6}}(\mathbb{R}^3)} \leqslant C\|u\|_{L^{\frac{3p}{3-p}}(\mathbb{R}^3)}^{\frac{2-\frac{3}{p}}{s-\frac{3}{2}+\frac{3}{p}}} \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}^{\frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{\frac{2-\frac{3}{p}}{s-\frac{3}{2}+\frac{3}{p}}} \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}^{\frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}}, \tag{4.9}$$

where we need $p \geqslant \frac{3}{2}, 0 \leqslant \frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}} \leqslant 1$, $s - \frac{3}{2} + \frac{3}{p} > 0$ and $\frac{s}{s+1} < \frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}$.

On one hand, we can examine $\frac{2-\frac{3}{p}}{s-\frac{3}{2}+\frac{3}{p}} \leqslant 1$ via $q \geqslant 4$ and $2 > p \geqslant \frac{3}{2}$. On the other, direct calculation ensures that $q > 2, p > \frac{3}{2}$ yields that $s - \frac{3}{2} + \frac{3}{p} > 0$. Moreover, $\frac{3}{2} < p < \frac{12}{7}$ means $\frac{s}{s+1} < \frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}$.

Inserting (4.9) into (4.8), we find

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s u\|^2_{L^2(\mathbb{R}^3)} + \|\Lambda^{s+1}u\|^2_{L^2(\mathbb{R}^3)} \leqslant C\|\nabla u\|_{L^p(\mathbb{R}^3)}^{\frac{s+\frac{1}{2}}{s-\frac{3}{2}+\frac{3}{p}}} \|\Lambda^{s+1}u\|_{L^2(\mathbb{R}^3)}^{\frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}+1},$$

which implies that

$$\|\Lambda^s u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \|\Lambda^{s+1}u\|^2_{L^2(0,T;L^2(\mathbb{R}^3))}$$
$$\leqslant C_0 + C\|\nabla u\|_{L^{\frac{2s+1}{2-\frac{3}{p}}}(0,T;L^p(\mathbb{R}^3))}^{\frac{s+\frac{1}{2}}{s-\frac{3}{2}+\frac{3}{p}}} \|\Lambda^{s+1}u\|_{L^2(0,T;L^2(\mathbb{R}^3))}^{\frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}}+1}.$$

We deduce from $p > \frac{3}{2}$ that $\frac{s-\frac{7}{2}+\frac{6}{p}}{s-\frac{3}{2}+\frac{3}{p}} + 1 < 2$. Hence, the Young inequality further allows us to get that

$$\|\Lambda^s u\|^2_{L^\infty(0,T;L^2(\mathbb{R}^3))} + \frac{1}{2}\|\Lambda^{s+1}u\|^2_{L^2(0,T;L^2(\mathbb{R}^3))} \leqslant C_0 + C\|\nabla u\|_{L^{\frac{2s+1}{2-\frac{3}{p}}}(0,T;L^p(\mathbb{R}^3))}^{\frac{2s+1}{s-\frac{3}{p}}}.$$

The proof of this lemma is completed. $\qquad\square$

Proof of corollary 1.7. Combining lemma 4.2, lemma 4.3 and theorem 1.5, we immediately finish the proof of corollary 1.7. □

Acknowledgements

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