MONOTONE FUNCTIONS ON LINEAR LATTICES

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1. Introduction. If R is a sequentially continuous linear lattice, a function $f(x)$, defined on $R^+ = \{x: 0 \leqslant x \in R\}$ with $0 \leqslant f(x) \leqslant +\infty$, will be called a monotone function if it satisfies

(M1) $\qquad f(x) \leqslant f(y) \qquad$ when $x \leqslant y$,

(M2) $\qquad f(x) = \sup_i f(x_i) \qquad$ when $0 \leqslant x_i \uparrow_{i=1}^{\infty} x$.

Monotone functions, subject to additional conditions, have been studied at length. For example a length function **(2)** is a monotone function on the sequentially continuous linear lattice of equivalence classes of measurable functions. On L^p, $1 \leqslant p < \infty$, the norms are monotone functions as are the positive continuous linear functionals. A modular on a universally continuous linear lattice is a monotone function **(3)**.

In the present paper we study the problems of extension of monotone functions from semi-normal manifolds of a sequentially continuous linear lattice R to R itself.

2. Definitions and notation. A semi-ordered vector space R in which each pair of elements x, y has a supremum $x \cup y$ and an infimum $x \cap y$ in R is called a *linear lattice* or *Riesz space.*

A linear lattice R is: (i) *sequentially continuous* (called continuous in **(3)**) if $a_n \in R^+$, $n = 1, 2, \ldots$, implies

$$\bigcap_{n=1}^{\infty} a_n \in R;$$

(ii) *universally continuous* if for every collection $a_\lambda \in R^+$ $(\lambda \in \Lambda)$, $\bigcap_\lambda a_\lambda \in R$; (iii) *superuniversally continuous* if it is universally continuous and if $a_\lambda \in R^+$ $(\lambda \in \Lambda)$ implies the existence of a subsequence a_{λ_i}, $i = 1, 2, \ldots$, such that

$$\bigcap_{i=1}^{\infty} a_{\lambda_i} = \bigcap_{\lambda \in \Lambda} a_\lambda .$$

A monotone function $f(x)$ is *convex* (*concave*) if

$$f(\alpha x + \beta y) \leqslant \alpha f(x) + \beta f(y) \qquad (f(\alpha x + \beta y) \geqslant \alpha f(x) + \beta f(y))$$

for $\alpha + \beta = 1$, $\alpha, \beta \geqslant 0$.

A monotone function is *linear* if $f(x + y) = f(x) + f(y)$, *sublinear* if $f(x + y) \leqslant f(x) + f(y)$, *superlinear* if $f(x + y) \geqslant f(x) + f(y)$; *homogeneous* if $f(\alpha x) = \alpha f(x)$ for $\alpha \geqslant 0$; and *additive* if $x \perp y$ implies that $f(x + y) = f(x) + f(y)$.

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A monotone function that is linear is homogeneous. The proof uses only (M1) (**1**, Proposition 1, p. 33). Sublinear and convex (superlinear and concave) coincide if and only if f is homogeneous. Sublinear implies that $f(nx) \leqslant nf(x)$, $n = 1, 2, \ldots$. Convex with $y = 0$, $\alpha = 1/n$, $\beta = (n-1)/n$ implies that $f(nx) \geqslant nf(x)$, $n = 1, 2, \ldots$. If f is both sublinear and convex it follows that $f(nx) = nf(x)$, $n = 1, 2, \ldots$, then for n replaced by an arbitrary rational number, and finally, using (M2), for arbitrary $\alpha > 0$.

A monotone function that is homogeneous and convex or sublinear will be called a *length function*. A monotone function that is convex and additive will be called *modular*.

A linear subset S of a linear lattice R will be called a semi-normal manifold of R if $x \in S$, $|a| \leqslant |x|$ implies that $a \in S$. In the terminology of Birkhoff this is an l-ideal. S is itself a linear lattice for the ordering induced by R. If R is sequentially continuous, universally continuous, or superuniversally continuous, so is S.

3. Extensions of monotone functions. If S is a semi-normal manifold of a sequentially continuous linear lattice R, a monotone function f defined on R^+ will be called an extension of a monotone function g defined on S^+ if $f(x) = g(x)$ in S^+.

THEOREM 3.1. *Let R be a sequentially continuous linear lattice, S a semi-normal manifold of R. Then each monotone function f on S has unique maximal and minimal extension f_M and f_m to R such that, for an arbitrary extension g of f from S to R,*

$$f_m(x) \leqslant g(x) \leqslant f_M(x)$$

for all x in R^+. If $f(x)$ is convex (concave, homogeneous, linear, sublinear, superlinear, or additive) so are f_M and f_m. Thus, if f is a length function (or modular) on S, f_M and f_m are length functions (modular) on R.

If $a \in R^+$ and there exists

$$x_i \uparrow_{i=1}^{\infty} a \quad (\text{that is, } 0 \leqslant x_1 \leqslant x_2 \ldots;\ \bigcup_{i=1}^{\infty} x_i = a) \quad x_i \in S^+,$$

then if (M2) is to hold for f_M and f_m we must define

$$f_M(a) = f_m(a) = \lim_{i\to\infty} f(x_i) \leqslant +\infty.$$

If $a \in R^+$ and there exists no sequence

$$x_i \uparrow_{i=1}^{\infty} a, \qquad x_i \in S^+,$$

we define

$$f_M(a) = +\infty,$$
$$f_m(a) = \sup_{\substack{x \leqslant a \\ x \in S^+}} f(x).$$

Let $\bar{S}$ denote the set of elements of R such that $x \in \bar{S}$ implies the existence of a sequence $x_i \in S^+$, $i = 1, 2, \ldots,$ with

$$x_i \uparrow_{i=1}^{\infty} |x|.$$

It is easily verified that $\bar{S}$ is a semi-normal manifold of R.

We next show that for $a \in \bar{S}^+$, $f_M(a)$ and $f_m(a)$ depend only on a and not on the choice of sequence $\{x_i\}$,

$$x_i \uparrow_{i=1}^{\infty} a.$$

LEMMA 3.1. *Let*

$$a \in \bar{S}^+, \qquad x_i \uparrow_{i=1}^{\infty} a, \qquad y_j \uparrow_{j=1}^{\infty} a, \qquad x_i, y_j \in S^+, \qquad i, j = 1, 2, \ldots .$$

Then

$$\lim_{i\to\infty} f(x_i) = \lim_{j\to\infty} f(y_j).$$

Proof of Lemma. Fix j. Since S is semi-normal, $x_i \cap y_j \in S^+$, $i = 1, 2, \ldots .$ Since $y_j \leqslant a$,

$$x_i \uparrow_{i=1}^{\infty} a, \qquad x_i \cap y_j \uparrow_{i=1}^{\infty} y_j$$

so that, by (M2) for S,

$$\lim_{i\to\infty} f(x_i \cap y_j) = f(y_j).$$

Thus

$$\lim_{i\to\infty} f(x_i) \geqslant \lim_{i\to\infty} f(x_i \cap y_j) = f(y_j), \qquad j = 1, 2, \ldots,$$

$$\lim_{i\to\infty} f(x_i) \geqslant \lim_{j\to\infty} f(y_j).$$

Reversing the roles of x and y leads to equality.

COROLLARY. *f_M and f_m are uniquely defined on R^+ and coincide on $\bar{S}^+$.*

We next prove that f_m is a monotone function on R^+. For $x \leqslant y$ we have

$$f_m(x) = \sup_{x \geqslant a \in S^+} f(a) \leqslant \sup_{y \geqslant a \in S^+} f(a) = f_m(y),$$

that is (M1) holds.

If

$$x_i \uparrow_{i=1}^{\infty} x,$$

(M1) implies that

$$f_m(x) \geqslant \lim_{i\to\infty} f_m(x_i).$$

For $x \geqslant a \in S^+$ we have

$$a \cap x_i \in S^+, \qquad a \cap x_i \uparrow_{i=1}^{\infty} a$$

and hence

$$\lim_{i\to\infty} f_m(x_i) \geqslant \lim_{i\to\infty} f(x_i \cap a) = f(a).$$

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Thus

$$\lim_{i\to\infty} f_m(x_i) \geqslant \sup_{x \geqslant a \in S^+} f(a) = f_m(x)$$

and (M2) holds. It follows from the corollary that f_m and f_M define the same monotone function on $\bar{S}^+$.

We prove that f_M is a monotone function on R^+. (M1) is trivially true for f_M when $y \notin \bar{S}^+$, and is therefore true for f_M on R^+. (M2) is trivially true if $x_i \notin \bar{S}^+$ for some i. We complete the proof by showing that if $x_i \in \bar{S}^+$, $i = 1, 2, \ldots$, and

$$x = \bigcup_{i=1}^{\infty} x_i \in R,$$

then $x \in \bar{S}^+$.

There exist sequences

$$x_{ij} \in S^+, \qquad x_{ij} \uparrow_{j=1}^{\infty} x_i, \qquad j = 1, 2, \ldots,$$

and

$$x = \bigcup_{i=1}^{\infty} x_i = \bigcup_{i,j} x_{ij}.$$

With

$$\bar{x}_n = \bigcup_{j=1}^{n} x_{nj} \in S^+,$$

we have

$$\bar{x}_n \uparrow_{n=1}^{\infty} \bigcup_{i,j} x_{ij} = x$$

and $x \in \bar{S}$.

Let g be an arbitrary monotone function extending f on S to R. Since g and f_M coincide in $\bar{S}^+$ and $f_M = +\infty$ in $R^+ - \bar{S}^+$, $g(x) \leqslant f_M(x)$ for all $x \in R^+$. For every $a \in S^+$ with $a \leqslant x$, (M1) implies that $g(x) \geqslant g(a) = f(a)$, whence $g(x) \geqslant f_m(x)$ for all x in R^+.

It is not difficult to show that if f on S has one of the additional properties in the theorem, then f_M and f_m have the same property on R. We give a proof for f sublinear.

If $x, y \in R^+$, $x + y \in R^+$. For $a \leqslant x + y$, $a \in S^+$, $a = a \cap (x + y) \leqslant a \cap x + a \cap y$ (**3**, 10, p. 10). Thus

$$f_m(x + y) = \sup_{\substack{a \in S^+ \\ a \leqslant x+y}} f(a) \leqslant f_m(x) + f_m(y).$$

Thus f_m is sublinear on R and both f_m and f_M are sublinear on $\bar{S}$. If one of x, y is not in $\bar{S}$, $f_M(x + y) \leqslant f_M(x) + f_M(y)$ trivially. This completes Theorem 3.1.

THEOREM 3.2. *Let S be an arbitrary semi-normal manifold of R. Then in order that $f_m(x) = f_M(x)$ for every monotone function f on S^+ and all $x \in R^+$, it*

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is necessary and sufficient that $R = \bar{S}$. *In order that* $f_m(x) = f_M(x)$ *for every* $x \in R^+$, f *a fixed monotone function on* S^+, *it is necessary and sufficient that* $f_m(x) = +\infty$ *on* $R^+ - \bar{S}^+$.

Proof. The proof is trivial, necessity in the first part being shown by the monotone function $g(x) = 0$, $x \in S^+$.

4. Full semi-normal manifolds of sequentially continuous linear lattices. Elements x, y of a linear lattice R are called mutually orthogonal if $|x| \cap |y| = 0$ and we then write $x \perp y$. The set of all elements $y \in R$ orthogonal to x is denoted by $x^\perp$ and the set of all elements of R orthogonal to every element of a set A is denoted by $A^\perp$. A subset N of R is called a normal manifold of R if to each $a \in R$ corresponds $x \in N$ and $y \in N^\perp$ with $a = x+y$. A normal manifold is necessarily a semi-normal manifold (l-ideal). If R is sequentially continuous, $z \in R$, then every $a \in R$ can be written uniquely as $a = x + y$ with $x \in z^{\perp\perp}$, $y \in z^\perp$. The mapping $a \to x$ is a projection determined by z. The corresponding linear operator, which is written as $[z]$, is called a *projector*. Projectors are studied in detail in **(3)**. The sets $z^{\perp\perp} = [z]R$ and $z^\perp$ are normal manifolds of R and $z \in z^{\perp\perp}$. As an example let R be the continuous linear lattice $L^1(X, S, \mu)$ of equivalence classes of integrable functions on the measure space (X, S, μ). Then if $f \in L^1$ and $e_f = [x \in X : f(x) \neq 0]$, $[f]$ means projection on e_f, modulo null sets.

For an arbitrary index set Λ and elements $x_\lambda \in R$ $(\lambda \in \Lambda)$ we say that the set $\{x_\lambda\}$ is filtering for $\leqslant$ and write $x_\lambda \uparrow_{\lambda\in\Lambda}$ if to any two elements x_λ, $x_{\lambda'}$ corresponds an element $x_{\lambda''}$ such that $x_\lambda \cup x_{\lambda'} \leqslant x_{\lambda''}$. If in addition $x = \bigcup_{\lambda\in\Lambda} x_\lambda$ we write $x_\lambda \uparrow_{\lambda\in\Lambda} x$. For projection operators on R, $[x_\lambda] \uparrow_{\lambda\in\Lambda} [x]$ means that for every $a \in R^+$, $[x_\lambda]a \uparrow_{\lambda\in\Lambda} [x]a$.

DEFINITION. *A semi-normal manifold* S *of* R *is full* (*in* R) *if* $x \in R$, $x \perp S$ *implies that* $x = 0$. *An orthogonal system* a_λ, $\lambda \in \Lambda$, *of elements of* R *is full in* R *if* $x \in R$, $x \perp a_\lambda$ *for all* $\lambda \in \Lambda$ *implies that* $x = 0$.

If R denotes the sequentially continuous linear lattice of Lebesgue measurable functions on $(-\infty, \infty)$, with the natural ordering if $S = L^1$, the corresponding set of Lebesgue integrable functions, without identifying equivalence classes, S is a semi-normal manifold of R that is full in R. If S_1 denotes the Lebesgue integrable functions vanishing outside $(0, 1)$, S_1 is a semi-normal manifold of R that is not full in R. If

$$f(x) = \int_0^1 x(t)dt,$$

f is a linear monotone function on S_1. In this case clearly f has many linear, monotone function extensions to R different from f_m and f_M. On the other hand, $\bar{S} = R$ so that

$$f(x) = \int_{-\infty}^{\infty} x(t)dt,$$

which is a linear monotone function on S^+, has a unique extension to R^+ as a monotone function. If $x \in R^+ - S^+$, $f(x) = +\infty$. In the sequel we study full, semi-normal manifolds of sequentially continuous linear lattices.

LEMMA 4.1. *If S is a full semi-normal manifold of a sequentially continuous linear lattice R, then, for each $x \in R^+$,*

$$x \cap s \uparrow_{s \in S^+} x.$$

Proof. We suppose that $y \geqslant x \cap s$ for all $s \in S^+$. Then $y \geqslant x \cap ns$ for all $s \in S^+$ and $n = 1, 2, \ldots$ because $s \in S^+$ implies $ns \in S^+$, $n = 1, 2, \ldots$. Thus we have $y \geqslant [s]x$ for all $s \in S^+$. This implies that $[s]y \geqslant [s]x$ for all $s \in S^+$ and thus

$$[s](x - y)^+ = ([s]x - [s]y)^+ = 0,$$

for all $s \in S^+$, whence $s \perp (x - y)^+$ and $x \leqslant y$.

THEOREM 4.0 (**3**, Theorem 13.2). *In order that a sequentially continuous linear lattice R be superuniversally continuous it is necessary and sufficient that for any orthogonal system a_λ ($\lambda \in \Lambda$) in R and for any $p \in R^+$ we have $[p][a_\lambda] = 0$ except for at most countable $\lambda \in \Lambda$.*

THEOREM 4.1. *Let R be a sequentially continuous linear lattice. In order that $f_m = f_M$ for every full semi-normal manifold S of R and every monotone function f on S^+, it is necessary and sufficient that R be superuniversally continuous.*

Proof. Necessity. We prove necessity by showing that, if R is not superuniversally continuous, there exists a full semi-normal manifold S of R with $R \neq \bar{S}$. Theorem 3.2, applied to S, completes the argument.

If R is not superuniversally continuous, Theorem 4.0 implies that there exists $a \in R^+$ and an orthogonal system $a_\lambda \in R^+$, $\lambda \in \Lambda$, with the index set Λ not countable. If the orthogonal system is not maximal in R it can be extended to a maximal system a_λ, $\lambda \in \Lambda$, by Zorn's lemma or transfinite induction and a maximal system is full in R by definition.

Let S denote the set of elements of R with

$$|x| < n \bigcup_{i=1}^{\infty} a_{\lambda_i},$$

for some positive integer n and some countable collection $\lambda_i \in \Lambda$, $i = 1, 2, \ldots$. It is easy to verify that S is a semi-normal manifold of S. Since $a_\lambda \in S$ for every $\lambda \in \Lambda$ and the orthogonal system a_λ, $\lambda \in \Lambda$, is full in R, S is full in R. Suppose that $a \in \bar{S}^+$. There is then a countable sequence $x_i \in S^+$ with

$$a = \bigcup_{1}^{\infty} x_i.$$

The definition of S implies that $[x_i][a_\lambda] = 0$ for all but a countable collection of values λ, $i = 1, 2, \ldots$. Thus $[a][a_\lambda] = 0$ for all but a countable collection of indices λ, giving a contradiction. Thus $a \notin \bar{S}$, $R \neq \bar{S}$.

Sufficiency. If $x \in R^+$, $x = \bigcup_{s \in S^+}(x \cap s)$ by Lemma 4.1, and the definition of superuniversal continuity implies that there is a sequence $s_i \in S^+$ (which may be assumed to be increasing since S is semi-normal) with

$$x = \bigcup_{s \in S^+} (x \cap s) = \bigcup_{i=1}^{\infty} (x \cap s_i).$$

Thus every x is in $\bar{\bar{S}}$ and Theorem 3.2 implies that $f_m = f_M$.

COROLLARY.* *In order that R be superuniversally continuous it is necessary and sufficient that for every full, semi-normal manifold S of R, $\bar{\bar{S}} = R$.*

If R is superuniversally continuous and S is a full semi-normal manifold of R, then $f_m(x) = f_M(x)$ for every monotone function f on S^+ and every $x \in R^+$ by Theorem 4.1. Thus $\bar{\bar{S}} = R$ by Theorem 3.2. If $\bar{\bar{S}} = R$, then for every monotone function f on S^+, $f_m(x) = f_M(x)$ for every $x \in R^+$. Thus if $\bar{\bar{S}} = R$ for every full, semi-normal manifold of R, R is superuniversally continuous by Theorem 4.1.

We note that S need not coincide with R. For example if R is the space of equivalence classes of Lebesgue measurable functions on $(0, 1)$ with the natural partial ordering modulo null sets, R is superuniversally continuous by Theorem 4.0. If $S = L^1$, S is a full, semi-normal manifold properly contained in R. Note that $\bar{\bar{S}} = R$.

DEFINITION. *The monotone function f on R^+ is semi-continuous if $x_\lambda \in R^+$, $x_\lambda \uparrow_{\lambda \in \Lambda} x$, implies that $f(x) = \sup_{\lambda \in \Lambda} f(x_\lambda)$.*

THEOREM 4.2. *Let f be an arbitrary monotone function on R^+, where R is a sequentially continuous linear lattice. Then in order that f be semi-continuous it is necessary and sufficient that $a \geqslant 0$, $[x_\lambda] \uparrow_{\lambda \in \Lambda} [x]$ imply*

$$f([x]a) = \sup_\lambda f([x_\lambda]a).$$

In **(3)** Nakano has defined semi-continuous norms on linear lattices in terms of countable sequences rather than filtering sets of elements. Noting that a norm on a sequentially continuous linear lattice is a subadditive monotone function, the present theorem is the analogue of his Theorem 30.5. The proof requires an extension to filtering sets of Theorem 6.18 which we give as

LEMMA 4.2. *Let R be a sequentially continuous linear lattice. If $p \in R^+$, $p_\lambda \uparrow_\lambda p$, then for every $a \in R^+$, $[p_\lambda]a \uparrow_\lambda [p]a$.*

Proof of the lemma. Since $p \leqslant q$ implies $[p] \leqslant [q]$, $[p_\lambda]a \uparrow_{\lambda \in \Lambda}$. Since $[p]a \geqslant [p_\lambda]a$ $(\lambda \in \Lambda)$, it remains to be shown that if $x \geqslant [p_\lambda]a$ $(\lambda \in \Lambda)$, then $x \geqslant [p]a$.

*The authors are indebted to the referee for suggesting this corollary and for comments improving several proofs.

Let $b = [p]a - ([p]a) \cap x$. Then, using **(3**, § 5**)**,

$$[p_\lambda]b = [p_\lambda]([p]a - ([p]a) \cap x) = [p_\lambda]a - ([p_\lambda]a) \cap x = 0,$$

for all $\lambda \in \Lambda$. Now for $q, r \in R$, $[q]r = 0$ if and only if $q \perp r$. Thus $b \perp p_\lambda$ $(\lambda \in \Lambda)$ and $b \perp \bigcup_\lambda p_\lambda = p$ by **(3**, Theorem 4.2**)**. Thus

$$0 = [p]b = [p]([p]a - ([p]a) \cap x) = [p]a - ([p]a) \cap x,$$
$$[p]a = ([p]a) \cap x \leqslant x.$$

Proof of Theorem 4.2. The proof of **(3**, Theorem 30.5**)** now applies with minor changes. Necessity follows immediately from the definitions. To prove sufficiency we suppose that $x_\lambda \uparrow_\lambda x$, $x, x_\lambda \in R^+$, and must show that the conditions of the theorem imply that $f(x) = \sup_\lambda f(x_\lambda)$. Since (M1) implies that $\geqslant$ holds we need only show that $\leqslant$ holds.

We fix ϵ, $0 < \epsilon < 1$, and define

$$p_\lambda = (x_\lambda - (1 - \epsilon)x)^+ \leqslant (x - (1 - \epsilon)x)^+ = (\epsilon x)^+ = \epsilon x,$$

for all $\lambda \in \Lambda$. Clearly $p_\lambda \uparrow_{\lambda \epsilon \Lambda}$. Suppose that $y \geqslant p_\lambda$ $(\lambda \in \Lambda)$. Since

$$(x_\lambda - (1 - \epsilon)x)^+ \geqslant x_\lambda - (1 - \epsilon)x,$$
$$y \geqslant \bigcup_{\lambda \epsilon \Lambda} x - (1 - \epsilon)x = \epsilon x.$$

Thus $p_\lambda \uparrow_{\lambda \epsilon \Lambda} \epsilon x$ and, since $[\epsilon x] = [x]$, if $x \geqslant 0$ Lemma 4.2 implies that $[p_\lambda] \uparrow_\lambda [x]$, whence $[p_\lambda]x \uparrow_\lambda [x]x = x$. By hypothesis then

$$f(x) = \sup_\lambda f([p_\lambda]x).$$

Since $[a^+]a = a^+$, $[p_\lambda](x_\lambda - (1 - \epsilon)x) = (x_\lambda - (1 - \epsilon)x)^+$ and

$$(1 - \epsilon)[p_\lambda]x \leqslant [p_\lambda]x_\lambda \leqslant x_\lambda.$$

Let

$$\epsilon_i \downarrow_{i=1}^{\infty} 0.$$

Then

$$(1 - \epsilon_i)([p_\lambda]x) \uparrow_{i=1}^{\infty} [p_\lambda]x$$

and, by (M2), $\sup_i f[(1 - \epsilon_i)([p_\lambda]x)] = f([p_\lambda]x)$. Thus

$$f(x_\lambda) \geqslant f[(1 - \epsilon_i)([p_\lambda]x)], \qquad i = 1, 2, \ldots,$$
$$f(x_\lambda) \geqslant f([p_\lambda]x),$$
$$\sup_\lambda f(x_\lambda) \geqslant \sup_\lambda f([p_\lambda]x) = f(x).$$

Let S be a semi-normal manifold of R. Then the restriction of an arbitrary monotone function f on R^+ to S^+ is a monotone function on S^+ and has minimal and maximal extensions to R^+ which we shall write $f_{m(S)}$ and $f_{M(S)}$ respectively.

THEOREM 4.3. *Let R be a sequentially continuous linear lattice, f a monotone function on R^+. Then in order that $f = f_{m(S)}$ for every full semi-normal manifold S of R, it is necessary and sufficient that f be semi-continuous.*

Proof. Necessity. Assume that f on R^+ is not semi-continuous. Then Theorem 4.2 implies that there exists $a \in R^+$ and a sequence $[a_\lambda] \uparrow_\lambda [a]$ such that

$$\sup_\lambda f([a_\lambda]a) < f(a).$$

Let S denote the collection of elements $x \cup y$ with $x \in a^\perp$, y in $[a_\lambda]R$ for some $\lambda \in \Lambda$. It is easily verified that S is a semi-normal manifold of R. If $b \perp S$, then $b \perp a^\perp$ and $b \perp [a_\lambda]R$ $(\lambda \in \Lambda)$. If $x \in R^+$, since $[a_\lambda] \uparrow_\lambda [a]$

$$[a_\lambda]x \uparrow_\lambda [a]x, \qquad [a]x = \cup_\lambda [a_\lambda]x,$$
$$b \cap [a]x = b \cap (\cup_\lambda [a_\lambda]x) = \cup_\lambda (b \cap [a_\lambda]x) = 0$$

(**3**, 7, p. 9). Thus $b \perp R$, $b = 0$ and we have shown that S is full in R.

By definition

$$f_m(a) = \sup_{\substack{x \in S \\ x \leqslant a}} f(x).$$

If $x \in S$, $x \leqslant a$ implies that $x \in [a_\lambda]R$ for some $\lambda \in \Lambda$. Since $[a_\lambda]x \leqslant [a_\lambda]a$,

$$f_m(a) = \sup_{\substack{x \in S \\ x \leqslant a}} f(x) = \sup_{\substack{x \leqslant a \\ \lambda \in \Lambda}} f([a_\lambda]x) \leqslant \sup_{\lambda \in \Lambda} f([a_\lambda]a) < f(a).$$

Sufficiency. Let $x \in R^+$. By Lemma 4.1 fulness implies that $x = \cup_{s \in S} (x \cap s)$. Since $\{x \cap s\}$ is filtering for $\leqslant$,

$$(x \cap s) \uparrow_{s \in S} x,$$

$$f(x) = \sup_{s \in S} f(x \cap s) = \sup_{\substack{s \in S \\ s \leqslant x}} f(s) = f_m(x),$$

if f is semi-continuous.

THEOREM 4.4. *Let R be a sequentially continuous linear lattice and suppose that R contains a superuniversally continuous full semi-normal manifold S. Then in order that a monotone function f on R^+ be semi-continuous it is necessary and sufficient that for each $a \in R^+$,*

$$f(a) = \sup_{s \in S} f([s]a). \tag{4.1}$$

Proof. Necessity. Since S is full in R, if $a \in R^+$,

$$s \cap a \uparrow_{s \in S} a,$$

by Lemma 4.1. Lemma 4.2 then implies that $[s \cap a] \uparrow_{s \in S} [a]$ and in particular that $[s \cap a]a = [s]a \uparrow_{s \in S} [a]a = a$. If the monotone function f is semi-continuous, (4.1) holds.

Sufficiency. We first note that if S is superuniversally continuous and $s_0 \in S$, then $[s_0]R$ is superuniversally continuous. Suppose it is not and choose a_λ $(\lambda \in \Lambda)$, $y \in ([s_0]R)^+$ with $[y][a_\lambda] \neq 0$ for uncountably many $\lambda \in \Lambda$. Now $s_0 \cap a_\lambda$ $(\lambda \in \Lambda)$ and $y \cap s_0 \in S$ (since S is semi-normal). Since $[s_0 \cap a_\lambda] = [a_\lambda]$, $[s_0 \cap y] = [y]$, this contradicts the fact that S is superuniversally continuous.

We now assume that a_λ $(\lambda \in \Lambda)$, $a \in R^+$ and that $a_\lambda \uparrow_{\lambda\in\Lambda} a$. By (M1),

$$f(a) \geqslant \sup_\lambda f(a_\lambda),$$

and it is sufficient to prove that $\leqslant$ holds when (4.1) is assumed.

Since $[s_0]R$ is superuniversally continuous, Theorem 4.1 implies that, for every full semi-normal manifold S of $[s_0]R$ and every monotone function f on S^+, $f_m = f_M$ on $([s_0]R)^+$. Thus on $([s_0]R)^+$, $f = f_{m(S)}$ for every full, semi-normal manifold S of $[s_0]R$ and, by Theorem 4.3, f is semi-continuous on $[s_0]R$. Since $a_\lambda \uparrow_\lambda a$ implies that $[s_0]a_\lambda \uparrow_\lambda [s_0]a$, Theorem 4.2 implies that

$$f([s_0]a) = \sup_\lambda f([s_0]a_\lambda).$$

Given $\epsilon > 0$, (4.1) implies that there exists $s_0 \in S$ with

$$f(a) \leqslant f([s_0]a) + \epsilon = \sup_\lambda f([s_0]a_\lambda) + \epsilon \leqslant \sup_\lambda f(a_\lambda) + \epsilon.$$

Since ϵ is arbitrary, the proof is complete.

DEFINITION. *R is locally superuniversally continuous if to each $x \in R^+$, $x \neq 0$, corresponds $p \in R^+$, $p \neq 0$, such that $[p] \leqslant [x]$ and $[p]R$ is superuniversally continuous.*

THEOREM 4.5. *If R is a locally superuniversally continuous, sequentially continuous linear lattice and S denotes the set of elements s of R for which $[s]R$ is superuniversally continuous, then S is a superuniversally continuous, full, semi-normal manifold of R.*

Proof. If $p \in S$, since $[p] = [|p|] = [\alpha p]$ for every real number $\alpha \neq 0$, $|p|$ and αp are in S. For each p, $[p]R$ is a subspace of R. If $|q| \leqslant |p|$, $x \in [q]R$, $y \in [p]R$, $p \in S$, with $|y| \leqslant |x|$, then $|y| = y_1 + y_2$ with $y_1 \in ([q]R)^+$, $y_2 \in q^{\perp\perp}$. Since $y_2 \leqslant |y| \leqslant |x|$, $y_2 \in q^{\perp\perp}$, $y_2 = 0$ and $y, |y| \in [q]R$. Since $[p]R$ is assumed to be superuniversally continuous it now follows easily from the definition that $[q]R$ is superuniversally continuous and $q \in S$. Thus S is a semi-normal manifold of R if it is linear and this will follow if we show that $|p| + |q| \in S$ if $p, q \in S$.

Using (**3**, Theorems 6.7 and 6.15), if $a \in R^+$,

$$(|p| + |q|)a = [p, q]a = [p]a \cup [q]a.$$

Let $a, a_\lambda \in ([|p| + |q|]R)^+$. Now $[a]\,[a_\lambda] \neq 0$ if and only if $a \cap a_\lambda \neq 0$. Assuming $a \cap a_\lambda \neq 0$,

$$a \cap a_\lambda = [|p| + |q|]\,(a \cap a_\lambda) = [p](a \cap a_\lambda) \cup [q](a \cap a_\lambda),$$

whence $[p](a \cap a_\lambda) \neq 0$ and/or $[q](a \cap a_\lambda) \neq 0$.

With Theorem 4.0 this shows that if $[|p| + |q|]R$ is not superuniversally continuous, then at least one of $[p]R$, $[q]R$ is not superuniversally continuous, giving a contradiction. We conclude that $|p| + |q| \in S$. An alternative proof of this part, independent of Theorem 4.0, can be based on the fact that $[p \cup q]R$ is a direct product of $[p]R$ and

$$[q - \bigcup_{n=1}^{\infty} (np \cap q)]R = (1 - [p])[q]R.$$

We next show that S is full in R. Suppose that $h \perp S$ and assume that $h \neq 0$. Since R is locally superuniversally continuous, there then exists $p \neq 0$, $p \in R$ with $0 < [p] \leqslant [h]$, $p \in S$, contradicting the hypothesis that $h \perp S$. We conclude that $h = 0$.

Finally we show that S is superuniversally continuous. Let a_λ, $\lambda \in \Lambda$, be any orthogonal set of elements of S, x an arbitrary element of S. Since $[x]R$ is superuniversally continuous

$$[x][a_\lambda] = [x][a_\lambda \cap x] = 0$$

for all but at most countably many λ, showing that S is superuniversally continuous.

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