Glasgow Math. J. **62** (2020) 109–122. © Glasgow Mathematical Journal Trust 2019.
doi:10.1017/S0017089518000563.

A CLASS OF CRITICAL KIRCHHOFF PROBLEM ON THE HYPERBOLIC SPACE $\mathbb{H}^n$

PAULO CESAR CARRIÃO

Departamento de Matemática, Universidade Federal de Minas Gerais, 31270-901 Belo Horizonte, Minas Gerais, Brazil
e-mail: pauloceca@gmail.com

AUGUSTO CÉSAR DOS REIS COSTA

Faculdade de Matemática, Instituto de Ciências Exatas e Naturais, Universidade Federal do Pará, 66075-110 Belém, PA, Brazil
e-mail: aug@ufpa.br

OLIMPIO HIROSHI MIYAGAKI

Departamento de Matemática, Universidade Federal de Juiz de Fora, 36036-330 Juiz de Fora, Minas Gerais, Brazil
e-mail: ohmiyagaki@gmail.com

(Received 12 September 2017; revised 3 August 2018; accepted 5 December 2018; first published online 21 January 2019)

Abstract. We investigate questions on the existence of nontrivial solution for a class of the critical Kirchhoff-type problems in Hyperbolic space. By the use of the stereographic projection the problem becomes a singular problem on the boundary of the open ball $B_1(0) \subset \mathbb{R}^n$. Combining a version of the Hardy inequality, due to Brezis–Marcus, with the mountain pass theorem due to Ambrosetti–Rabinowitz are used to obtain the nontrivial solution. One of the difficulties is to find a range where the Palais Smale converges, because our equation involves a nonlocal term coming from the Kirchhoff term.

2000 *Mathematics Subject Classification*. Primary 58J05; Secondary 35R01, 35J60, 35B33

1. Introduction. In this paper, we investigate questions on the existence of nontrivial solution for the following Kirchhoff-type equation

$$-\left(a + b\int_{\mathbf{B}^3} |\nabla_{\mathbf{B}^3} u|^2 dV_{\mathbf{B}^3}\right) \Delta_{\mathbf{B}^3} u = \lambda |u|^{q-2}u + |u|^4 u \quad \text{in} \quad u \in H^1(\mathbf{B}^3) \tag{1.1}$$

in Hyperbolic space $\mathbf{B}^3$, where a, b, and λ are positive constants, $4 < q < 6$, $H^1(\mathbf{B}^3)$ is the usual Sobolev space on the disc model of the Hyperbolic space $\mathbf{B}^3$, and $\Delta_{\mathbf{B}^3}$ denotes the Laplace Beltrami operator on $\mathbf{B}^3$. For the hyperbolic space $\mathcal{H}^n$, we make use of the stereographic projection $E : \mathcal{H}^n \to \mathbb{R}^n$, where each point $P' \in \mathcal{H}^n$ is projected to $P \in \mathbb{R}^n$, where P is the intersection of the straight line connecting P' and the point $(0, ..., 0, -1)$. More exactly, we have explicitly the projection operator $G : \mathbb{R}^n \to \mathcal{H}^n$ and $G^{-1} : \mathcal{H}^n \to \mathbb{R}^n$ given by

$$G(x) = (x.p(x), (1 + |x|^2)p/2) \text{ and } G^{-1}(y) = \frac{1}{y_{n+1}} y, x, y \in \mathbb{R}^n,$$

where $p(x) = \dfrac{2}{1 - |x|^2}$.

This projection takes $\mathcal{H}^n$ onto the open ball $B_1(0) \subset \mathbb{R}^n$, and we denote by $D \subset B_1(0)$ the stereographic projection of $D' \subset \mathcal{H}^n$. See more details in the excellent books [54, 58].

We will consider the metric

$$ds = p(x)|dx|, \text{ where } p(x) = \frac{2}{1 - |x|^2},$$

and we denote by $\mathbf{B}^n$ the ball $B_1(0)$ endowed with the above metric.

The gradient, the Dirichlet integral, and the Laplace–Beltrami operator corresponding to this metric are

$$\nabla_{\mathbf{B}^n} u = \frac{\nabla u}{p^2}, \qquad Du = \int_{D'} |\nabla_{\mathbf{B}^n} u|^2 dV_{\mathbf{B}^n} = \int_D |\nabla u|^2 p^{n-2} dx,$$
$$\Delta_{\mathbf{B}^n} u = p^{-n} div(p^{n-2} \nabla u).$$

See [13, 37, 57].

Elliptic problems, in an euclidean space, involving Sobolev's critical exponent, that is, when the non-linearity behaves as a polynomial function of degree $2^* = \frac{2N}{N-2}$, $N \geq 3$, were studied in a pioneering and remarkable article due to Brezis and Nirenberg [19]. In that paper, the lack of compactness was overcome by analyzing the critical level set of the functional associated with the problem. From this work, several authors have been working on the theme trying to extend or complement existing results in several directions, in that sense, we would like to mention some articles, and we apologize for not mentioning all the authors. For such problems modeled in a bounded domain, we cite [3, 5, 10, 14, 16, 20, 23, 24, 29, 32, 56, 62], while we mention [33, 51, 50, 63] for problems in unbounded domains. For the problems involving the p -Laplacian operator or more general degenerate operators, the following works have treated these subjects, [21, 38–40, 51]. The authors in [8, 9, 12] and also in [11, 13, 15, 22, 36, 37, 41, 49, 57] have treated some critical problem in a sphere and in a hyperbolic space, respectively. We cite [15, 41] for the related problems in the cases linear or supercritical. See references therein, as well as the book [61] for additional remarks and results. On the other hand, our equation in an euclidean space is related to a stationary Kirchhoff equation [45], namely,

$$u_{tt} - M\left(\int_\Omega |\nabla_x u|^2 dx\right) \Delta_x u = f(x, t), \ (x, t) \in \Omega \times \mathbb{R},$$

where Ω is bounded domain, $M(s) = a + bs$, $a, b > 0$, and f is a suitable function, which is an extension of the classical D'Alembert's wave equation, since in this case, the model considers the effects of the changes in the length of the strings during the vibrations. See [46]. The main difficulty is because the term containing M in the equation makes this equation nonlocal, that is, the equation does not satisfy a pointwise identity any longer.

The above equation has been received special attention after the work by Lions [46], where a non-linear functional analysis approach was proposed. Up to our knowledge, Ma and Rivera [48] were the pioneers to study this problem by variational methods, more exactly, by using the minimization method. In [1], was employed the mountain pass theorem, while in [53] a topologic argument was used, more specifically, the Yang index and critical groups, and in [44] is studied the equation by using the minimization arguments and fountain theorem. We would like to cite [26, 35, 60] for more multiplicity results. For the Kirchhoff equation involving critical exponents we refer to [2, 34, 42, 43, 47] and references therein. See [4, 7, 28, 30] and [25, 27, 55] for some related results.

Returning to our subject, restricted to $n=3$, if u is a solution of equation (1.1), putting $v := p^{\frac{1}{2}}u$, then v satisfies the following problem:

$$\begin{cases} \left(a+b\|v\|^2\right)\left(-\Delta v+(3/4)p^2 v\right) = \lambda p^{\alpha}|v|^{q-2}v+|v|^4 v, & \text{in } B_1(0) \\ v=0, & \text{on } \partial B_1(0), \end{cases} \tag{1.2}$$

where $\alpha=(6-q)/2$ and $\|v\|^2 = \displaystyle\int_{B_1(0)} \left(|\nabla v|^2+(3/4)p^2v^2\right)$.

From now on, we will consider $\Omega := B_1(0)$. We will denote by $H^1_{0,r}(\Omega)$ the subspace of $H^1_0(\Omega)$ of the radial functions which is endowed with the norm given by

$$\|v\|^2 = \int_\Omega \left(|\nabla v|^2+(3/4)p^2v^2\right).$$

Since the euclidean sphere with center at the origin $0\in\mathbb{R}^N$ is also a hyperbolic sphere with center at the origin $0\in\mathbf{B}^n$, $H^1_{0,r}(\Omega)$ can also be seen as the subspace of $H^1_0(\Omega)$ consisting of the hyperbolic radial functions. See this characterization as well as others remarks in [**13**, Appendix]. We observe that in [**13**, Theorem 3.1], $H^1_{0,r}(\Omega)$ is embedded compactly in $L^q(\Omega)$ for $2<q<2^*$. Note also that here $2^*=6$.

We have the following functional $J : H^1_{0,r}(\Omega)\to\mathbb{R}$ associated with problem (1.2)

$$J(v) = \frac{a}{2}\|v\|^2+\frac{b}{4}\|v\|^4-\frac{\lambda}{q}\int_\Omega p^\alpha|v|^q-\frac{1}{6}\int_\Omega|v|^6, \tag{1.3}$$

whose Gateaux derivative is given by

$$J'(v)w = \left(a+b\|v\|^2\right)\int_\Omega\left(\nabla v\nabla w+\frac{3}{4}p^2vw\right)-\lambda\int_\Omega p^\alpha|v|^{q-2}vw-\int_\Omega|v|^4vw. \tag{1.4}$$

Now, we present our main result.

THEOREM 1.1. *Suppose* $4<q<6$. *Then, for every* $\lambda>0$ *the problem* (1.1) *has a nontrivial solution* $u\in H^1(\mathbf{B}^3)$.

This result of existence of nontrivial solution, in the hyperbolic space, extends results presented in [**22**, **43**].

2. Proof of the main result.

The proof is made by applying the mountain pass theorem (See Willem [**61**] for a reference). To this end, we have the following

LEMMA 2.1. (*Mountain pass geometry*).

(a) There exist $\beta>0$ *and* $\rho>0$ *such that* $J(v)\geq\beta$ *when* $\|v\|=\rho$.
(b) There exists an element $e\in H^1_{0,r}(\Omega)$ *with* $\|e\|>\rho$ *such that* $J(e)<0$.

Proof. For item (a), we observe that by [**17**, **18**], there exists a constant $C>0$, such that

$$\int_\Omega p^\alpha v^q \leq C\left(\int_\Omega|\nabla v|^2\right)^{\frac{q}{2}} \leq C\left[\int_\Omega\left(|\nabla v|^2+(3/4)p^2v^2\right)\right]^{\frac{q}{2}}.$$

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

Thus,

$$J(u) \geq \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{C\lambda}{q}\left[\int_\Omega \left(|\nabla v|^2 + (3/4)p^2v^2\right)\right]^{\frac{q}{2}} - \frac{1}{6}\int_\Omega |v|^6,$$

and by the Sobolev continuous embedding, there is a constant $\widetilde{C} > 0$, verifying

$$J(u) \geq \frac{a}{2}\|v\|^2 + \frac{b}{4}\|v\|^4 - \frac{C\lambda}{q}\|v\|^q - \frac{\widetilde{C}}{6}\|v\|^6 \geq \beta,$$

where the conclusion follows by making $\|v\| = \rho$ sufficiently small.

Now for item (b), take $0 < v \in H^1_{0,r}(\Omega)$ and $0 < t$. Note that

$$J(tv) = \frac{at^2}{2}\|v\|^2 + \frac{bt^4}{4}\|v\|^4 - \frac{\lambda t^q}{q}\int_\Omega p^\alpha |v|^q - \frac{t^6}{6}\int_\Omega |v|^6.$$

Therefore, $J(tv) \to -\infty$ as $t \to +\infty$. Consequently, J satisfies the geometry of the mountain pass theorem. □

Lemma 2.1 and Ekeland's Variational Principle [6] allow us to use the general minimax principle, see [61, Theorem 2.9], which gives us a Palais-Smale sequence, $(v_k) \subset H^1_{0,r}(\Omega)$, at the level c, i.e.,

$$J(v_k) \to c \ \text{ and } \ \|J'(v_k)\|_{H^1_{0,r}(\Omega)^*} \to 0, \ \text{ as } k \to \infty, \tag{2.5}$$

where

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)),$$

and $\Gamma = \left\{\gamma \in C([0, 1], H^1_{0,r}(\Omega));\ \gamma(0) = 0, J(\gamma(1)) < 0\right\}$.

LEMMA 2.2. *The sequence $(v_k) \subset H^1_{0,r}(\Omega)$ defined above is bounded.*

Proof. Since (v_k) is a Palais-Smale sequence at the level c,

$$c + 1 + \|v_k\| \geq J(v_k) - \frac{1}{q}J'(v_k)v_k.$$

Thus, by Sobolev continuous embedding, there is a constant $\widetilde{C}' > 0$, such that

$$c + 1 + \|v_k\| \geq \left(\frac{a}{2} - \frac{a}{q}\right)\|v_k\|^2 + \left(\frac{b}{4} - \frac{b}{q}\right)\|v_k\|^4 + \widetilde{C}'\left(\frac{1}{q} - \frac{1}{6}\right)\|v_k\|^6.$$

Therefore, the sequence is bounded. □

LEMMA 2.3. *We have $c < \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{3/2}$, where*

$$S := \inf_{u \in H^1_{0,r}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\left(\displaystyle\int_\Omega u^6\right)^{1/3}}.$$

Proof. In this proof, we will follow some of the arguments made in [19], see also, for instance, [43, 42, 50]. First, we observe that it suffices to show that there exists a $v_0 \in H^1_{0,r}(\Omega)$, $v_0 \neq 0$ such that

$$\sup_{t\geq 0} J(tv_0) < \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{3/2}. \tag{2.6}$$

Indeed, observing that $J(tv_0) \to -\infty$ as $t \to \infty$, there exists $R > 0$ such that $J(Rv_0) < 0$. Now, we write $u_1 := Rv_0$, and from Lemma 2.1, we have

$$0 < \beta \leq c = \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} J(\gamma(\tau)) \leq \sup_{t\geq 0} J(tv_0) < \frac{1}{4}abS^3 + \frac{1}{24}b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{3/2}.$$

Therefore, we are going to prove the existence of a function v_0 such that (2.6) holds.

Let $0 < R < \frac{1}{2}$ be fixed, and let $\varphi \in C_0^\infty(\Omega)$ be a cut-off function with support at B_{2R}, such that φ is identically 1 on B_R and $0 \leq \varphi \leq 1$ on B_{2R}, where B_r denotes the ball in $\mathbb{R}^3$ with center at the origin and radius r.

Given $\varepsilon > 0$, we set $\psi_\varepsilon(x) := \varphi(x)\omega_\varepsilon(x)$, where

$$\omega_\varepsilon(x) = (3\varepsilon)^{\frac{1}{4}} \frac{1}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

and ω_ε satisfies (see[**59**])

$$\int_{\mathbb{R}^3} |\nabla\omega_\varepsilon|^2 = \int_{\mathbb{R}^3} |\omega_\varepsilon|^6 = S^{1/2}. \tag{2.7}$$

From the definition of ω_ε, it can be shown that

$$\int_{B_R} |\nabla\omega_\varepsilon|^2 \leq \int_{B_R} |\omega_\varepsilon|^6, \tag{2.8}$$

and

$$\int_{B_1 - B_R} |\nabla\psi_\varepsilon|^2 = O(\varepsilon^{\frac{1}{2}}) \text{ as } \varepsilon \to 0. \tag{2.9}$$

Now, we define

$$v_\varepsilon := \frac{\psi_\varepsilon}{\left(\int_{B_{2R}} \psi_\varepsilon^6\right)^{1/6}}$$

and also $X_\varepsilon := \int_{B_1} |\nabla v_\varepsilon|^2$. Then, we have

$$X_\varepsilon = \int_\Omega |\nabla v_\varepsilon|^2 = \int_\Omega \frac{|\nabla\psi_\varepsilon|^2}{B^2}$$
$$= \int_{B_R} \frac{|\nabla\psi_\varepsilon|}{B^2} + \int_{B_{2R}-B_R} \frac{|\nabla\psi_\varepsilon|}{B^2},$$

where $B := \left(\int_{B_{2R}} \psi_\varepsilon^6\right)^{1/6}$. Then, since, $\varphi \equiv 1$ and consequently $\nabla\varphi \equiv 0$ on B_R, we have

$$X_\varepsilon = \frac{1}{B^2}\int_{B_R} |\nabla\psi_\varepsilon|^2 + \frac{1}{B^2}\int_{B_{2R}-B_R} |\nabla\psi_\varepsilon|^2$$
$$= \frac{1}{B^2}\int_{B_R} |\nabla\omega_\varepsilon|^2 + \int_{B_{2R}-B_R} |\nabla\psi_\varepsilon|^2.$$

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

By equations (2.8) and (2.9) and considering $\delta = \frac{1}{2}$ we obtain

$$X_\varepsilon \leq \frac{1}{B^2}\int_{B_R} |\omega_\varepsilon|^6 + O(\varepsilon^\delta) = \frac{\displaystyle\int_{B_R} |\omega_\varepsilon|^6}{\left(\displaystyle\int_{B_R} |\omega_\varepsilon|^6 + \int_{B_{2R}-B_R} |\psi_\varepsilon|^6\right)^{1/3}} + O(\varepsilon^\delta)$$

$$\leq \frac{\displaystyle\int_{B_R} |\omega_\varepsilon|^6}{\left(\displaystyle\int_{B_R} |\omega_\varepsilon|^6\right)^{1/3}} + O(\varepsilon^\delta) \leq \left(\int_{B_R} |\omega_\varepsilon|^6\right)^{2/3} + O(\varepsilon^\delta)$$

$$\leq \left(\int_{\mathbb{R}^3} |\omega_\varepsilon|^6\right)^{2/3} + O(\varepsilon^\delta) = S + O(\varepsilon^\delta)$$

Therefore, we have

$$X_\varepsilon \leq S + O(\varepsilon^\delta). \tag{2.10}$$

On the other hand, we have

$$\lim_{t\to+\infty} J(tv_\varepsilon) = -\infty, \ \forall\ \varepsilon > 0.$$

This implies that there exists $t_\varepsilon > 0$ such that $\sup_{t\geq 0} J(tv_\varepsilon) = J(t_\varepsilon v_\varepsilon)$. Now, we will prove an estimate for this t_ε.

$$J'(tv_\varepsilon)\big|_{t=t_\varepsilon} = 0.$$

Thus,

$$at_\varepsilon \|v_\varepsilon\|^2 + bt_\varepsilon^3 \|v_\varepsilon\|^4 - \lambda t_\varepsilon^{q-1} \int_\Omega p^\alpha |v_\varepsilon|^q - t_\varepsilon^5 \int_\Omega |v_\varepsilon|^6 = 0,$$

which implies

$$a\|v_\varepsilon\|^2 + bt_\varepsilon^2 \|v_\varepsilon\|^4 - \lambda t_\varepsilon^{q-2} \int_\Omega p^\alpha |v_\varepsilon|^q - t_\varepsilon^4 \int_\Omega |v_\varepsilon|^6 = 0,$$

Since $\displaystyle\int_\Omega |v_\varepsilon|^6 = 1$, we have

$$-a\|v_\varepsilon\|^2 - bt_\varepsilon^2 \|v_\varepsilon\|^4 + t_\varepsilon^4 \leq 0.$$

Hence

$$0 \leq t_\varepsilon^2 \leq \frac{b\|v_\varepsilon\|^4 + \left[(b\|v_\varepsilon\|^4)^2 + 4a\|v_\varepsilon\|^2\right]^{1/2}}{2} := t_0.$$

Since the function $t \mapsto \dfrac{a}{2}t^2\|v_\varepsilon\|^2 + \dfrac{b}{4}t^4\|v\|^4 - \dfrac{t^6}{6}$ is increasing on $[0, t_0)$, denoting $C_1 = a\|v_\varepsilon\|^2$ and $C_2 = b\|v_\varepsilon\|^4$, we have

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

$$
\begin{aligned}
J(t_\varepsilon v_\varepsilon) &\leq \frac{at_0}{2}\|v_\varepsilon\|^2 + \frac{bt_0^2}{4}\|v_\varepsilon\|^4 - \frac{\lambda t_\epsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q - \frac{t_0^3}{6} \\
&\leq \frac{t_0 C_1}{2} + \frac{t_0^2 C_2}{4} - \frac{\lambda t_\epsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q - \frac{t_0^3}{6} \\
&\leq \frac{1}{2}\left[\frac{C_2 + (C_2^2 + 4C_1)^{1/2}}{2}\right] C_1 + \frac{1}{4}\left[\frac{C_2 + (C_2^2 + 4C_1)^{1/2}}{2}\right]^2 C_2 \\
&\quad - \frac{1}{6}\left[\frac{C_2 + (C_2^2 + 4C_1)^{1/2}}{2}\right]^3 - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q \\
&\leq \frac{C_1 C_2}{4} + \frac{C_2^3}{24} + \frac{1}{24}(C_2^2 + 4C_1)^{3/2} - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q.
\end{aligned}
$$

Considering $A = 3/4 \int_\Omega p^2 v_\varepsilon^2$, by definition of the norm, and the inequality (2.10), we obtain

$$
\begin{aligned}
J(t_\varepsilon v_\varepsilon) &\leq \frac{ab}{4}(X_\varepsilon + A)^3 + \frac{b^3}{24}(X_\varepsilon + A)^6 + \frac{1}{24}\left[b^2(X_\varepsilon + 4)^4 + 4a(X_\varepsilon + A)\right]^{3/2} \\
&\quad - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q \\
&\leq \frac{ab}{4}\big(S + O(\varepsilon^{1/2}) + A\big)^3 + \frac{b^3}{24}\big(S + O(\varepsilon^{1/2}) + A\big)^6 \\
&\quad + \frac{1}{24}\Big[b^2\big(S + O(\varepsilon^{1/2}) + A\big)^4 + 4a\big(S + O(\varepsilon^{1/2}) + A\big)\Big]^{3/2} - \frac{\lambda t_\varepsilon^q}{q}\int_\Omega p^\alpha v_\varepsilon^q.
\end{aligned}
$$

By using several times the standard inequality (see, e.g., [50, p. 778])

$$
(a+b)^\beta \leq a^\beta + \beta(a+b)^{\beta-1}b, \quad \forall\, \beta \geq 1, \forall\, a, b > 0,
$$

we infer that

$$
J(t_\varepsilon v_\varepsilon) \leq \frac{abS^3}{4} + \frac{b^3 S^6}{24} + \frac{1}{24}(b^2 S^4 + 4aS)^{3/2} + O(\varepsilon^{1/2}) + \int_{B_{2R}}\left(\frac{3C}{4}p^2 v_\varepsilon^2 - \lambda C_\varepsilon p^\alpha v_\varepsilon^q\right), \tag{2.11}
$$

for some constant $C > 0$, where $C_\varepsilon = \frac{t_\varepsilon^q}{q}$.

At this point, we can assume that there exists a positive constant C_0 such that $C_\varepsilon \geq C_0 > 0$, $\forall\, \varepsilon > 0$. If that was not the case, we could find a sequence $\varepsilon_k \to 0$ as $k \to \infty$, such that $t_{\varepsilon_k} \to 0$ as $k \to \infty$, since $C_\varepsilon \geq 0$. Now, passing to a subsequence, if necessary, which still denoted by ε_k, we have $t_{\varepsilon_k} v_{\varepsilon_k} \to 0$ as $k \to \infty$.

Therefore,

$$
0 < c \leq \sup_{t \geq 0} J(t v_{\varepsilon_k}) = J(t_{\varepsilon_k} v_{\varepsilon_k}) = J(0) = 0,
$$

which is a contradiction.

Observing that $\int_{B_{2R}} p^2 v_\varepsilon^2 < \infty$, we claim

Claim: $\displaystyle\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{1/2}} \int_{B_{2R}} \left(\frac{3C}{4}p^2 v_\varepsilon^2 - C_\varepsilon \lambda p^\alpha v_\varepsilon^q\right) = -\infty.$

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

Assuming the Claim is proved, from equation (2.11), we have

$$J(t_\varepsilon v_\varepsilon) < \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{1}{24}(b^2S^4 + 4aS)^{3/2},$$

for some $\varepsilon > 0$ sufficiently small, and the proof is complete.

Now, we are going to prove the Claim. For this, it is sufficient to show that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{1/2}} \left(\int_{B_R} \left(\frac{3C}{4} p^2 \omega_\varepsilon^2 - C_\varepsilon \lambda p^\alpha \omega_\varepsilon^q \right) \right) = -\infty \tag{2.12}$$

and

$$\int_{B_{2R}-B_R} \left(\frac{3C}{4} p^2 v_\varepsilon^2 - C_\varepsilon \lambda p^\alpha v_\varepsilon^q \right) = O(\varepsilon^{1/2}). \tag{2.13}$$

First, we will consider

$$\begin{aligned} J_\varepsilon &= \frac{1}{\varepsilon^{1/2}} \int_{B_R} \left(\frac{3C}{4} p^2 \omega_\varepsilon^2 - C_\varepsilon \lambda p^\alpha \omega_\varepsilon^q \right) \\ &= \frac{3C}{4\varepsilon^{1/2}} \int_{B_R} \left(\frac{2}{1-|x|^2} \right)^2 \frac{(3\varepsilon)^{1/2}}{(\varepsilon + |x|^2)} - \frac{\lambda C_\varepsilon}{\varepsilon^{1/2}} \int_{B_R} \left(\frac{2}{1-|x|^2} \right)^\alpha \frac{(3\varepsilon)^{q/4}}{(\varepsilon + |x|^2)^{q/2}} \\ &= \tilde{C} \int_{B_R} \left(\frac{2}{1-|x|^2} \right)^2 \frac{1}{(\varepsilon + |x|^2)} - \lambda \tilde{C}_\varepsilon \, \varepsilon^{\frac{(q-2)}{4}} \int_{B_R} \left(\frac{2}{1-|x|^2} \right)^\alpha \frac{1}{(\varepsilon + |x|^2)^{q/2}} \\ &= J_1 - J_2, \text{ for some constant } \widetilde{C} > 0. \end{aligned} \tag{2.14}$$

We observe that on B_R,

$$2 < \frac{2}{1-|x|^2} \leq \frac{2}{1-R^2}. \tag{2.15}$$

Therefore, making the change of variables $x = \varepsilon^{1/2} y$ and then using the polar coordinates, we obtain, for some constant $\widetilde{C} > 0$,

$$\begin{aligned} J_1 &\leq \frac{4\tilde{C}}{(1-R^2)^2} \int_{B_R} \frac{1}{(\varepsilon + |x|^2)} = \frac{4\tilde{C}}{(1-R^2)^2} \int_{B_{R\varepsilon^{-1/2}}} \frac{\varepsilon^{3/2}}{(\varepsilon + \varepsilon|y|^2)} \\ &= \frac{4\tilde{C}}{(1-R^2)^2} \omega \varepsilon^{1/2} \int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)} dr. \end{aligned} \tag{2.16}$$

Now, for J_2, we have, considering again equation (2.15), the change of variables $x = \varepsilon^{1/2} y$ and then using the polar coordinates that, we get for some constant $\widetilde{C}_\epsilon > 0$,

$$\begin{aligned} J_2 &\geq \lambda \tilde{C}_\varepsilon \, \varepsilon^{\frac{(q-2)}{4}} \int_{B_R} \left(\frac{2}{1-|x|^2} \right)^\alpha \frac{1}{(\varepsilon + |x|^2)^{q/2}} \\ &\geq \lambda \tilde{C}_\varepsilon \, \varepsilon^{\frac{(q-2)}{4}} 2^\alpha \int_{B_{R\varepsilon^{-1/2}}} \frac{\varepsilon^{3/2}}{(\varepsilon + \varepsilon|y|^2)^{q/2}} \\ &= \lambda \tilde{C}_\varepsilon 2^\alpha \, \varepsilon^{\frac{(q-2)}{4}} \frac{\varepsilon^{3/2}}{\varepsilon^{q/2}} \int_{B_{R\varepsilon^{-1/2}}} \frac{1}{(1+|y|^2)^{q/2}} \\ &= \lambda \tilde{C}_\varepsilon 2^\alpha w \, \varepsilon^{-\frac{q}{4}+1} \int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{q/2}} dr. \end{aligned} \tag{2.17}$$

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

Thus, combining equations (2.14), (2.16), and (3.24), we obtain

$$J_\varepsilon \leq \frac{4\tilde{C}}{(1-R^2)^2}\omega\varepsilon^{1/2}\int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)}dr - \lambda\tilde{C}_\varepsilon 2^\alpha w\,\varepsilon^{-\frac{q}{4}+1}\int_0^{R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)^{q/2}}dr.$$

Observing that

$$\int_0^{R\varepsilon^{-1/2}} \frac{r^2}{1+r^2}dr = R\varepsilon^{-1/2} - \tan^{-1}(R\varepsilon^{-1/2})$$

and $\lim_{\varepsilon\to 0^+} \varepsilon^{-\frac{q}{4}+1} = \infty$ as $4 < q$, we conclude that equation (2.12) holds.

Now, we will prove equation (2.13). First, we observe that we can find and fix an $\varepsilon > 0$ sufficiently small such that $O(\varepsilon^\delta) + \varepsilon^\delta I_\varepsilon < 0$. As in [**19**], we obtain

$$\int_{B_{2R}} |\psi_\varepsilon|^6 = 3^{3/2}\int_{\mathbb{R}^3} \frac{dx}{(1+|x|^2)^3} + O(\varepsilon^{3/2}). \tag{2.18}$$

From equation (2.18), we obtain

$$\frac{1}{\varepsilon^{1/2}}\int_{B_{2R}-B_R}\left(\frac{3C}{4}p^2v_\varepsilon^2 - \lambda C_\varepsilon p^\alpha v_\varepsilon^q\right) \leq \frac{C'}{\varepsilon^{1/2}}\int_{B_{2R}-B_R} p^2\varphi^2\omega_\varepsilon^2.$$

We define $\Theta = B_{2R} - B_R$. Since $R \leq |x| \leq 2R$, we have

$$\frac{2}{1-R^2} \leq p(x) \leq \frac{2}{1-4R^2},$$

therefore,

$$I_1 := \frac{C'}{\varepsilon^{1/2}}\int_\Theta p^2\varphi^2\omega_\varepsilon^2 \leq \frac{4C'}{\varepsilon^{1/2}(1-4R^2)^2}\int_\Theta \varphi^2\frac{\varepsilon^{1/2}}{(\varepsilon+|x|^2)}.$$

Making the change of variables $x = \varepsilon^{1/2}y$ and later changing to polar coordinates we obtain

$$\begin{aligned} I_1 &\leq \frac{4C'}{(1-4R^2)^2}\int_{\Theta'} \varphi^2(\varepsilon^{1/2}y)\frac{\varepsilon^{3/2}}{(\varepsilon+\varepsilon|y|^2)} \\ &\leq \frac{4C'\omega\varepsilon^{1/2}}{(1-4R^2)^2}\int_{R\varepsilon^{-1/2}}^{2R\varepsilon^{-1/2}} \frac{r^2}{(1+r^2)}dr, \end{aligned}$$

where $\Theta' = B_{2R\varepsilon^{-1/2}} - B_{R\varepsilon^{-1/2}}$.

By the Mean Value Theorem for integrals, there exists $r_0 \in [R\varepsilon^{-1/2}, 2R\varepsilon^{-1/2}]$ such that

$$\begin{aligned} I_1 &\leq \left[\frac{4C'\omega\varepsilon^{1/2}}{(1-4R^2)^2}\right]\frac{r_0^2}{(1+r_0^2)}\left(2R\varepsilon^{-1/2} - R\varepsilon^{-1/2}\right) = \left[\frac{4C'\omega\varepsilon^{1/2}}{(1-4R^2)^2}\right]\frac{Rr_0^2\varepsilon^{-1/2}}{\left(1+r_0^2\right)} \\ &\leq \left[\frac{4C'\omega\varepsilon^{1/2}}{(1-4R^2)^2}\right]\frac{R(2R\varepsilon^{-1/2})^2\varepsilon^{-1/2}}{\left(1+(R\varepsilon^{-1/2})^2\right)} = \left[\frac{4C'\omega}{(1-4R^2)^2}\right]\frac{2^2R^3\varepsilon^{-1}}{\left(1+\frac{R^2}{\varepsilon}\right)}. \end{aligned}$$

Therefore,

$$I_1 \quad \leq \quad \frac{C(R)}{\varepsilon+R^2}.$$

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

Since $0 < \varepsilon \leq 1$, then

$$\frac{1}{1+R^2} \leq \frac{1}{\varepsilon + R^2} \leq \frac{1}{R^2}.$$

Therefore,

$$I_1 \leq \frac{C(R)}{R^2}.$$

□

3. Proof of the theorem. Taking the sequence $\{v_n\}$ given by equation (2.5), by Lemma 2.2 this sequence $\{v_n\}$ is bounded in $H^1_{0,r}(\Omega)$. So that, we can assume, passing to a subsequence, that $v_n \rightharpoonup v$, weakly in $H^1_{0,r}(\Omega)$, as $n \to \infty$ and

$$J'(v_n)w = o(1), \forall\, w \in H^1_{0,r}(\Omega). \tag{3.19}$$

Now, note that

$$|J'(v_n)w - J'(v)w| \to 0, \tag{3.20}$$

as $n \to \infty$, for all $w \in C^\infty_{c,rad}(\Omega)$. From this, it follows that $J'(v)w = 0$, for all $w \in C^\infty_{c,rad}(\Omega)$. By density we conclude that

$$J'(v)w = 0, \forall w \in H^1_{0,r}(\Omega), \tag{3.21}$$

and v is a critical point of the functional J restricted to the space $H^1_{0,r}(\Omega)$.

Now, we will follow the ideas of [**14**, **21**, **31**] (see also [**52**]). Since $H^1_{0,r}(\Omega)$ is a closed subspace of $H^1_0(\Omega)$, we can write

$$H^1_0(\Omega) = H^1_{0,r}(\Omega) \oplus H^1_{0,r}(\Omega)^\perp,$$

where $\cdot^\perp$ denotes the orthogonal complement of the space. Therefore, for each $w \in H^1_0(\Omega)$, there exist $\vartheta \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$ such that

$$w = \vartheta + \vartheta^\perp. \tag{3.22}$$

As $H^1_{0,r}(\Omega)$ is a Hilbert space and $J'(v) \in H^1_{0,r}(\Omega)^*$, from the Riesz Representation Theorem there exists $z \in H^1_{0,r}(\Omega)$ such that

$$J'(v)w = \int_\Omega \nabla z \cdot \nabla w, \quad \text{for all } w \in H^1_{0,r}(\Omega).$$

Thus, as $z \in H^1_{0,r}(\Omega)$ and $\vartheta^\perp \in H^1_{0,r}(\Omega)^\perp$, we have

$$J'(v)\vartheta^\perp = 0. \tag{3.23}$$

From equations (3.21)–(3.23), for each $w \in H^1_0(\Omega)$, we obtain

$$J'(v)w = J'(v)\vartheta + I'(v)\vartheta^\perp = 0.$$

This allows us to conclude that v is a critical point of the functional J in $H^1_0(\Omega)$ and consequently v is a weak solution for the problem (1.1).

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

If $v \neq 0$ we are done.

Suppose now that $v \equiv 0$. Considering $v_n \rightharpoonup 0$, as $n \to \infty$, we have

$$J'(v_n)v_n = a\|v_n\|^2 + b\|v_n\|^4 - \lambda \int_\Omega p^\alpha |v_n|^q - \int_\Omega |v_n|^6 = o_n(1). \tag{3.24}$$

But

$$\lambda \int_\Omega p^\alpha |v_n|^q \to 0, \text{ as } n \to \infty. \tag{3.25}$$

Let $L_1 > 0$, $L_2 > 0$ be such that

$$a\|v_n\|^2 \to L_1 \quad \text{and} \quad b\|v_n\|^4 \to L_2, \text{ as } n \to \infty. \tag{3.26}$$

By equations (3.24)–(3.26)

$$\int_\Omega |v_n|^6 \to L_1 + L_2, \text{ as } n \to \infty. \tag{3.27}$$

But

$$S\left(\int_\Omega v_n^6\right)^{1/3} \le \int_\Omega |\nabla v_n|^2, \tag{3.28}$$

which implies

$$aS\left(\int_\Omega v_n^6\right)^{1/3} \le a\int_\Omega |\nabla v_n|^2 \le a\int_\Omega \left(|\nabla v_n|^2 + (3/4)p^2 v_n^2\right) = a\|v_n\|^2, \tag{3.29}$$

and

$$bS^2\left(\int_\Omega v_n^6\right)^{2/3} \le b\left[\int_\Omega |\nabla v_n|^2\right]^2 \le b\left[\int_\Omega \left(|\nabla v_n|^2 + (3/4)p^2 v_n^2\right)\right]^2 = b\|v_n\|^4. \tag{3.30}$$

Thus, by equations (3.26), (3.27), (3.29), and (3.30)

$$L_1 \ge aS\,(L_1 + L_2)^{1/3} \quad \text{and} \quad L_2 \ge bS^2\,(L_1 + L_2)^{2/3}. \tag{3.31}$$

On the other hand, $J(v_n) = c + o(1)$. So

$$c = \frac{L_1}{2} + \frac{L_2}{4} - \frac{1}{6}(L_1 + L_2) = \frac{L_1}{3} + \frac{L_2}{12}. \tag{3.32}$$

By equation (3.31), we have

$$(L_1 + L_2)^{1/3} \ge \frac{bs^2 + (b^2 s^4 + 4as)^{1/2}}{2}. \tag{3.33}$$

Hence by equations (3.31)–(3.33)

$$\begin{aligned} c &\ge \frac{1}{3}L_1 + \frac{1}{12}L_2 \ge \frac{1}{3}aS(L_1 + L_2)^{1/3} + \frac{1}{12}bS^2\left[(L_1 + L_2)^{1/3}\right]^2 \\ &\ge \frac{1}{4}abS^3 + \frac{1}{24}b^3 S^6 + \frac{1}{24}(b^2 S^4 + 4aS)^{3/2}, \end{aligned}$$

which is a contradiction with Lemma 2.3. Therefore, we conclude that $v \neq 0$.

https://doi.org/10.1017/S0017089518000563 Published online by Cambridge University Press

ACKNOWLEDGMENTS. This paper was performed while P. C. C. was visiting the Department of Mathematics of UFJF, under financial support by FAPEMIG CEX APQ 00063/15. A. C. R. C. was supported in part by PNPD/CAPES 2017—PGM/UFJF. O. H. M. is the corresponding author and he received research grants from CNPq/Brazil 304015/2014-8, FAPEMIG CEX APQ 00063/15 and INCTMAT/CNPQ/Brazil.

REFERENCES

1. C. O. Alves, F. J. S. A. Corrêa and T. F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.* **49** (2005), 85–93.

2. C. O. Alves and G. M. Figueiredo, Nonlinear perturbations of a periodic Kirchhoff equation in $\mathbb{R}^N$, *Nonlinear Anal.* **75** (2012), 2750–2759.

3. A. Ambrosetti and M. Struwe, A note on the problem $\Delta u = \lambda u + u|u|^{2^*-2}$, *Manuscripta Math.* **54** (1986), 373–379.

4. A. Arosio, and S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.* **348** (1996), 305–330.

5. F. V. Atkinson and A. Peletier, Emden–Fowler equations involving critical exponents, *Nonlinear Anal.* **10** (1986), 755–176.

6. T. Aubin and I. Ekeland, *Applied nonlinear analysis* (Dover Publication, New York, 1984).

7. G. Autuori and P. Pucci, Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$, *J. Differ. Equ.* **255** (2013), 2340–2362.

8. C. Bandle and R. Benguria, The Brezis–Nirenberg problem on $\mathcal{S}^N$, *J. Differ. Equ.* **178**(1) (2002), 264–279.

9. C. Bandle and Y. Kabeya, On the positive, "radial" solutions of a semilinear elliptic equation in $\mathbb{H}^N$, *Adv. Nonlinear Anal.* **1**(1) (2012), 1–25.

10. V. Benci and G. Cerami, Existence of positive solutions of the equation $\Delta u + a(x)u = u^{\frac{n+2}{n-2}}$ in $\mathbb{R}^N$, *J. Funct. Anal.* **86** (1996), 90–117.

11. S. Benguria, The solution gap of the Brezis–Nirenberg problem on the hyperbolic space, *Monatsh. Math.* **181**(3) (2016), 537–559.

12. R. Benguria and S. Benguria, The Brezis–Nirenberg problem on $\mathcal{S}^N$ in spaces of fractional dimension, arXiv:1503.06347 (2015).

13. M. Bhakta and K. Sandeep, Poincaré–Sobolev equations in the hyperbolic spaces, *Calc. Var. Partial Differ. Equ.* **44** (2012), 247–269.

14. G. Bianchi, J. Chabrowski and A. Szulkin, On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev expoent, *Nonlinear Anal. TMA* **25**(1) (1995), 41–59.

15. L. P. Bonorino and P. K. Klaser, Existence and nonexistence results for eigenfunctions of the Laplacian in unbounded domains of $\mathcal{H}^N$, arXiv:1310.3133 (2013).

16. H. Brezis, Nonlinear elliptic equations involving the critical Sobolev exponent: survey and perspectives, in *Directions in partial differential equations* (Crandall M. G., Rabinowitz P. H. and Turner R. E. L., Editors) Academic Press, New York, 1987, pp. 17–36.

17. H. Brezis and M. Marcus, Hardy's inequalities revisited, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4), **25**(1–2) (1997), 217–237.

18. H. Brezis, M. Marcus and I. Shafrir, Extremal functions for Hardy's inequality with weight, *J. Func. Anal.* **171**(1) (2000), 177–191.

19. H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.* **36** (1983), 437–477.

20. A. Capozzi, D. Fortunato and G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, *Ann. Inst. H. Poincaré: Analyse non Lineaire* **2**(6) (1985), 463–470.

21. P. C. Carrião, O. H. Miyagaki and J. C. Pádua, Radial solutions of elliptic equations with critical exponents in $\mathbb{R}^N$, *Differ. Integral Equ.* **11**(1) (1998), 61–68.

22. P. C. Carrião, R. Lehrer, O. H. Miyagaki and A. Vicente, A nonhomogeneous Brezis–Nirenberg problem on the hyperbolic space $\mathbb{H}^n$, submitted.

23. G. Cerami, D. Fortunato and M. Struwe, Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents, *Ann Inst. H. Poincaré: Analyse non Lineaire* **l**(5) (1985), 341–350.

24. G. Cerami, S. Solimini and M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents, *J. Funct. Anal.* **69** (1986), 289–306.

25. C. Y. Chen, Y. C. Kuo and T. F. Wu, The Nehari manifold for a Kirchhoff type problem involving signchanging weight functions, *J. Differ. Equ.* **250** (2011), 1876–1908.

26. B. Cheng, S. Wu and J. Liu, Multiplicity of nontrivial solutions for Kirchhoff type problems, *Bound. Value Probl.* **2010** (2010), 268946. doi: 10.1155/2010/268946

27. F. Colasuonno and P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations. *Nonlinear Anal.* **74**(17) (2011), 5962–5974.

28. F. J. S. A. Corrêa and R. G. Nascimento, On a nonlocal elliptic system of p-Kirchhoff type under Neumann boundary condition, *Math. Comput. Modell.* **49** (2009), 598–604.

29. E. N. Dancer, A note on an equation with critical exponent, *Bull. London Math. Soc.* **20** (1988), 600–602.

30. P. D'Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **108** (1992), 247–262.

31. Y. B. Deng, H. S. Zhong and X. P. Zhu, On the existence and $L^p(R^N)$ bifurcation for the semilinear elliptic equation, *J. Math. Anal. Appl.* **154** (1991), 116–133.

32. W. Y. Ding and W. M. Ni, On the elliptic equation $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ and related topics, *Duke Math. J.* **52** (1985), 485–506.

33. H. Egnell, Existence and nonexistence results for m-Laplace equations involving critical Sobolevc exponents, *Arch. Ration. Mech. Anal.* **104** (1988), 57–77.

34. G. M. Figueiredo, Existence of positive solution for a Kirchhoff problem type with critical growth via truncation argument. *J. Math. Anal. Appl.* **401** (2013), 706–713.

35. G. M. Figueiredo and J. Santos Junior, Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth, *Differ. Integral Equ.* **25** (2012), 853–868.

36. D. Ganguly and K. Sandeep, Sign changing solutions of the Brezis–Nirenberg problem in the hyperbolic space, *Calc. Var. Partial Differ. Equ.* **50**(1–2) (2014), 69–91.

37. D. Ganguly and K. Sandeep, Nondegeneracy of positive solutions of semilinear elliptic problems in the hyperbolic space, *Commun. Contemp. Math.* **17**(1) (2015), 1450019.

38. J. Garcia Azorero and I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, *Trans. Am. Math. Soc.* **323**(2) (1991), 877–895.

39. M. García-Huidobro and C. S. Yarur, On quasilinear Brezis–Nirenberg type problems with weights, *Adv. Differ. Equ.* **15**(5–6) (2010), 401–436.

40. M. Guedda and L. Veron, Quasilinear elliptic equations involving critical Sobolev exponents, *Nonlinear Anal.* **13**(8) (1989), 879–902.

41. H.-Y. He, Supercritical elliptic equation in hyperbolic space, *J. Partial Differ. Equ.* **28**(2) (2015), 120–127.

42. H.-Y. He and G.-B. Li, Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, *Calc. Var.* **54** (2015), 3067–3106.

43. H.-Y. He, G.-B. Li and S.-J. Peng, Concentrationg bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, *Adv. Nonl. Stud.* **14** (2014), 483–510.

44. X. He and W. Zou, Infinitely many positive solutions for Kirchhoff-type problems, *Nonlinear Anal.* **70** (2009), 1407–1414.

45. G. Kirchhoff, *Mechanik* (Teubner, Leipzig, 1883).

46. J. Lions, On some questions in boundary value problems of mathematical physics, in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations. Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro (1997)*. vol. 30 (North-Holland Mathematics Studies, Amsterdam, 1978), 284–346.

47. Z. Liu and S. Guo, Existence and concentration of positive ground states for a Kirchhoff equation involving critical Sobolev exponent, *Z. Angew. Math. Phys.* **66** (2015), 747–769.

48. T. Ma and J. Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, *Appl. Math. Lett.* **16** (2003), 243–248.

49. G. Mancini, and K. Sandeep, On a semilinear elliptic equation in $\mathcal{H}^N$, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **7**(5) (2008), 635–671.

50. O. H. Miyagaki, On a class of semilinear elliptic problems in $\mathbb{R}^N$ with critical growth, *Nonlinear Anal. Theory, Meth. Appl.* **29**(7) (1997), 773–781.

51. E. S. Noussair, C. A. Swanson and J. Yang, Positive finite energy solutions of critical semilinear elliptic problems, *Can. J. Math.* **44**(5) (1992), 1014–1029.

52. R. S. Palais, The Principle of Symmetric Criticality, *Commun. Math. Phys.* **69** (1979) 19–30.

53. K. Perera and Z. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, *J. Differ. Equ.* **21** (2006), 246–255 .

54. J. G. Ratcliffe, *Foundations of hyperbolic manifolds*. Graduate Texts in Mathematics, vol. 149 (Springer, New York, 1994).

55. B. Ricceri, On an elliptic Kirchhoff-type problem depending on two parameters, *J. Global Optim.* **46** (2010), 543–549.

56. M. Schechter and W.-M. Zou, On the Brezis–Nirenberg problem, *Arch. Ration. Mech. Anal.* **197**(1) (2010), 337–356.

57. S. Stapelkamp, The Brezis–Nirenberg problem on $\mathbf{B}^N$: existence and uniqueness of solutions, in *Elliptic and Parabolic Problems, Rolduc and Gaeta, 2001* (World Scientific, Singapore, 2002), 283–290.

58. S. Stoll, Harmonic function theory on real hyperbolic space, Preliminary draft, http: citeseerx.ist.psu.edu.

59. G. Talenti, Best constants in Sobolev inequality, *Ann. Math. Pura Appl.* **110** (1976), 353–372.

60. J. Wang, L. Tian, J. Xu and F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differ. Equ.* **253** (2012), 2314–2351.

61. M. Willem, *Minimax theorems* (Birkhäuser Boston, Basel, Berlin, 1996).

62. X.-R. Yue and W.-M. Zou, Remarks on a Brezis–Nirenbergś result, *J. Math. Anal. Appl.* **425**(2) (2015), 900–910.

63. X.-P. Zhu and J. Yang, The quasilinear elliptic equations on unbounded domain involving critical Sobolev exponent, *J. Partial Differ. Equ.* **2**(2) (1989), 53–64.