A series identity

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I have constructed two simple proofs of the result established by J. C. P. Miller, *Mathematical Notes*, No. 29 (1935), pp. vi-viii.

Miller's result may be stated in the following form:

Let

$$A_0 = 1, \quad A_n = \left(-\frac{1}{2}\cdot\frac{1}{4}\cdots\frac{2n-3}{2n}\right)^2, \qquad (n > 0)$$

and write

$$S_1 = \sum_{n=0}^{\infty} \frac{A_n}{2x+2n}, \quad S_2 = \sum_{n=0}^{\infty} \frac{A_n}{2x-2n+1}.$$

Then $$S_1 = S_2,$$

whenever x is a positive integer.

Taking x to be unrestricted, combine like terms of S_1 and S_2; we get

$$S_1 - S_2 = \sum_{n=0}^{\infty} \frac{A_n(1-4n)}{(2x+2n)(2x-2n+1)}$$
$$= \frac{1}{2x(2x+1)} \cdot {}_5F_4\begin{pmatrix} -\tfrac{1}{2}, & \tfrac{3}{4}, & -\tfrac{1}{2}, & x, & -x-\tfrac{1}{2} \\ & -\tfrac{1}{4}, & 1, & x+1, & \tfrac{1}{2}-x \end{pmatrix}.$$

Now, by a theorem due to Dougall,

$${}_5F_4\begin{pmatrix} a, & 1+\tfrac{1}{2}a, & c, & d, & e \\ & \tfrac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e \end{pmatrix}$$
$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-e-c)\,\Gamma(1+a-c-d)}.$$

In Dougall's theorem write

$$a = c = -\tfrac{1}{2}, \quad d = x, \quad e = -x-\tfrac{1}{2},$$

and we find at once that

$$S_1 - S_2 = \frac{1}{2x(2x+1)} \, \frac{\Gamma(1)\,\Gamma(\tfrac{1}{2}-x)\,\Gamma(1+x)\,\Gamma(\tfrac{3}{2})}{\Gamma(\tfrac{1}{2})\,\Gamma(1)\,\Gamma(\tfrac{3}{2}+x)\,\Gamma(1-x)},$$

i.e.

$$S_1 - S_2 = -\frac{\Gamma(x)\Gamma(-x-\tfrac{1}{2})}{8\,\Gamma(\tfrac{3}{2}+x)\,\Gamma(1-x)},$$

whence Miller's result is obvious. A proof of Dougall's theorem is to be found in W. N. Bailey's *Generalised hypergeometric series* (Cambridge, 1935), pp. 27-28, with complete references to the relevant literature.

The result given above for unrestricted x suggests my second proof, which is considerably simpler. From the asymptotic expansion of the Gamma-function it follows that the function

$$-\frac{\Gamma(x)\,\Gamma(-x-\frac{1}{2})}{8\Gamma(\frac{3}{2}+x)\,\Gamma(1-x)}$$

satisfies the conditions (*Cf. Modern Analysis*, § 7·4) which admit of its being expressed as a sum of partial fractions. Since the residues of the function at the poles $x=-n$, $x=n-\frac{1}{2}$ are respectively

$$\tfrac{1}{2}A_n, \quad -\tfrac{1}{2}A_n,$$

we obtain anew the result

$$-\frac{\Gamma(x)\,\Gamma(-x-\frac{1}{2})}{8\,\Gamma(\frac{3}{2}+x)\,\Gamma(1-x)}=\sum_{n=0}^{\infty}\frac{A_n}{2x+2n}-\sum_{n=0}^{\infty}\frac{A_n}{2x-2n+1}$$

for unrestricted x.

It is, of course, possible to write down any number of generalisations of this result, the simplest perhaps being the expression for

$$\frac{\Gamma(x)\,\Gamma(-x-a)}{\Gamma(1+a+x)\,\Gamma(1-x)}$$

as a sum of partial fractions, where a is a suitably restricted constant.

On the configuration known as a double-six of lines

By H. W. Richmond, King's College, Cambridge.

In geometry of three dimensions it is well known that, when two quadrics Q_1, Q_2 are given, if one set of four points exists having the properties that each point lies on Q_1, and each two points are conjugate with respect to Q_2, an infinity of such sets of points can be found. The quadrics Q_1, Q_2 stand in a special relation to one another[1], expressed by the vanishing of the coefficient of λ in the discriminant of $Q_2+\lambda Q_1$, an invariant of Q_1, Q_2. Two quadrics Q_1, Q_2 are thus related if the equation of Q_1 contains no squares

[1] See Salmon, *Analytic Geometry of Three Dimensions*, Rogers' revised edition, Vol. 1, p. 204; or Sommerville, *Analytical Geometry of Three Dimensions*, p. 309.

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