The Co-apolars of a Cubic Curve.

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1. It is a well-known fact and easy to prove that if Γ and Γ' be any two class-cubics and P a variable point, such that the pencil of lines from P to Γ apolarly separate the pencil of lines from P to Γ', then the locus of P is a cubic curve G called the "apolar locus" of Γ and Γ'. Also, Γ and Γ' are said to be "co-apolar class-cubics" of G, or simply "co-apolars" of G. The problem of finding the general system of co-apolars of a given cubic curve has not yet been completely solved, but particular and more important cases have been investigated by me in several papers contributed to the London Mathematical Society. In the present communication I propose to deal with the most general solution of the above problem.

Let the assigned cubic curve be taken in the form

$$G \equiv ax^3 + by^3 + cz^3 + 3a_2 x^2 y + 3a_3 x^2 z + 3b_3 y^2 z + 3b_1 y^2 x \\ + 3c_1 z^2 x + 3c_2 z^2 y + 6k\, xyz = 0, \quad \ldots\ldots(1)$$

and let Γ be a class-cubic where

$$\Gamma \equiv \lambda\,(l^3 + m^3 + n^3) + 6\rho\, lmn = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We wish to find the system of class-cubics Γ' co-apolar with Γ with respect to G.

Let

$$\Gamma' \equiv Al^3 + Bm^3 + Cn^3 + 3A_2 l^2 m + 3A_3 l^2 n + 3B_3 m^2 n + 3B_1 m^2 l \\ + 3C_1 n^2 l + 3C_2 n^2 m + 6K\, lmn = 0. \quad \ldots\ldots(3)$$

Then the Apolar Locus of Γ and Γ' is

$$\lambda\,(\overline{C-B}x^3 + \overline{A-C}y^3 + \overline{B-A}z^3 + 3B_1 x^2 y - 3C_1 x^2 z + 3C_2 y^2 z \\ - 3A_2 y^2 x + 3A_3 z^2 x - 3B_3 z^2 y) \\ + 6\rho\,(C_2 x^2 y - B_3 x^2 z + A_3 y^2 z - C_1 y^2 x + B_1 z^2 x - A_2 z^2 y) = 0 \quad (4)$$

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Identifying (1) and (4), we obtain

$$\lambda(C-B)=a \qquad \text{..........} (5)$$

$$\lambda(A-C)=b \qquad \text{..........} (6)$$

$$\lambda(B-A)=c \qquad \text{..........} (7)$$

$$\lambda B_1+2\rho C_2=a_2 \qquad \text{..........} (8)$$

$$-\lambda C_1-2\rho B_3=a_3 \qquad \text{..........} (9)$$

$$\lambda C_2+2\rho A_3=b_3 \qquad \text{..........} (10)$$

$$-\lambda A_2-2\rho C_1=b_1 \qquad \text{..........} (11)$$

$$\lambda A_3+2\rho B_1=c_1 \qquad \text{..........} (12)$$

$$-\lambda B_3-2\rho A_2=c_2 \qquad \text{..........} (13)$$

$$0=k \qquad \text{..........} (14)$$

(5), (6), (7) taken together show that

$$a+b+c=0. \qquad \text{..........} (15)$$

Hence (14) and (15) taken together show that (1) is apolar to every member of the system (2) where λ and ρ are supposed to vary. We thus see that *a necessary condition of* Γ *being used as a co-apolar generator of* G *is that* G *be apolar to* Γ, *and also to the Hessian of* Γ.

If we therefore suppose the two conditions (14) and (15) satisfied, and if in addition we put

$$A+B+C=3\sigma, \qquad \text{..........} (16)$$

we obtain, on solving the above system of equations,

$$A=\frac{b-c}{3\lambda}+\sigma$$

$$B=\frac{c-a}{3\lambda}+\sigma$$

$$C=\frac{a-b}{3\lambda}+\sigma$$

$$\left.\begin{aligned} -A_2&=\lambda^2 b_1-2\rho\lambda a_3+4\rho^2 c_2\\ A_3&=\lambda^2 c_1-2\rho\lambda a_2+4\rho^2 b_3\\ -B_3&=\lambda^2 c_2-2\rho\lambda b_1+4\rho^2 a_3\\ B_1&=\lambda^2 a_2-2\rho\lambda b_3+4\rho^2 c_1\\ -C_1&=\lambda^2 a_3-2\rho\lambda c_2+4\rho^2 b_1\\ C_2&=\lambda^2 b_3-2\rho\lambda c_1+4\rho^2 a_2 \end{aligned}\right\} \div (\lambda^3+8\rho^3). \qquad \text{..........} (17)$$

$$K=\tau.$$

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Hence the co-apolars of Γ belong to the system

$$\Gamma_0' + \sigma(l^3 + m^3 + n^3) + 6\tau\, lmn = 0 \quad \text{.................}(18)$$

where

$$\Gamma_0' \equiv \frac{b-c}{3\lambda} l^3 + \frac{c-a}{3\lambda} m^3 + \frac{a-b}{3\lambda} n^3$$

$$+ \frac{3}{\lambda^3 + 8\rho^3} \left\{ \begin{array}{r} -(\lambda^2 b_1 - 2\rho\lambda a_3 + 4\rho^2 c_2)\, l^2 m + (\lambda^2 c_1 - 2\rho\lambda a_2 + 4\rho^2 b_3)\, l^2 n \\ -(\lambda^2 c_2 - 2\rho\lambda b_1 + 4\rho^2 a_3)\, m^2 n \\ +(\lambda^2 a_2 - 2\rho\lambda b_3 + 4\rho^2 c_1)\, m^2 l - (\lambda^2 a_3 - 2\rho\lambda c_2 + 4\rho^2 b_1)\, n^2 l \\ +(\lambda^2 b_3 - 2\rho\lambda c_1 + 4\rho^2 a_2)\, n^2 m \end{array} \right\} = 0.$$

(19)

***Hence the co-apolars of* Γ *with respect to G form a doubly infinite linear system of class-cubics. In fact, if* Γ' *be any one of the co-apolars of* Γ *with respect to G, the complete system of co-apolars of* Γ *consists of the net of class-cubics defined by* Γ_0', Γ *and the Hessian of* Γ.** (20)

N.B.—Γ_0' is that particular member of the co-apolar system which is apolar to $(x^3 + y^3 + z^3) + 6\mu\, xyz = 0$ for all values of μ, *i.e.* to the Cayleyan and the Hessian of the Cayleyan of Γ.

2. Let us now consider the degenerate members of the above net of co-apolar class-cubics. Two conditions are necessary in order that a class-cubic should degenerate into a point and a conic. We should therefore expect that a finite number of the co-apolars of Γ consist each of a point and a conic, because we have the two parameters σ and τ at our disposal.

Let

$$\begin{aligned}\Gamma_1' &\equiv (lX + mY + nZ)(\alpha l^2 + \beta m^2 + \gamma n^2 + \delta\, mn + \epsilon\, nl + \zeta\, lm) \\ &\equiv \alpha X l^3 + \beta Y m^3 + \gamma Z n^3 + (\alpha Y + \zeta X)\, l^2 m + (\alpha Z + \epsilon X)\, l^2 n \\ &\qquad + (\beta Z + \delta Y)\, m^2 n \\ &\quad + (\beta X + \zeta Y)\, m^2 l + (\gamma X + \epsilon Z)\, n^2 l + (\gamma Y + \delta Z)\, n^2 m \\ &\qquad + (\delta X + \epsilon Y + \zeta Z)\, lmn. \quad (21)\end{aligned}$$

Hence, since (21) must belong to the net of class-cubics described in (20), Γ_1' must be of the form

$$\Gamma_0' + \mu(l^3 + m^3 + n^3) + 6\nu\, lmn = 0 \quad \text{...............}\ (22)$$

where Γ' is defined in (3). Identifying (21) and (22), we obtain

$$\alpha X = A + \mu \qquad \text{.................................(23)}$$

$$\beta Y = B + \mu \qquad \text{.................................(24)}$$

$$\gamma Z = C + \mu \qquad \text{.................................(25)}$$

$$\alpha Y + \zeta X = 3A_2 \qquad \text{.................................(26)}$$

$$\alpha Z + \epsilon X = 3A_3 \qquad \text{.................................(27)}$$

$$\beta Z + \delta Y = 3B_3 \qquad \text{.................................(28)}$$

$$\beta X + \zeta Y = 3B_1 \qquad \text{.................................(29)}$$

$$\gamma X + \epsilon Z = 3C_1 \qquad \text{.................................(30)}$$

$$\gamma Y + \delta Z = 3C_2 \qquad \text{.................................(31)}$$

$$\delta X + \epsilon Y + \zeta Z = 6(K + \nu). \qquad \text{..........................(32)}$$

If now we attempt to eliminate α, β, γ, δ, ϵ, ζ, μ, ν, and to solve for X, Y, Z, we shall find that there are *9* sets of values for (X, Y, Z) defined by the intersections of the apolar loci of Γ' and Γ, Γ' and the Hessian of Γ.

Eliminating δ, ϵ, ζ from (26), (27), ... (31), we get

$$\beta Z^2 - \gamma Y^2 = 3(B_3 Z - C_2 Y) \qquad \text{....................(33)}$$

$$\gamma X^2 - \alpha Z^2 = 3(C_1 X - A_3 Z) \qquad \text{....................(34)}$$

$$\alpha Y^2 - \beta X^2 = 3(A_2 Y - B_1 X) \qquad \text{....................(35)}$$

Also, multiplying (33), (34), (35) by YZ, ZX, XY respectively, and substituting for α, β, γ from (23), (24), (25), we obtain

$$(B + \mu) Z^3 - (C + \mu) Y^3 = 3\,YZ(B_3 Z - C_2 Y) \qquad \text{.........(36)}$$

$$(C + \mu) X^3 - (A + \mu) Z^3 = 3ZX(C_1 X - A_3 Z) \qquad \text{.........(37)}$$

$$(A + \mu) Y^3 - (B + \mu) X^3 = 3XY(A_2 Y - B_1 X). \qquad \text{.........(38)}$$

Adding (36), (37), (38), we obtain

$$(B - C) X^3 + (C - A) Y^3 + (A - B) Z^3 + 3YZ(B_3 Z - C_2 Y)$$
$$+ 3ZX(C_1 X - A_3 Z) + 3XY(A_2 Y - B_1 X) = 0,$$

which is the Apolar Locus of Γ and $l^3 + m^3 + n^3 = 0$ by (4).

If, again, we multiply (36), (37), (38) by X^3, Y^3, Z^3 respectively, add and divide by XYZ, we see that (X, Y, Z) lies on

$$X^2(B_3 Z - C_2 Y) + Y^2(C_1 X - A_3 Z) + Z^2(A_2 Y - B_1 X) = 0,$$

i.e. the Apolar Locus of Γ' and $lmn = 0$ by (4).

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Hence there are nine co-apolars of Γ *with respect to* G *which each consist of a point and a conic. The points are the* 9 *intersections of the Apolar Loci of* Γ' *and* Γ, Γ' *and the Hessian of* Γ, *where* Γ' *is any one of the co-apolars of* Γ *with respect to* G. *These nine points therefore constitute an "associated system of points" on* G *itself.*

We next wish to show that if (X, Y, Z) be any one of the above 9 points, the conic which along with it forms a co-apolar of Γ is uniquely determined.

Eliminating α, β, γ between (26), (27), ... (31), we deduce

$$X(\zeta Z - \epsilon Y) = 3(A_2 Z - A_3 Y) \quad \text{.............. (39)}$$

$$Y(\delta X - \zeta Z) = 3(B_3 X - B_1 Z) \quad \text{.............. (40)}$$

$$Z(\epsilon Y - \delta X) = 3(C_1 Y - C_2 X). \quad \text{.............. (41)}$$

Hence, by the subtraction of (40) from (41) and the use of (32), the following results appear :—

$$\delta X = \frac{(B_3 X - B_1 Z)}{Y} - \frac{(C_1 Y - C_2 X)}{Z} + 2(K+\nu) \quad \text{...... (42)}$$

$$\epsilon Y = \frac{(C_1 Y - C_2 X)}{Z} - \frac{(A_2 Z - A_3 Y)}{X} + 2(K+\nu) \quad \text{...... (43)}$$

$$\zeta Z = \frac{(A_2 Z - A_3 Y)}{X} - \frac{(B_3 X - B_1 Z)}{Y} + 2(K+\nu). \quad \text{...... (44)}$$

If we substitute from (23), (24), (25), (42), (43), (44), in (26), (27), (28), (29), (30), (31), we obtain in addition to the Apolar Loci of Γ, $l^3+m^3+n^3$ and lmn, linear equations for μ and ν in terms of X, Y, Z.

Hence there exist nine *conics which correspond each to each to one of the* nine *intersections of the Apolar Loci of* Γ' *and* Γ, Γ' *and the Hessian of* Γ *where* Γ' *is any one of the co-apolars of* Γ *with respect to* G. *These nine points and their corresponding conics constitute the* nine *degenerate co-apolar members of the* Γ' *system.*

3. Corresponding to every value of $\lambda : \rho$ in the system

$$\Gamma \equiv \lambda(l^3+m^3+n^3) + 6\rho\, lmn = 0$$

we get a determinate system of nine "associated points" on the original cubic G. By what has been proved above, we see that

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these nine points may be regarded as the intersection of G with the cubic

$$x^2(B_3 z - C_2 y) + y^2(C_1 x - A_3 z) + z^2(A_2 y - B_1 x) = 0,$$

i.e. in virtue of (17)

$$\begin{aligned} &\lambda^2(b_3 x^2 y + c_2 x^2 z + c_1 y^2 z + a_3 y^2 x + a_2 z^2 x + b_1 z^2 y) \\ &- 2\rho\lambda(c_1 x^2 y + b_1 x^2 z + a_2 y^2 z + c_2 y^2 x + b_3 z^2 x + a_3 z^2 y) \\ &+ 4\rho^2(a_2 x^2 y + a_3 x^2 z + b_3 y^2 z + b_1 y^2 x + c_1 z^2 x + c_2 z^2 y) = 0. \end{aligned}$$

This is a much simpler system of cubic curves than one would have expected from the nature of the problem, and would probably yield interesting results on further investigation.

4. The above investigation sheds a good deal of light indirectly on a paper published by me in the *Proceedings of the Edinburgh Mathematical Society* (Session 1911–12), on "The system of cubic curves circumscribing two triangles and apolar to them." We can show that *if ABC and DEF be the two triangles, and if* $\Sigma_1 \equiv 6\,lmn = 0$ *and* $\Sigma_2 \equiv Al^3 + \text{etc.} = 0$ *be respectively the tangential equations to the vertices of ABC and DEF, the system of cubic curves circumscribing ABC, DEF and apolar to them is the Apolar Loci of* $\lambda\Sigma_1 + \Sigma_2$ *and the Hessian of* $\mu\Sigma_1 + \Sigma_2$, λ *and* μ *being arbitrary.*

For the circumscribing system was shown in the above paper to consist of the pencil defined by the two cubic curves,

$$C_2 x^2 y - B_3 x^2 z + A_3 y^2 z - C_1 y^2 x + B_1 z^2 x - A_2 z^2 y = 0 \quad .. \ (45)$$

and

$$CB_1 x^2 y - BC_1 x^2 z + AC_2 y^2 z - CA_2 y^2 x + BA_3 z^2 x - AB_3 z^2 y = 0. \ (46)$$

(45) is plainly the Apolar Locus of Σ_1 (or Σ_2) and $\lambda\Sigma_1 + \Sigma_2 = 0$.

Again, the Hessian of $\lambda\Sigma_1 + \Sigma_2$ is

$$\begin{aligned} &\{AB_1C_1 - A(K+\lambda)^2 + 2A_2A_3(K+\lambda) - B_1A_3^2 - C_1A_2^2\}\,l^3 + \text{etc.} \\ &+ \{ABC_1 - 2AB_3(K+\lambda) + AB_1C_2 - BA_3^2 + A_2(K+\lambda)^2 \\ &\qquad - B_1C_1A_2 + 2A_2A_3B_3 - C_2A_2^2\}\,l^2 m + \text{etc.} \\ &+ \{ABC - (AB_3C_2 + BC_1A_3 + CA_2B_1) + 2(K+\lambda)^3 \\ &\qquad - 2(K+\lambda)(B_1C_1 + C_2A_2 + A_3B_3) + 3(A_2B_3C_1 + A_3B_1C_2)\}\,lmn = 0, \end{aligned}$$

i.e. on expansion of powers of λ (I being a certain Invariant),

$$I\Sigma_2 + \lambda\,\{(-2AK+2A_2A_3)\,l^3 + (-2BK+2B_3B_1)\,m^3$$
$$+(-2CK+2C_1C_2)\,n^3 + (-2AB_3+2A_2K)\,l^2m$$
$$+(-2AC_2+2A_3K)\,l^2n + (-2BC_1+B_3K)\,m^2n$$
$$+(-2BA_3+2B_1K)\,m^2l + (-2CA_2+2C_1K)\,n^2l$$
$$+(-2CB_1+2C_2K)\,n^2m + (6K^2 - 2\overline{B_1C_1+C_2A_2+A_3B_3})\,lmn\}$$
$$+\lambda^2\{-Al^3 - Bm^3 - Cn^3 + A_2l^2m + A_3l^2n + B_3m^2n + B_1m^2l$$
$$+C_1n^2l + C_2n^2m + 6K\,lmn\}$$
$$+2\lambda^3\,lmn = 0 \qquad\qquad (47)$$

(since the Hessian of the points is three points).

Now the Apolar Locus of Σ_2 and lmn is (45). Also the Apolar Locus of the coefficient of λ^2 and lmn is (45), while the Apolar Locus of the coefficient of λ^3 and lmn vanishes identically.

Finally, the Apolar Locus of the coefficient of λ and lmn is

$$(-CB_1x^2y + BC_1x^2z - AC_2y^2z + CA_2y^2x - BA_3z^2x + AB_3z^2y)$$
$$+K(C_2x^2y - B_3x^2z + A_3y^2z - C_1y^2x + B_1z^2x - A_2z^2y) = 0.$$

Hence the Apolar Locus of (47) and lmn is

$$(C_2x^2y - B_3x^2z + A_3y^2z - C_1y^2x + B_1z^2x - A_2z^2y)$$
$$+\kappa(-CB_1x^2y + BC_1x^2z - AC_2y^2z + CA_2y^2x - BA_3z^2x + AB_3z^2y) = 0 \qquad (48)$$

which gives the required result in virtue of (45) and (46), κ being easily found in terms of λ.

Hence the Apolar Loci of any member $\lambda(ABC)+(DEF)$ and the Hessian of any other member $\mu(ABC)+(DEF)$ constitute the complete system of cubic curves circumscribing ABC and DEF and apolar to them.

5. Reverting once more to the results of § 2, let P_1, P_2 be two of the points forming part of the 9 degenerate members of the

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co-apolar system Γ', and let C_1, C_2 be the tangential equations to their corresponding (or complementary) conics. Then by (18),

$$P_1 C_1 \equiv \Gamma_0' + \sigma_1 (l^3 + m^3 + n^3) + 6\tau_1 lmn$$

$$P_2 C_2 \equiv \Gamma_0' + \sigma_2 (l^3 + m^3 + n^3) + 6\tau_2 lmn$$

$$\therefore \quad P_1 C_1 - P_2 C_2 \equiv (\sigma_1 - \sigma_2)(l^3 + m^3 + n^3) + (\tau_1 - \tau_2) lmn.$$

Hence the class-cubic of the Γ*-system which touches the line joining* P_1 *and* P_2 *can be regarded as generated by the four-line system of conics touching the common tangents of the conics* C_1 *and* C_2, *and having* P_1 *corresponding to* C_2 *and* P_2 *to* C_1. *In fact, a class-cubic touching the 9 lines (viz. the common tangents of* C_1 *and* C_2, *the join of* P_1 *and* P_2, *the tangents from* P_1 *to* C_2 *and* P_2 *to* C_1*) passes through the 9 fundamental points of the* Γ*-system.*