For quintic numbers (perfect fifth powers), we have the first few

$$\begin{array}{ccccccccc} 1 & 32 & 243 & 1024 & 3125 & 7776 & 16807 & 32768 & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 1^5 & 2^5 & 3^5 & 4^5 & 5^5 & 6^5 & 7^5 & 8^5 & \dots \end{array}$$

but clearly in the first steps, we observe that $2^4 < 2^5 < 3^4 < 3^5 < 4^4$. Actually, we have for any integer $n \geqslant 5$,

$$2^{n-1} < 2^n < 3^{n-1} < 3^n < 4^{n-1}$$

since easily and inductively we can show that

$$2 < \left(1 + \frac{1}{2}\right)^{n-1} \qquad \text{and} \qquad 3 < \left(1 + \frac{1}{3}\right)^{n-1}$$

for any integer $n \geqslant 3$ and for any integer $n \geqslant 5$, respectively, and these inequalities yield $2^n < 3^{n-1}$ and $3^n < 4^{n-1}$; thus in $[2^n, 3^n]$ there is only one perfect $(n - 1)$-th power which is 3^{n-1}, for any integer $n \geqslant 5$.

(iii) By Bernoulli's Inequality, for $a \geqslant 1$ and $r = \dfrac{n}{m} > 1$, we have

$$(a + 1)^r - a^r = a^r\left(\left(1 + \frac{1}{a}\right)^r - 1\right) \geqslant a^r\left(1 + \frac{r}{a} - 1\right) = ra^{r-1}$$

and now for $a \geqslant \left(\dfrac{k}{r}\right)^{1/(r-1)}$, we get $(a + 1)^r - a^r \geqslant k$.

Therefore, put $a_0 = a_0(k, m, n) = \left(\dfrac{km}{n}\right)^{m/(n-m)}$. This completes the proof.

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108.04 Digital root analysis of Smith numbers

Introduction

A composite integer N whose digit sum $S(N)$ is equal to the sum of the digits of its prime factors $S_p(N)$ is called *a Smith number*. For example 636 is a Smith number because the digit sum of 636 i.e. $S(636) = 6 + 3 + 6 = 15$, which is equal to the sum of the digits of its prime factors i.e. $S_p(636) = S_p(2 \times 2 \times 3 \times 53) = 2 + 2 + 3 + 5 + 3 = 15$.

Albert Wilansky [1] named Smith numbers from his brother-in-law Harold Smith's telephone number 4937775 with this property i.e. $4937775 = 3 \times 5 \times 5 \times 65837$, since

$$4 + 9 + 3 + 7 + 7 + 7 + 5 = 3 + 5 + 5 + (6 + 5 + 8 + 3 + 7) = 42.$$

Let N be a composite number whose factorisation is

$$p_1^{a_1} \times p_2^{a_2} \times p_3^{a_3} \times \ \dots \times p_k^{a_k},$$

the p_i being distinct primes and the a_i being positive integers. The sum of the digits of the prime factors of N i.e. $S_p(N) = \sum_{i=1}^{k} a_i S(p_i)$, then for N to be a Smith number, $S(N) = S_p(N)$.

In 1987, Wayne McDaniel proved that there are infinitely many Smith numbers [2]. Computations of large Smith numbers are time-consuming. A new approach to speed up the computations of Smith numbers using digital root properties is demonstrated in this paper. Surprising results have been obtained regarding the minimum number of prime factors required for a Smith number with a known digital root. An especially exciting example is that a Smith number with a digital root of 1 or 7 must have at least five prime factors (not necessarily distinct).

There are 25154060 Smith numbers below 10^9, [3]. Smith numbers below 1000 are:

4, 22, 27, 58, 85, 94, 121, 166, 202, 265, 274, 319, 346, 355, 378, 382, 391, 438, 454, 483, 517, 526, 535, 562, 576, 588, 627, 634, 636, 645, 648, 654, 663, 666, 690, 706, 728, 729, 762, 778, 825, 852, 861, 895, 913, 915, 922, 958 and 985.

Digital roots of Smith numbers

The digital root of a number x is obtained by summing the digits of x until a single digit is obtained. For example, the digital root of 287 is 8 as $2 + 8 + 7 = 17$ and $1 + 7 = 8$. The following properties of digital roots can easily be verified. Let the digital root of x be denoted by $d[x]$. So, $d[x] = 9$, if $x \equiv 0 \pmod 9$, or else $x \equiv d[x] \pmod 9$. Also, $d[x + y] = d[d[x] + d[y]]$, $d[x \times y] = d[d[x] \times d[y]]$ and $d[d[x]] = d[x]$.

The digital root of the first 49 Smith numbers below 1000 are 4, 4, 9, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 9, 4, 4, 6, 4, 6, 4, 4, 4, 4, 9, 3, 6, 4, 6, 6, 9, 6, 6, 9, 6, 4, 8, 9, 6, 4, 6, 6, 6, 4, 4, 6, 4, 4 and 4.

Why are there so many Smith numbers with digital root 4? Why is there no Smith number with a digital root of 1, 2, 5 or 7 in the above list? Before understanding this, look at the distribution of digital roots of all Smith numbers below 10^4, 10^7, 10^8 and 10^9 as given in Table 1.

It can be seen that out of 376 Smith numbers below 10^4, the digital root of only one Smith number is 7. Similarly, the number of Smith numbers with digital roots 1, 2 and 5 are 4, 5 and 4 respectively. It can also be noted that the percentage of Smith numbers up to n, with digital roots 4, 6 and 9 is decreasing with increasing value of n, whereas the percentage of Smith numbers up to n, with digital roots 1, 2, 3, 5, 7 and 8 is increasing with increasing value of n. The smallest Smith numbers with digital roots from 1 to 9 respectively are 2944, 2576, 588, 4, 5936, 438, 9880, 728 and 27.

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d	$m = 4$	$m = 7$	$m = 8$	$m = 9$
1	4	8534	94788	1029826
2	5	14597	159416	1691224
3	15	27913	285768	2884078
4	180	88467	762255	6712304
5	4	10936	125397	1374638
6	91	56797	500543	4509948
7	1	6509	76737	865744
8	17	24177	246882	2496804
9	59	40481	380972	3589494
Total	376	278411	2632758	25154060

TABLE 1: Number of Smith numbers below 10^m with digital root d

Let N be a number whose factorisation is $p_1 \times p_2 \times p_3 \times \ldots \times p_n$, where the p_i are primes, not necessarily distinct. Then the definition of N being a Smith number is that $S(N) = S(p_1) + S(p_2) + S(p_3) + \ldots + S(p_n)$, and this implies $d[S(N)] = d[S(p_1) + S(p_2) + S(p_3) + \ldots + S(p_n)]$, and, by the properties of digital roots established earlier, this is

$$d[N] = d[d[S(p_1)] + d[S(p_2)] + d[S(p_3)] + \ldots + d[S(p_n)]]$$

i.e.

$$d[d[p_1] \times d[p_2] \times d[p_3] \times \ldots \times d[p_n]] = d[d[p_1] + d[p_2] + d[p_3] + \ldots + d[p_n]],$$

so this is a necessary condition for N to be a Smith number.

For example, consider the previous example of the Smith number 4937775 with factorisation $3 \times 5 \times 5 \times 65837$. To verify the necessary condition above:

$$d[d[3] + d[5] + d[5] + d[65873]] = d[3 + 5 + 5 + 2] = d[15] = 6$$

and

$$d[d[3] \times d[5] \times d[5] \times d[65873]] = d[3 \times 5 \times 5 \times 2] = d[150] = 6.$$

Let us study the number of prime factors vis-à-vis the digital roots of Smith numbers.

Smith numbers with two prime factors

For a Smith number $N = p_1 \times p_2$, $S(N) = S(p_1) + S(p_2)$, the necessary condition is, $d[d[p_1] + d[p_2]] = d[d[p_1] \times d[p_2]]$.

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Theorem 1: A Smith number with a digital root other than 4 must have more than two prime factors.

Proof: Partition 4 as the digital root of the sum of the digital roots of two prime numbers and see whether it equals the digital root of the product of these two numbers.

$4 = d[1 + 3]$ but $d[1 + 3] \neq d[1 \times 3]$.

$4 = d[2 + 2]$ and $d[2 + 2] = d[2 \times 2]$. (It is not ruled out by the necessary condition).

$4 = d[4 + 9]$ but $d[4 + 9] \neq d[4 \times 9]$. (Moreover, 9 cannot be taken as it cannot be the digital root of a prime, being always divisible by 9). $4 = d[5 + 8]$ and $d[5 + 8] = d[5 \times 8]$. (It is not ruled out by the necessary condition).

$4 = d[6 + 7]$ but $d[6 + 7] \neq d[6 \times 7]$. (Moreover, 6 cannot be taken as it cannot be the digital root of a prime, being always divisible by 3).

So, a Smith number with digital root 4 can have two prime factors. The digital root of these two prime factors must either be 2 and 2, or 5 and 8, as shown above.

It can be easily checked that no other number with a digital root from 1 to 9 except 4 can be partitioned in the digital roots of two possible primes, such that the digital root of sum and product of these two is the same. Though the digital root of 9 can be partitioned into two digital roots as follows, these cannot be the digital root of primes.

$9 = d[3 + 6]$ and $d[3 + 6] = d[3 \times 6]$. (But 6 cannot be taken as it cannot be the digital root of a prime, being always divisible by 3).

$9 = d[9 + 9]$ and $d[9 + 9] = d[9 \times 9]$. (But 9 cannot be taken as it cannot be the digital root of a prime, being always divisible by 9).

Similarly, it can be shown that any Smith number with digital roots 1, 2, 3, 5, 6, 7, 8 or 9 must have more than two prime factors. So, a Smith number with two prime factors can only have a digital root 4 and any Smith number with a digital root other than 4 must have more than two prime factors.

This is also the reason why Smith numbers with digital root 4 are more frequent initially and decrease subsequently.

For Smith numbers with three or more prime factors, we state the following theorems along with a summary of the results. The proofs of these theorems are left as an exercise for the readers as it can be easily done on similar lines to the proof of Theorem 1.

Smith numbers with three prime factors

For a Smith number $N = p_1 \times p_2 \times p_3$, the necessary condition is $d[d[p_1] + d[p_2] + d[p_3]] = d[d[p_1] \times d[p_2] \times d[p_3]]$.

Theorem 2: A Smith number with a digital root 6 or 9 must have at least three prime factors.

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Out of the three prime factors, one prime factor is 3 and the digital roots of the other two must be (1, 2), (4, 8) or (5, 7). However, a Smith number with digital root 9 must have more than three prime factors except for Smith number 27 with three prime factors as 3, 3 and 3.

Smith numbers with four prime factors

For a Smith number $N = p_1 \times p_2 \times p_3 \times p_4$, the necessary condition is $d[d[p_1] + d[p_2] + d[p_3] + d[p_4]] = d[d[p_1] \times d[p_2] \times d[p_3] \times d[p_4]]$.

Theorem 3: A Smith number with a digital root 2, 3, 5 or 8 must have at least four prime factors.

For Smith numbers with digital root 2, the digital roots of four prime factors must be (1, 1, 4, 5), (1, 5, 7, 7) or (4, 4, 5, 7). Out of four prime factors for Smith numbers with digital root 3, one prime factor is 3 and the digital roots of the other three must be (1, 1, 7), (1, 4, 4) or (4, 7, 7). The digital roots of four prime factors for Smith numbers with digital root 5, must be (1, 1, 4, 8), (1, 7, 7, 8) or (4, 4, 7, 8). Similarly, the digital roots of four prime factors for Smith numbers with digital root 8, must be (1, 1, 2, 4), (1, 2, 7, 7) or (2, 4, 4, 7).

Smith numbers with five prime factors

For a Smith number $N = p_1 \times p_2 \times p_3 \times p_4 \times p_5$, the necessary condition is,

$$d[d[p_1] + d[p_2] + d[p_3] + d[p_4] + d[p_5]] \\ = d[d[p_1] \times d[p_2] \times d[p_3] \times d[p_4] \times d[p_5]].$$

Theorem 4: A Smith number with a digital root 1 or 7 must have at least five prime factors.

For Smith numbers with digital root 1, the digital roots of these five prime factors must be (1, 1, 1, 2, 5), (1, 1, 1, 8, 8), (1, 2, 4, 5, 7), (1, 4, 7, 8, 8), (2, 4, 4, 4, 5), (2, 5, 7, 7, 7), (4, 4, 4, 8, 8) or (7, 7, 7, 8, 8). Similarly, the digital roots of five prime factors for Smith numbers with digital root 7, must be (1, 1, 2, 5, 7), (1, 1, 7, 8, 8), (1, 2, 4, 4, 5), (1, 4, 4, 8, 8), (2, 4, 5, 7, 7) or (4, 7, 7, 8, 8).

This is the reason why Smith numbers with digital root 1 or 7 are less frequent initially and increase subsequently.

It is important to note that necessary conditions for Smith numbers given above are not sufficient, as can be seen from the example given below.

Example: Let us examine three numbers each with digital root 7 and five prime factors i.e. 71890, 75922 and 76570, to find out whether these satisfy the necessary condition and whether or not these are Smith numbers.

For N to be a Smith number, $S(N) = S_p(N)$ and necessary condition for Smith numbers with five prime factors is

$$d\left[d[p_1] + d[p_2] + d[p_3] + d[p_4] + d[p_5]\right] = d\left[d[p_1] \times d[p_2] \times d[p_3] \times d[p_4] \times d[p_5]\right].$$

$71890 = 2 \times 5 \times 7 \times 13 \times 79$, $d[2 + 5 + 7 + 4 + 7] = d[25] = 7$ and $d[2 \times 5 \times 7 \times 4 \times 7] = d[1960] = 7$, so necessary condition is satisfied.

$S(71890) = 25$ and $S_p(71890) = 34$. So, 71890 is not a Smith number. It confirms that necessary condition is not sufficient.

$75922 = 2 \times 7 \times 11 \times 17 \times 29$, $d[2 + 7 + 2 + 8 + 2] = d[21] = 3$ and $d[2 \times 7 \times 2 \times 8 \times 2] = d[448] = 7$, so necessary condition is not satisfied.

$S(75922) = 25$ and $S_p(75922) = 30$. So, obviously 75922 is not a Smith number, because the necessary condition is not satisfied.

$76570 = 2 \times 5 \times 13 \times 19 \times 31$, $d[2 + 5 + 4 + 1 + 4] = d[16] = 7$ and $d[2 \times 5 \times 4 \times 1 \times 4] = d[160] = 7$, so the necessary condition is satisfied.

$S(76570) = 25$ and $S_p(76570) = 25$. So, 76570 is a Smith number.

Results

Based on the digital root analysis, the following results are obtained:

(i) A Smith number with a digital root other than 4 must have more than two prime factors.

(ii) A Smith number with a digital root 6 or 9 must have at least three prime factors. Except for the Smith number $27 = 3 \times 3 \times 3$, any other Smith number with a digital root 9 must have at least four prime factors. If a Smith number with digital root 6 consists of three prime factors, then one of the prime factors is 3.

(iii) A Smith number with a digital root 2, 3, 5 or 8 must have at least four prime factors.

(iv) A Smith number with a digital root 1 or 7 must have at least five prime factors.

(v) A Smith number with digital root 4 cannot have three or four prime factors.

(vi) A Smith number with digital root 6 cannot have five prime factors.

The concepts and results obtained above can certainly be used to speed up the computations of Smith numbers.

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108.05 Ramanujan's proof of Bertrand's postulate

Introduction

In this Note we adhere closely to Ramanujan's original paper [1]. We think it should be inspirational for mathematics students to see an accurate reproduction of a short but significant work by a great mathematician with perhaps some of the pitfalls of trying to understand that work smoothed over. Our main contribution is to remove any mention of the gamma function or Stirling's formula. Simply to invoke a technical device without explaining how it can be used in a proof is insufficient. Instead of referring to Stirling's formula we give a direct proof in Lemma 3 of two inequalities which are unique and central to Ramanujan's proof. The assertions of Lemma 3 are essential for the validity of Ramanujan's argument and conclusions, but the proof of Lemma 3 bears no relation to the rest of the paper. It would be feasible just to assume the conclusions of Lemma 3, essentially as Ramanujan has done, but we have chosen to give a proof. The binomial coefficent $\binom{2n}{n}$ first occurred in a proof of Bertrand's postulate in Ramanujan's paper. In his proof of Bertrand's postulate [2, 3], Erdős also used this binomial coefficient. Aside from our direct proof of the two inequalities of Lemma 3 and our preliminaries, which prepare the reader for Ramanujan's context, we do not change Ramanujan's argument. Perhaps interested readers will note that Ramanujan comes back to [4], connecting to asymptotic distributions of primes, whereas Erdős, following his proof of Bertrand's postulate, turns toward Sylvester's Theorem [5], which generalises Bertrand's postulate in another direction.

The following are the opening sentences of Ramanujan's paper [1] (or google “Ramanujan's Proof of Bertrand's Postulate” to find Ramanujan's article scanned into the net.):

“Landau in his *Handbuch* [4, pp 89–92], gives a proof of a theorem the truth of which was conjectured by Bertrand: namely that there is at least one prime p such that $x < p \leqslant 2x$, if $x \geqslant 1$. Landau's proof is substantially the same as that given by Tschebyschef. The following is a much simpler one.”