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On Radicals of Green's Relations in Ordered Semigroups

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Abstract. In this paper, we give a new definition of radicals of Green's relations in an ordered semigroup and characterize left regular (right regular), intra regular ordered semigroups by radicals of Green's relations. We also characterize the ordered semigroups that are unions and complete semilattices of t-simple ordered semigroups.

1 Introduction

L. N. Shevrin [12] characterized the radical of a relation on a semigroup (without order). Using radicals of Green's relations, regularity of semigroups (without order) have been investigated by several mathematicians, such as A. H. Clifford and G. B. Preston [3], J. T. Sedlock [11], B. Pondelicek [10], and D. W. Miller [9]. S. Bogdanović and M. Ćirić [2] used the idea of Shevrin [12] to introduce radicals of Green's relations on a semigroup (without order) to characterize semilattice decompositions of semigroups (without order). Q.-S. Zhu characterized the property of radical of Green's relations in ordered semigroups in [13]. In that paper, the author concentrated mainly on the work of radical of an ideal (specially prime and primary) rather than a radical of a relation.

Here we extend the notion of radicals of Green's relations on a semigroup (without order) defined by Bogdanović and Ćirić [2] to ordered semigroups and characterize different particular classes of ordered semigroups by the radicals of Green's relations. We show that an ordered semigroup S is intra-regular if and only if $\sqrt{\mathcal{J}} = \mathcal{J}$. Completely regular ordered semigroups have been characterized by the radicals of Green's $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$-relations. Also we have characterized the ordered semigroups that are complete semilattices of left simple ordered semigroups and t-simple ordered semigroups.

The article is organized as follows. The basic definitions and properties of ordered semigroups are presented in Section 2. Section 3 is devoted to characterize ordered semigroups which can be decomposed into complete semilattice of t-simple (simple) ordered semigroups by using radicals of Green's relations. Furthermore, a necessary and sufficient condition via radicals of Green's relations, for an ordered semigroup that is a union of t-simple ordered semigroups (equivalently a completely regular ordered semigroup) is also given in this section.

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2 Preliminaries

In this paper $\mathbb{N}$ is the set of all natural numbers. An ordered semigroup S is a partially ordered set $(S, \leq)$, and at the same time a semigroup $(S, \cdot)$ such that $a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in S$. It is denoted by $(S, \cdot, \leq)$. For an ordered semigroup S and a nonempty subset $H \subseteq S$, denote

$$(H] = \{t \in S : t \leq h, \text{ for some } h \in H\}.$$

Let I be a nonempty subset of an ordered semigroup S. I is called a left (resp. right) ideal of S, if $SI \subseteq I$ (resp. $IS \subseteq I$) and $(I] = I$. I is an ideal of S if it is both a left and a right ideal of S. The principal left ideal, principal right ideal, and principal ideal of S, respectively, generated by $a (a \in S)$, are denoted by

$$L(a) = \{t \in S \mid t \leq a \text{ or } t \leq ya \text{ for some } y \in S\},$$
$$R(a) = \{t \in S \mid t \leq a \text{ or } t \leq ay \text{ for some } y \in S\},$$
$$I(a) = \{t \in S \mid t \leq a \text{ or } t \leq ya \text{ for some } y \in S \text{ or } t \leq ax \text{ for some } x \in S \text{ or } t \leq xay \text{ for some } x, y \in S\}.$$

An ordered semigroup S is left (resp. right) simple if it has no non-trivial proper left (resp. right) ideal. Similarly we define simple ordered semigroups. S is called t-simple ordered semigroup if it is both left and right simple. Thus, an ordered semigroup S is t-simple if and only if for all $a, b \in S$ there are $x, y \in S$ such that $a \leq xb$ and $a \leq by$.

An element $a \in S$ is called ordered regular if $a \leq axa$ for some $x \in S$. If every element of S is ordered regular, then S is called a regular ordered semigroup. S is called left regular (resp. right regular) if for every $a \in S, a \in (Sa^2]((a^2S])$. Kehayopulu [5] defined a completely regular ordered semigroup as an ordered semigroup S such that $a \in (a^2Sa^2]$ for all $a \in S$. As in [1] a regular ordered semigroup S is a Clifford ordered semigroup if for all $a, b \in S$ there is $x \in S$ such that $ab \leq bxa$. An ordered semigroup S is called intra-regular if for every $a \in S$, $a \in (Sa^2S]$. S is said to be weakly commutative [8] (resp. right weakly commutative [4]) if for any $a, b \in S, (ab)^m \in (bSa]$ (resp. $(ab)^m \in (Sa]$) for some $m \in \mathbb{N}$.

Kehayopulu [6] defined Green's relations on an ordered semigroup S as follows: $a\mathcal{L}b$ if $L(a) = L(b)$, $a\mathcal{R}b$ if $R(a) = R(b)$, $a\mathcal{J}b$ if $I(a) = I(b)$, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. These four relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{H}$ are equivalence relations on S.

A congruence ρ on S is called a semilattice congruence if for all $a, b \in Sa\rho a^2$ and $ab\rho ba$. A semilattice congruence ρ on S is called complete if $a \leq b$ implies $(a, ab) \in \rho$. Let $\mathcal{C}$ be a class of ordered semigroups and let us call the members of $\mathcal{C}$ C-ordered semigroups. Then an ordered semigroup S is called a complete semilattice of C-subsemigroups if there exists a complete semilattice congruence ρ on S such that each ρ-class is a C-ordered semigroup. For the sake of convenience, we collect fa ew auxiliary results of [1].

Lemma 2.1 ([1]) *In an ordered semigroup S the following conditions are equivalent:*

(i) *S is completely regular;*
(ii) *S is union of t-simple ordered semigroups;*
(iii) *S is regular ordered semigroup and $a \in (a^2S] \cap (Sa^2]$ for all $a \in S$.*

Lemma 2.2 *Let S be complete semilattice of t-simple ordered semigroups. Then the following statements hold for S:*

(i) *S is complete regular.*
(ii) *S is intra-regular.*
(iii) $\mathcal{L} = \mathcal{R}$.

Proof Both (i) and (ii) follow trivially.

(iii) Let ρ be the corresponding complete semilattice congruence on S. Let $a, b \in S$ be such that $a\mathcal{L}b$. Then there are $x, y \in S$ such that $a \leq xb$, and $b \leq ya$. Then by the completeness of ρ, we have that $(a)_\rho = (axb)_\rho = (bax)_\rho$. This implies that $a, bax \in (a)_\rho$. By the right simplicity of $(a)_\rho$, it follows that $a \in (bS]$. Similarly, we can show that $b \in (aS]$, and evidently $a\mathcal{R}b$. Thus, $\mathcal{L} \subseteq \mathcal{R}$. Similarly, $\mathcal{R} \subseteq \mathcal{L}$, and hence $\mathcal{L} = \mathcal{R}$. ∎

3 Radicals of Green's Relations

By radical of a subset A of an ordered semigroup S, we shall mean the set $\sqrt{A}$ defined by

$$\sqrt{A} = \left\{ x \in S \mid (\exists m \in \mathbb{N}) x^m \in A \right\}.$$

Following Shevrin [12], let us denote the radical of a relation ρ in an ordered semigroup by $\sqrt{\rho}$ and defined by:

$$a\sqrt{\rho}b \text{ if and only if there exists } m, n \in \mathbf{N} \text{ such that } (a^m, b^n) \in \rho.$$

It is obvious that $\rho \subseteq \sqrt{\rho}$.

Thus, the radical of the Green's relation $\mathcal{L}$ is denoted by $\sqrt{\mathcal{L}}$ and defined by

$$(a, b) \in \sqrt{\mathcal{L}} \text{ if and only if there exists } m, n \in \mathbf{N} \text{ such that } (a^m, b^n) \in \mathcal{L}.$$

Similarly, we define $\sqrt{\mathcal{R}}$, $\sqrt{\mathcal{J}}$, and $\sqrt{\mathcal{H}}$.

Kehayolpulu [7] showed that every intra-regular ordered semigroup is a semilattice of simple ordered semigroups. Now we show that on an intra-regular ordered semigroup $\mathcal{J}$ is a complete semilattice congruence. As it is routine to check, we omit the proof.

Lemma 3.1 *Let S be an intra-regular ordered semigroup. Then $\mathcal{J}$ is complete semilattice congruence on S.*

Also we state only the following useful result.

Lemma 3.2 *Let S be an intra-regular ordered semigroup. Then for every $\kappa \in \{\mathcal{L}, \mathcal{R}, \mathcal{H}\}$, $a\sqrt{\kappa}a^2$ for all $a \in S$.*

We now characterize the ordered semigroups that are intra-regular ordered semigroups via radicals of Green's relations.

Theorem 3.3 *In an ordered semigroup S the following conditions are equivalent:*

(i) *S is intra-regular;*
(ii) $\sqrt{\mathcal{J}} = \mathcal{J}$;
(iii) $\sqrt{\mathcal{L}} \subseteq \mathcal{J}$;
(iv) $\sqrt{\mathcal{H}} \subseteq \mathcal{J}$.

Proof (i)⇒(ii): Suppose that S is intra-regular. Let $a, b \in S$ be such that $a\sqrt{\mathcal{J}}b$. Then there are $m, n \in \mathbb{N}$ such that $a^m \mathcal{J} b^n$. Now $a\mathcal{J}a^m\mathcal{J}b^n\mathcal{J}b$, by Lemma 3.1. So $a\mathcal{J}b$. Hence $\sqrt{\mathcal{J}} = \mathcal{J}$.

(ii)⇒(iii) This follows from the fact that $\mathcal{L} \subseteq \mathcal{J}$.

(iii)⇒(iv): Since $\sqrt{\mathcal{H}} \subseteq \sqrt{\mathcal{L}}$ this follows directly from the condition (iii).

(iv)⇒(i): Let $a \in S$. Then $a\sqrt{\mathcal{H}}a^2$, by Lemma 3.2. So by the given condition $a\mathcal{J}a^2$. Thus S is intra-regular. ∎

In the following theorem we find a necessary and sufficient condition for an ordered semigroup to be left regular (right regular), via radicals of Green's relation.

Theorem 3.4 *An ordered semigroup S is left regular (right regular) if and only if $\sqrt{\mathcal{L}} = \mathcal{L}$ $(\sqrt{\mathcal{R}} = \mathcal{R})$.*

Proof First suppose that S is a left regular semigroup. Let $a, b \in S$ be such that $a\sqrt{\mathcal{L}}b$. Then there are $m, n \in \mathbb{N}$ such that $a^m\mathcal{L}b^n$. Since $\mathcal{L}$ is a right congruence, by the left regularity of S it follows that $x\mathcal{L}x^r$ for all $r \in \mathbb{N}$, and $x \in S$. Then $a\mathcal{L}a^m\mathcal{L}b^n\mathcal{L}b$ and so $a\mathcal{L}b$. Thus $\sqrt{\mathcal{L}} \subseteq \mathcal{L}$. Also we have $\mathcal{L} \subseteq \sqrt{\mathcal{L}}$. Thus, $\mathcal{L} = \sqrt{\mathcal{L}}$.

Conversely, assume that $\sqrt{\mathcal{L}} = \mathcal{L}$. Let $a \in S$. Then $a\mathcal{L}a^2$, by Lemma 3.2. Then there are $x \in S^1$ such that $a \leq xa^2 \leq (xa)xa^2 \leq (xax)a^2$ for some $xax \in S$. This shows that S is left regular ordered semigroup. ∎

Theorem 3.5 *The following conditions are equivalent in a regular ordered semigroup S:*

(i) *S is completely regular;*
(ii) $\sqrt{\mathcal{L}} = \mathcal{L}$ *and* $\sqrt{\mathcal{R}} = \mathcal{R}$;
(iii) $\sqrt{\mathcal{H}} = \mathcal{H}$.

Proof (i)⇒(ii): Let S be completely regular. Then by Lemma 2.1 we have $a \in (Sa^2] \cap (a^2S]$ for every $a \in S$. So S is left as well as right regular. Thus $\sqrt{\mathcal{L}} = \mathcal{L}$ and $\sqrt{\mathcal{R}} = \mathcal{R}$ by Theorem 3.4.

(ii)⇒(iii): Let $a, b \in S$ be such that $a\sqrt{\mathcal{H}}b$. Then there are $m, n \in \mathbb{N}$ such that $a^m\mathcal{H}b^n$, which implies that $a^m\mathcal{L}b^n$. Thus, $a\sqrt{\mathcal{L}}b$ and so $a\mathcal{L}b$. Similarly, $a\mathcal{R}b$. Thus, $a\mathcal{H}b$ and hence $\mathcal{H} = \sqrt{\mathcal{H}}$.

(iii)⇒(i): By Lemma 3.2 we have $a\sqrt{\mathcal{H}}a^2$ for every $a \in S$. Then $a\mathcal{H}a^2$, which implies that $a\mathcal{L}a^2$ and $a\mathcal{R}a^2$. Hence $a \in (Sa^2] \cap (a^2S]$. Thus, (i) follows from Lemma 2.1. ∎

Lemma 3.6 *In an ordered semigroup S, the following hold:*

(i) *If $\sqrt{\mathcal{L}} = \mathcal{J}$, then S is right weakly commutative.*
(ii) *If $\sqrt{\mathcal{H}} = \mathcal{J}$, then S is weakly commutative.*

Proof (i) Suppose that $\sqrt{\mathcal{L}} = \mathcal{J}$. By Lemma 3.2, we have $a\sqrt{\mathcal{L}}a^2$ for every $a \in S$. So S is intra-regular, and thus $ab\mathcal{J}ba$ by Lemma 3.1. This yields that $ab\sqrt{\mathcal{L}}ba$. Then there are $m', n' \in \mathbb{N}$ such that $(ab)^{m'} \le x_1(ba)^{n'} \le x_1 b(ab)^{n'-1}a$ for some $x_1 b(ab)^{n'-1} \in S$. This shows that S is right weakly commutative.

(ii) This is similar to (i). ∎

Theorem 3.7 *An ordered semigroup S is a complete semilattice of left simple semigroups if and only if $\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{J}$.*

Proof Let S be a complete semilattice $Y = S/\rho$ of left simple semigroups $\{S_\alpha\}_{\alpha\in Y}$. Let $a \in S$. Then there is $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left simple and $a, a^2 \in S_\alpha$, we have $a\mathcal{L}a^2$. Then there is $x \in S$ such that $a \le xa^2$, that is, $a \le x^2a^2a$. Thus, S is intra-regular and hence $\sqrt{\mathcal{L}} \subseteq \mathcal{J}$, by Theorem 3.3. Now let $a, b \in S$ be such that $a\mathcal{J}b$. Then there are $x, y \in S^1$ such that $a \le xby$. Since ρ is a complete semilattice congruence on S we have $(a)_\rho = (axby)_\rho = (axyb)_\rho$. Thus, $a, axyb \in (a)_\rho$. Since $(a)_\rho$ is left simple, there is $t \in S$ be such that $a \le taxyb$ for some $taxy \in S$. Similarly $b \le t'a$ for some $t' \in S$ and so $a\mathcal{L}b$, which implies that $\mathcal{J} \subseteq \mathcal{L}$. Thus, $\mathcal{L} \subseteq \sqrt{\mathcal{L}} \subseteq \mathcal{J} \subseteq \mathcal{L}$, and hence $\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{J}$.

Conversely, suppose that $\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{J}$. Then S is intra-regular, by Theorem 3.3 and so by Lemma 3.1, $\mathcal{J}$ is a complete semilattice congruence on S. Hence every $\mathcal{J}-$class of S is a subsemigroup. Now consider a $\mathcal{J}-$class J and $a, b \in J$. Then $a\mathcal{J}b$ and hence $a\sqrt{\mathcal{L}}b$. So there are $m, n \in \mathbb{N}, x \in S^1$ such that $a^m \le xb^n$. Again $a, a^{m+1} \in J$ implies that $a\mathcal{L}a^{m+1}$ and hence $a \le ya^{m+1}$ for some $y \in S^1$. Thus $a \le yaxb^n = (yaxb^{n-1})b$. Since $\mathcal{J}$ is a complete semilattice congruence, $a\mathcal{J}ayaxb^n\mathcal{J}yaxb^{n-1}$, which implies that $s = yaxb^{n-1} \in J$. Then $a \le sb$ shows that J is left simple. Hence, S is a complete semilattice of left simple semigroups. ∎

Characterization of ordered semigroups that are complete semilattice of t-simple ordered semigroups via radicals of Green's relation has been given in the following theorem.

Theorem 3.8 *In an ordered semigroup S following conditions are equivalent:*

(i) *S is complete semilattice of t-simple ordered semigroups;*
(ii) $\sqrt{\mathcal{H}} = \mathcal{J}$;
(iii) $\sqrt{\mathcal{L}} = \mathcal{R}$;
(iv) $\sqrt{\mathcal{R}} = \mathcal{L}$;
(v) $\sqrt{\mathcal{L}} = \sqrt{\mathcal{R}} = \mathcal{J}$.

Proof Here we show that each of (ii), (iii), (iv), and (v) is equivalent to (i). Also we omit the proofs for (iii)⇒ (i), (iv)⇒(i), and (v)⇒(i), as these are similar to (ii)⇒(i), since $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$.

(i)⇒(ii): Let ρ be the complete semilattice congruence on S. Consider $a, b \in S$ such that $a\mathcal{J}b$. Then there are $x, y, z, w \in S$ such that $a \le xby$ and $b \le zaw$. Since ρ is complete semilattice congruence on S we have that $(a)_\rho = (axby)_\rho = (baxy)_\rho$. Thus, $a, axyb \in (a)_\rho$, and so the left simplicity of $(a)_\rho$ yields that $a \le z_1axyb$ for some $z_1 \in S$. Thus $a \le s_1b$ where $s_1 = z_1axy$. Similarly, $b \le s_2a$ for some $s_2 \in S$. Therefore, $a\mathcal{L}b$, and so $\mathcal{J} \subseteq \mathcal{L}$. Also, it is obvious that $\mathcal{L} \subseteq \mathcal{J}$. Thus, $\mathcal{L} = \mathcal{J}$. Now by Lemma 2.2, $\mathcal{L} = \mathcal{R}$, and $\sqrt{\mathcal{H}} = \mathcal{H}$, by Lemma 2.2 and Theorem 3.5. Therefore, $\sqrt{\mathcal{H}} = \mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} = \mathcal{J}$.

(ii)⇒(i): Suppose that $\sqrt{\mathcal{H}} = \mathcal{J}$. Then by Theorem 3.3, S is intra-regular and so $\mathcal{J}$ is complete semilattice congruence on S, by Lemma 3.1. Also S is weakly commutative, by Lemma 3.6. Consider a $\mathcal{J}$–class, $(g)_\mathcal{J}$ of S for some $g \in S$. Clearly $(g)_\mathcal{J}$ is a subsemigroup. Let $a, b \in (g)_\mathcal{J}$, then $a\sqrt{\mathcal{H}}b$, and so $a\sqrt{\mathcal{R}}b$. Then there are $m_1, n_1 \in \mathbb{N}$ such that $a^{m_1} \le b^{n_1}t_1 = b(b^{n_1-1}t_1)$. Also by the intra-regularity of S we have $a\mathcal{J}a^l$ for all $l \in \mathbb{N}$. Then there are $x_1, y_1 \in S^1$ such that $a \le x_1a^{m_1}y_1 \le x_1b^{n_1}y_1$. Again for $x_1b \in S$ there are $t_2, t_3 \in S^1$ such that

$$x_1b \le t_2x_1bx_1bt_3 \le t_2x_1bt_2x_1bx_1bt_3^2 = (t_2x_1b)^2x_1bt_3^2 \le \cdots \le (t_2x_1b)^rx_1bt_3^r$$

for all $r \in \mathbb{N}$. Since S is weakly commutative, there is $r_1 \in \mathbb{N}$ and $t_4 \in S$ such that $(t_2x_1b)^{r_1} \le bt_4t_2x_1$. So $x_1b \le bt_4t_2x_1^2bt_3^{r_1}$, so finally $a \le bt_4t_2(x_1)^2bt_3^{r_1}b^{n_1-1}t_1y^1$. Now as $\mathcal{J}$ is congruence on S,

$$(a)_\mathcal{J} = \left(abt_4t_2(x_1)^2bt_3^{r_1}b^{n_1-1}t_1y^1\right)_\mathcal{J} = \left(t_4t_2(x_1)^2bt_3^{r_1}b^{n_1-1}t_1y^1\right)_\mathcal{J}.$$

Thus, $a \le bs$, where $s = t_4t_2(x_1)^2bt_3^{r_1}b^{n_1-1}t_1y^1 \in (g)_\mathcal{J}$. A similar approach to above allows one to obtain $a \le tb$ for some $t \in (g)_\mathcal{J}$. Hence, $(g)_\mathcal{J}$ is a t-simple ordered semigroup. So S is complete semilattice of t-simple ordered semigroups.

(i)⇒(iii): Suppose that S is complete semilattice of t-simple ordered semigroups. Then S is complete regular, by Lemma 2.2(1). This shows that S is left regular. So by Theorem 3.4 it follows that $\sqrt{\mathcal{L}} = \mathcal{L}$, and hence $\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{R}$ by Lemma 2.2(iii).

(i)⇒(iv): This is similar to (i)⇒(iii).

(i)⇒(v): This is similar to (i)⇒(ii). ■

In the following theorem we have characterized the complete semilattice decomposition of ordered semigroups that are regular and left simple via radicals of Green's relation.

Theorem 3.9 *An ordered semigroup S is a complete semilattice of regular and left simple ordered semigroups if and only if S is regular and $\sqrt{\mathcal{L}} = \mathcal{J}$.*

Proof First suppose that S is a complete semilattice of regular and left simple ordered semigroups. Then S is regular, and by Theorem 3.7, it follows that $\sqrt{\mathcal{L}} = \mathcal{J}$.

Conversely, assume that S is regular and $\sqrt{\mathcal{L}} = \mathcal{J}$. Since $\sqrt{\mathcal{L}} = \mathcal{J}$, S is a complete semilattice Y of left simple semigroups $S_\alpha (\alpha \in Y)$, by Theorem 3.7. Let ρ be the corresponding complete semilattice congruence on S. We now show that, each ρ-class S_α is regular.

Let $a \in S_\alpha$. Since S is regular, there is $x \in S$ such that $a \le axa \le a(xax)a$. Since ρ is complete semilattice congruence on S, $(a)_\rho = (a^2xaxa)_\rho = (xax)_\rho$. Thus, $xax \in S_\alpha$, and so S_α is regular. ∎

Theorem 3.10 *In a regular ordered semigroup S following, the conditions are equivalent:*

(i) *S is right regular and $\sqrt{\mathcal{L}} = \mathcal{J}$;*
(ii) *$\sqrt{\mathcal{L}} = \mathcal{L} = \mathcal{J}$ and $\sqrt{\mathcal{R}} = \mathcal{R}$;*
(iii) *S is completely regular and complete semilattice of left simple semigroups.*

Proof (i)⇒(ii): Suppose that S is a right regular ordered semigroup and $\sqrt{\mathcal{L}} = \mathcal{J}$. Then $\sqrt{\mathcal{R}} = \mathcal{R}$ follows from Theorem 3.4. Also by Theorem 3.7, S is complete semilattice of left simple ordered semigroups. Let σ be the corresponding complete semilattice congruence on S. Then for every $a \in S$, $(a)_\sigma = (a^2)_\sigma$. Since $(a)_\sigma$ is left simple there is $x \in S$ be such that $a \le xa^2$, which shows that S is left regular, and hence $\sqrt{\mathcal{L}} = \mathcal{L}$ by Theorem 3.4.

(ii)⇒(iii): This follows from Theorems 3.5 and 3.7.

(iii)⇒(i): This follows from Theorems 3.7 and 3.4 ∎

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