HOMOMORPHISMS OF DISTRIBUTIVE LATTICES AS RESTRICTIONS OF CONGRUENCES

GEORGE GRÄTZER AND HARRY LAKSER

1. Introduction. Given a lattice L and a convex sublattice K of L, it is well-known that the map $\mathrm{Con}\, L \to \mathrm{Con}\, K$ from the congruence lattice of L to that of K determined by restriction is a lattice homomorphism preserving 0 and 1. It is a classical result (first discovered by R. P. Dilworth, unpublished, then by G. Grätzer and E. T. Schmidt [2], see also [1], Theorem II.3.17, p. 81) that any finite distributive lattice is isomorphic to the congruence lattice of some finite lattice. Although it has been conjectured that any algebraic distributive lattice is the congruence lattice of some lattice, this has not yet been proved in its full generality. The best result is in [4]. The conjecture is true for ideal lattices of lattices with 0; see also [3].

In this paper we modify the proof of the characterization of congruence lattices of finite lattices to show that any 0, 1-preserving homomorphism of finite distributive lattices can be realized by restricting the congruence lattice of some finite lattice L to the congruence lattice of a convex sublattice K, where K can actually be chosen to be an ideal of L, and L can be chosen to be sectionally complemented.

2. Categoric preliminaries. In our proof we shall make use of several algebraic structures. It will be useful to use the language of category theory. (P. Pudlák [3] also uses a categorical approach; his line of attack is completely unrelated to ours.) We list here the various categories and some of the associated functors that appear. Note that all the structures are finite.

1. **D** denotes the category of finite distributive lattices. The morphisms are lattice homomorphisms that preserve the 0 and the 1.

2. **L** denotes the category whose objects are the finite lattices. The morphisms are embeddings as an ideal.

There is a contravariant functor $\mathrm{Con}\colon \mathbf{L} \to \mathbf{D}$ that associates with each lattice L its congruence lattice $\mathrm{Con}\, L$. If K, L are finite lattices and $i\colon K \to L$ is an embedding with $i(K)$ an ideal of L, that is, if i is a

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morphism in **L**, then Con i denotes the restriction map

Con i:Con $L \to$ Con K

defined by setting, for $x, y \in K$ and $\theta \in$ Con L,

$x \equiv y \,(\,(\text{Con } i)\theta)$ if and only if $ix \equiv iy(\theta)$.

Con i is then a morphism in **D**.

3. **P** denotes the category whose objects are finite posets and whose morphisms are isotone maps.

If P is a finite poset with partial order $\leqq$, then a *hereditary subset H of P* is a (possibly empty) subset H of P such that $x \in H$ and $y \leqq x$ imply $y \in H$. The set $\mathscr{H}P$ of hereditary subsets of P is a distributive lattice; the join is set union, the meet is set intersection, the 0 is $\emptyset$, and the 1 is P. We then have a contravariant functor $\mathscr{H}:\mathbf{P} \to \mathbf{D}$; if $f:P \to Q$ is an isotone map of posets then the map $\mathscr{H}f:\mathscr{H}Q \to \mathscr{H}P$ that sets $(\mathscr{H}f)H = f^{-1}(H)$ for each hereditary $H \subseteq Q$ is a morphism in **D**.

4. **S** denotes the category whose objects are finite partial lattices that are meet-semilattices. More specifically, an object S of **S** is a poset such that, for each pair $x, y \in S$, $\inf\{x, y\}$ exists, yielding the *total* operation

$\wedge{:}x \wedge y = \inf\{x, y\};$

$\vee$ is a partial operation: $x \vee y$ is defined only if $\sup\{x, y\}$ exists, and then

$x \vee y = \sup\{x, y\}.$

To describe the morphisms in **S** we need the concept of an "ideal" of a partial lattice. A nonempty subset $I \subseteq S$ is said to be an *ideal* if the following two conditions are satisfied:

(a) $x \in I$ and $y \leqq x$ imply that $y \in I$;
(b) $x, y \in I$ and $x \vee y$ defined imply that $x \vee y \in I$.

The morphisms of **S** are then embeddings $i:S \to T$ of S as an ideal of T. By an "embedding" i we mean that i is one-to-one, preserves $\wedge$, and preserves $\vee$ in the strong sense: $ix \vee iy$ exists if and only if $x \vee y$ exists and in this event

$i(x \vee y) = ix \vee iy.$

By a *congruence relation θ* on an object S of **S** we mean an equivalence relation θ on S that preserves $\wedge$ and all existing $\vee$. That is, an equivalence relation θ is a congruence relation if

(a) θ is a congruence relation of the underlying meet-semilattice;
(b) $x, y, u, v \in S$, $x \equiv y(\theta)$, $u \equiv v(\theta)$, $x \vee u$ defined, and $y \vee v$ defined imply that

$x \vee u \equiv y \vee v(\theta)$.

We shall show that the functor Con:$\mathbf{S} \rightarrow \mathbf{D}$ that associates with each object S of $\mathbf{S}$ its congruence lattice Con S is a contravariant functor. As in the case of $\mathbf{L}$, if $i:S \rightarrow T$ is an embedding as an ideal, then Con i:Con $T \rightarrow$ Con S is the map with

$x \equiv y((\text{Con } i)\theta)$ if and only if $ix \equiv iy(\theta)$.

Of course, $\mathbf{L}$ is a subcategory of $\mathbf{S}$ and Con:$\mathbf{L} \rightarrow \mathbf{D}$ is the restriction of Con:$\mathbf{S} \rightarrow \mathbf{D}$; there should thus not be any confusion in using the same notation "Con" in the two places.

5. The last category $\mathbf{Q}$ that we consider is of a more technical nature. Its significance will become clear in the subsequent discussion.
    An object of $\mathbf{Q}$ is a finite nonempty set Q with a binary relation ρ satisfying the following three conditions:
    a) ρ is antireflexive, that is, $x \not\rho x$ for all $x \in Q$;
    b) ρ is antisymmetric;
    c) for each $x \in Q$ there is a $y \in Q$ with $y \rho x$.
    If Q and R are objects of $\mathbf{Q}$ a morphism $f:Q \rightarrow R$ in $\mathbf{Q}$ is a one-to-one map that preserves ρ in the following strong sense:

$(fx) \rho (fy)$ if and only if $x \rho y$.

Let Q be an object of $\mathbf{Q}$; a subset H of Q is *hereditary* if $x \in H$ and $x \rho y$ imply that $y \in H$. The set $\mathscr{H}Q$ of hereditary subsets of Q is then a distributive lattice with $\emptyset$ the 0 and Q the 1. We have a contravariant functor $\mathscr{H}:\mathbf{Q} \rightarrow \mathbf{D}$ such that if $f:Q \rightarrow R$ is a morphism in $\mathbf{Q}$, then $\mathscr{H}f:\mathscr{H}R \rightarrow \mathscr{H}Q$, given by $(\mathscr{H}f)H = f^{-1}(H)$, is a morphism in $\mathbf{D}$. There should be no confusion in using the same symbol $\mathscr{H}$ for the functors $\mathscr{H}:\mathbf{P} \rightarrow \mathbf{D}$ and $\mathscr{H}:\mathbf{Q} \rightarrow \mathbf{D}$.

3. From D to P. We recall the very fundamental theorem of duality between finite distributive lattices and finite posets.
    Given an object D of $\mathbf{D}$, that is, a finite distributive lattice D, we consider the poset PD of join-irreducible members of D. Then P is a contravariant functor $P:\mathbf{D} \rightarrow \mathbf{P}$. If $\varphi:D \rightarrow E$ is a morphism in $\mathbf{D}$ then $P\varphi:PE \rightarrow PD$ with $P\varphi x = \wedge \varphi^{-1}(x)$ is an isotone map. We then have the following result of G. Birkhoff.

    LEMMA 1 ([1], Corollary II.1.10, p. 62). *There is a natural equivalence* $\psi:\text{id}_{\mathbf{D}} \rightarrow \mathscr{H}P$ *that associates with each object D of $\mathbf{D}$ the isomorphism* $\psi_D:D \rightarrow \mathscr{H}PD$ *given by*

$\psi_D x = \{ y \leqq x \mid y \text{ join-irreducible} \}$.

More specifically, Lemma 1 states that, given finite distributive lattices

D, E and a 0, 1-preserving homomorphism $\varphi:D \to E$, the diagram in Fig. 1 commutes, and ψ_D, ψ_E are lattice isomorphisms.
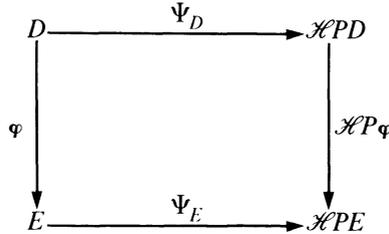

Figure 1

4. From P to Q. Let P, Q be objects in **P** and let $f:Q \to P$ be a **P**-morphism. We construct objects $B(Q)$, $A(f)$ in **Q** and define a **Q**-morphism $i_f:B(Q) \to A(f)$. Parenthetically, the notation reflects the fact that $B(Q)$ depends only on Q while $A(f)$ and i_f depend on Q, P, and $f:Q \to P$.

Set $B(Q) = Q \times \{L, M, R\}$, and for $a \in Q$ denote the ordered pairs by a_L, a_M, a_R rather than $\langle a, L \rangle$ etc. Define ρ on $B(Q)$ by setting:

a) $a_L\ \rho\ a_M\ \rho\ a_R\ \rho\ a_L$ for $a \in Q$;

b) $a_M\ \rho\ b_M$ whenever $a \succ b$ in Q, "$\succ$" denoting the cover relation in the poset Q.

Set

$$A(f) = (P \times \{L, M, R\}) \cup (Q \times \{L, M, R\}),$$

where we assume that P and Q are disjoint. Define ρ on $A(f)$ by setting:

a) $a_L\ \rho\ a_M\ \rho\ a_R\ \rho\ a_L$ if $a \in P \cup Q$;

b) $a_M\ \rho\ b_M$ if $a \succ b$, $a, b \in P$ or $a, b \in Q$;

c) $a_L\ \rho\ (fa)_L$ for all $a \in Q$;

d) $(fa)_R\ \rho\ a_R$ for all $a \in Q$.

Define $i_f:B(A) \to A(f)$ by setting $i_f a_K = a_K$ for $a \in Q$, $K \in \{L, M, R\}$. Then i_f is a **Q**-morphism.

We define maps $u:B(Q) \to Q$ and $v:A(f) \to P$ by setting

$$ua_K = a \quad \text{for } a \in Q, K \in \{L, M, R\};$$
$$va_K = fa \quad \text{for } a \in Q, K \in \{L, M, R\};$$
$$va_K = a \quad \text{for } a \in P, K \in \{L, M, R\}.$$

These then determine maps $u':\mathcal{H}Q \to \mathcal{H}B(Q)$ with $u'H = u^{-1}(H)$ and $v':\mathcal{H}P \to \mathcal{H}A(f)$ with $v'H = v^{-1}(H)$.

LEMMA 2. $u':\mathcal{H}Q \to \mathcal{H}B(Q)$ and $v':\mathcal{H}P \to \mathcal{H}A(f)$ are lattice isomorphisms and the diagram in Fig. 2 is commutative.

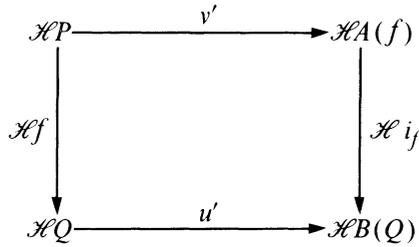$$\mathscr{H}P \xrightarrow{\;\;v'\;\;} \mathscr{H}A(f)$$

Figure 2

Proof. The computations for this proof are completely straight-forward.

We interrupt the formal presentation to discuss the raison-d'être for the constructions $B(Q)$ and $A(f)$. Given a **D**-morphism $\varphi : D \to E$, we shall construct lattices L and K, K an ideal in L, with Con $L \cong D$, Con $K \cong E$, and φ represented as congruence restriction from L to K. The elements of PD and PE will give rise to the join-irreducible congruence relations on L and K. We take the set $PD \cup PE$ and the subset PE, and construct lattices from these. The relation ρ determines the order relation on the join-irreducible congruences of the lattices derived from $PD \cup PE$ and PE. Essentially, $a \rho b$ forces the congruence relation corresponding to b to be included in that corresponding to a. Consequently, to get Con $K \cong E$, we set $a \rho b$ whenever $a \succ b$ in PE, and similarly for $PD \cup PE$. At this stage, however, we would get

Con $L \cong D \times E$;

to kill off the contribution of E, we must identify in $PD \cup PE$ each $a \in PE$ with $P\varphi a \in PD$. This requires that $a \rho P\varphi a$ and $P\varphi a \rho a$. However, our construction of the lattices works only if ρ is antisymmetric, although its closure ρ^* need not be. To take this difficulty into account, we triple each element, setting, for $a \succ b$, $a \rho b$ on the "middle" elements a_M, b_M, $a \rho P\varphi a$ on the "left" elements a_L, b_L, and $P\varphi a \rho a$ on the "right" elements a_R, b_R. The condition $a_L \rho a_M \rho a_R \rho a_L$ then identifies the congruence relations corresponding to a_L, a_M, and a_R, yielding only one congruence for the triple a_L, a_M, a_R and so cancelling the tripling on the congruence level.

5. Preliminaries on S. We present several more or less technical lemmas that will be useful in the sequel.

LEMMA 3. *Let S be a finite poset and let M be the set of maximal elements of S. If, for each $m \in M$, $(m]$ is a lattice and if, for $m, n \in M$, $m \wedge n$ exists, then S is an object of* **S**.

Proof. Indeed,

$$\inf\{x, y\} = (x \wedge_m a) \wedge (y \wedge_n a),$$

where $m, n \in M$, $x \leqq m$, $y \leqq n$, $a \geqq m \wedge n$, $\wedge_m$ (respectively, $\wedge_n$) denotes the meet in the lattice $(m]$ (respectively, in the lattice $(n]$) and $\wedge$ denotes $\wedge_m$ (or $\wedge_n$; they are the same) in $(m] \cap (n] = (a]$.

The proof of the next lemma is left to the reader.

LEMMA 4. *Let S be an object in* **S**. *If a pair* $x, y \in S$ *has an upper bound, then* $x \vee y$ *exists.*

LEMMA 5. *Let S be an object in* **S** *and let* θ *be an equivalence relation on S satisfying the following two conditions*:
1) *let* $x, y, z \in S$ *and* $x \equiv y(\theta)$; *then*

$$x \wedge z \equiv y \wedge z(\theta);$$

2) *let* $x, y, z \in S$, *let* $x \vee z$, $y \vee z$ *exist, and let* $x \equiv y(\theta)$; *then*

$$x \vee z \equiv y \vee z(\theta).$$

Then θ *is a congruence relation on S.*

Proof. Condition 1) states that θ preserves $\wedge$.

Now let $x, y, u, v \in S$ with $x \equiv y(\theta)$ and $u \equiv v(\theta)$; let $x \vee u$, $y \vee v$ exist. Then

$$x \equiv x \wedge y \equiv y(\theta)$$

and, by Lemma 4, $(x \wedge y) \vee u$ and $(x \wedge y) \vee v$ exist. Thus, by condition 2),

$$x \vee u \equiv (x \wedge y) \vee u \equiv (x \wedge y) \vee v \equiv y \vee v(\theta),$$

and the lemma follows.

LEMMA 6. Con *is a contravariant functor* **S** $\rightarrow$ **D**.

Proof. Since the objects S of **S** are meet-semilattices, congruence relations are determined by pairs x, y with $x \leqq y$. Since, by Lemma 4, $(y]$ is a sublattice of S, we get, exactly as in the case of lattices, that for $x \leqq y$,

$$x \equiv y(\theta_0 \vee \theta_1)$$

if and only if there is a sequence

$$x = z_0 \leqq z_1 \leqq \ldots \leqq z_n = y$$

with $z_i \equiv z_{i+1}(\theta_j)$, $j = 0$ or 1, for each i, $0 \leqq i < n$.

Then we can establish that Con S is a distributive lattice and that Con i is a **D**-morphism for any **S**-morphism i exactly as for the category **L** of finite lattices.

LEMMA 7. *Let S be an object in* **S** *and let M be the set of maximal elements of S. For each $m \in M$, let θ_m be a congruence relation on the lattice $(m]$. If, for $m, n \in M$,*

$$(*) \quad \theta_m|_{(m] \cap (n]} = \theta_n|_{(m] \cap (n]}$$

then there is a unique congruence relation θ on S with $\theta|_{(m]} = \theta_m$ for each $m \in M$.

Proof. θ is unique; this follows by observing that the congruence relations on S are determined by pairs x, y with $x < y$.

As for existence, define the relation θ on S by setting $\langle x, y \rangle \in \theta$ if and only if there are $m, n \in M$ with $x \leqq m$, $y \leqq n$, $x \wedge y \equiv x(\theta_m)$, and $x \wedge y \equiv y(\theta_n)$. By (*), θ is well defined.

Clearly, θ is reflexive and symmetric. θ is also transitive. Indeed, let $\langle x_0, x_1 \rangle \in \theta$ and $\langle x_1, x_2 \rangle \in \theta$, and let $m_i \in M$, $i = 0, 1, 2$, with $x_i \leqq m_i$. We then have

$$x_0 \equiv x_0 \wedge x_1(\theta_{m_0}),$$

$$x_2 \equiv x_1 \wedge x_2(\theta_{m_2}),$$

$$x_1 \equiv x_0 \wedge x_1(\theta_{m_1}),$$

$$x_1 \equiv x_1 \wedge x_2(\theta_{m_1}).$$

We get

$$x_0 \wedge x_1 = (x_0 \wedge m_1) \wedge x_1$$
$$\equiv (x_0 \wedge m_1) \wedge x_1 \wedge x_2$$
$$= x_0 \wedge x_1 \wedge x_2(\theta_{m_1})$$

and, since $x_0 \wedge x_1 \in (m_0] \wedge (m_1]$, it follows by (*) that

$$x_0 \wedge x_1 \equiv x_0 \wedge x_1 \wedge x_2 \, (\theta_{m_0}).$$

Thus,

$$x_0 \equiv x_0 \wedge x_1 \wedge x_2(\theta_{m_0}).$$

Similarly,

$$x_2 \equiv x_0 \wedge x_1 \wedge x_2(\theta_{m_2}).$$

Since $x_0 \wedge x_2 \in (m_0] \wedge (m_2]$, we join with $x_0 \wedge x_2$ and conclude that

$$x_0 \equiv x_0 \wedge x_2(\theta_{m_0}) \quad \text{and} \quad x_2 \equiv x_0 \wedge x_2(\theta_{m_2});$$

thus, $\langle x_0, x_2 \rangle \in \theta$, proving that θ is an equivalence relation.

We prove similarly that θ is a congruence relation, using 1) and 2) of Lemma 5, and noting that if $x \vee z$ is defined, then $x, z \in (m]$ for some $m \in M$.

6. From Q to S. We describe a covariant functor $S:\mathbf{Q} \to \mathbf{S}$. Let Q be an object in $\mathbf{Q}$ and set

$$SQ = \{0\} \cup \{a_1, a_2, a \mid a \in Q\} \cup \{b(a) \mid a, b \in Q \text{ with } b \rho a\},$$

where 0 is distinct from the other elements. Define a partial order $\leqq$ on SQ by setting

$$0 < a_i, \ i = 1, 2,$$

$$a_i < a, \ i = 1, 2,$$

$$b_1 < b(a) \text{ if } b \rho a,$$

$$a < b(a) \text{ if } b \rho a.$$

The maximal elements of SQ are then of the form $b(a)$ with $b \rho a$, $b, a \in Q$, since for each $a \in Q$ there is a $b \in Q$ with $b \rho a$. Each $(b(a)]$ is isomorphic to the lattice $M(b, a)$ depicted in Fig. 3.
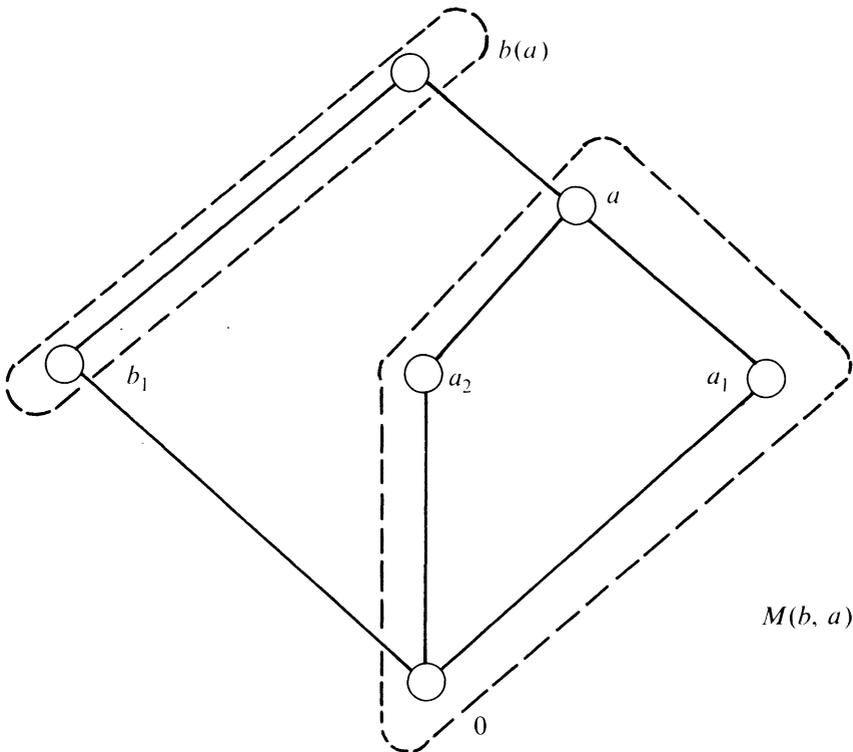

Figure 3

If $f:Q \to R$ is a **Q**-morphism, then define $Sf:SQ \to SR$ by setting

$$Sf0 = 0,$$

$$Sfa_i = (fa)_i, \; i = 1,2,$$

$$Sfa = fa,$$

$$Sfb(a) = (fb)((fa));$$

the last one makes sense, since $b \, \rho \, a$ implies that $fb \, \rho \, fa$.

LEMMA 8. *S is a covariant functor* $\mathbf{Q} \to \mathbf{S}$.

Proof. We use Lemma 3 to show that if Q is an object of $\mathbf{Q}$, then SQ is an object of $\mathbf{S}$. Note that in SQ

$d(c) \wedge b(a) = 0$ if $b \, \rho \, a$, $d \, \rho \, c$ and a, b, c, d are all distinct,

$c(b) \wedge b(a) = b_1$ if $c \, \rho \, b \, \rho \, a$, a, b, c distinct,

$c(a) \wedge b(a) = a$ if $c \, \rho \, a$, $b \, \rho \, a$, a, b, c distinct,

$c(a) \wedge c(b) = c_1$ if $c \, \rho \, a$, $c \, \rho \, b$, a, b, c distinct.

The only other possibility to check would be $a(b) \wedge b(a)$, but this cannot occur since ρ is antisymmetric. (Indeed, this is why in the definition of $\mathbf{Q}$ we required that ρ be antisymmetric; otherwise, the set $\{a(b), b(a)\}$ has two distinct maximal lower bounds, a_1 and b_1, and therefore has no inf.)

Clearly, Sf embeds SQ as an ideal in SR whenever $f : Q \to R$ is a $\mathbf{Q}$-morphism, completing the proof of the lemma.

Given an object Q of $\mathbf{Q}$ we define a map

$$\Phi_Q : \mathcal{H}Q \to \mathrm{Con} \; SQ$$

by setting

$$\Phi_Q H = \theta(H) = \bigvee(\theta(0, a) \mid a \in H),$$

where the element a in $\theta(0, a)$ is regarded in SQ.

We present a series of lemmas culminating in the theorem that Φ is a natural equivalence.

Given $a, b \in Q$ with $b \, \rho \, a$, the principal ideal $M(b, a)$ generated by $b(a)$ has exactly three congruence relations:

$\iota_{b,a}$, collapsing all of $M(b, a)$,

$\omega_{b,a}$, the identity relation,

$\alpha_{b,a}$, depicted in Fig. 3, with congruence classes $\{b_1, b(a)\}$ and $\{0, a_1, a_2, a\}$.

We first show that Φ_Q is surjective.

LEMMA 9. *Let Q be an object of $\mathbf{Q}$ and let θ be a congruence relation on SQ. Then*

$$H = \{a \in Q \mid a \equiv 0(\theta)\}$$

is a hereditary subset of Q and $\theta = \theta(H)$.

Proof. Let $b \in H$ and let $b \rho a$. Then $b \equiv 0(\theta)$ and so $b_1 \equiv 0(\theta)$. Thus

$$\theta|_{M(b,a)} = \iota_{b,a},$$

and we conclude that $a \in H$. Consequently, H is a hereditary subset of Q.

Since SQ is sectionally complemented, θ is determined by its congruence class containing 0. However, as is clear by referring to the congruence relations on $M(b, a)$, $a_1 \equiv 0$ if and only if $a_2 \equiv 0$ if and only if $a \equiv 0$, and $b(a) \equiv 0$ if and only if $b \equiv 0$ for any congruence relation. Thus θ is determined by

$$H = \{a \in Q \mid a \equiv 0(\theta)\},$$

that is, $\theta = \theta(H)$, concluding the proof.

We now characterize $\theta(H)$.

LEMMA 10. *Let Q be an object of* **Q** *and let H be a hereditary subset of Q. Let $a, b \in Q$ with $b \rho a$. Then*

$$\theta(H)|_{M(b,a)} = \begin{cases} \iota_{b,a} & \text{if } b \in H \\ \alpha_{b,a} & \text{if } b \notin H \text{ and } a \in H \\ \omega_{b,a} & \text{if } a \notin H. \end{cases}$$

Proof. The set M of maximal elements of SQ consists of all elements of the form $b(a)$ with $b \rho a$. For each $b(a) \in M$ define the congruence relation $\theta_{b(a)}$ on the ideal $M(b, a)$ by setting

$$\theta_{b(a)} = \begin{cases} \iota_{b,a} & \text{if } b \in H \\ \alpha_{b,a} & \text{if } b \notin H \text{ and } a \in H \\ \omega_{b,a} & \text{if } a \notin H. \end{cases}$$

Then, using the meet formulas in the proof of Lemma 8, it follows that

$$\theta_{b(a)}|_{M(b,a) \cap M(d,c)} = \theta_{d(c)}|_{M(b,a) \cap M(d,c)}$$

for any $b(a), d(c) \in M$. Thus, by Lemma 7 there is a unique congruence relation θ on SQ with

$$\theta|_{M(b,a)} = \theta_{b(a)} \quad \text{for all } b(a) \in M.$$

By Lemma 9,

$$\theta = \theta(H_1) \quad \text{with } H_1 = \{a \in Q \mid a \equiv 0(\theta)\}.$$

For each $a \in Q$ there is a $b \in Q$ with $b \rho a$. Then $a \in H_1$ if and only if $a \equiv 0(\theta)$ if and only if $a \equiv 0(\theta_{b(a)})$ if and only if $a \in H$, the last following by the definition of $\theta_{b(a)}$. Thus $H = H_1$, proving the lemma.

We are now ready for the main result of this section.

THEOREM 1. Φ *is a natural equivalence* $\mathcal{H} \to \mathrm{Con}\ S$. *That is, for each object Q in* **Q**,

$$\Phi_Q : \mathcal{H}Q \to \mathrm{Con}\ SQ$$

is an isomorphism and, given a **Q**-*morphism* $f : Q \to R$, *the diagram in* **D** *depicted in Fig.* 4 *commutes.*

$$
\begin{array}{ccc}
\mathcal{H}R & \xrightarrow{\ \Phi_R\ } & \mathrm{Con}\ SR \\
\Big\downarrow{\mathcal{H}f} & & \Big\downarrow{\mathrm{Con}\ SF} \\
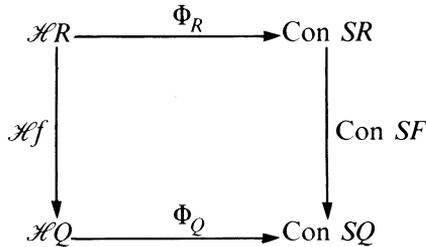\mathcal{H}Q & \xrightarrow{\ \Phi_Q\ } & \mathrm{Con}\ SQ
\end{array}
$$

Figure 4

Proof. We first show that, for each object Q of **Q**,

$$\Phi_Q : \mathcal{H}Q \to \mathrm{Con}\ SQ$$

is an isomorphism. Clearly, $H_1 \subseteq H_2$ implies that $\theta(H_1) \subseteq \theta(H_2)$ and, by Lemma 9, Φ_Q is surjective. We need thus only show that Φ_Q is an embedding, that is, that $\theta(H_1) \subseteq \theta(H_2)$ implies that $H_1 \subseteq H_2$. Let $\theta(H_1) \subseteq \theta(H_2)$ for $H_1, H_2 \in \mathcal{H}Q$. To show that $H_1 \subseteq H_2$, take $a \in H_1$, and let $b \in Q$ with $b\ \rho\ a$. Then $a \equiv 0(\theta(H_2))$ and so

$$\theta(H_2)|_{M(b,a)} \neq \omega_{b,a};$$

so, by Lemma 10, $a \in H_2$. Thus $H_1 \subseteq H_2$, concluding the proof that Φ_Q is an isomorphism.

Finally, we show that Fig. 4 is commutative. Let $H \in \mathcal{H}R$. Then, by Lemma 9,

$$(\mathrm{Con}\ Sf) \circ \Phi_R H = \theta(H_1) \quad \text{for some } H_1 \in \mathcal{H}Q.$$

But $a \in H_1$ if and only if $a \equiv 0((\mathrm{Con}\ Sf) \circ \Phi_R H)$ if and only if $Sfa \equiv 0(\Phi_R H)$, that is, $fa \equiv 0(\theta(H))$ if and only if $fa \in H$. Thus $H_1 = \mathcal{H}fH$. But then

$$(\mathrm{Con}\ Sf) \circ \Phi_R H = \Phi_Q \circ \mathcal{H}fH,$$

completing the proof of the theorem.

7. From S to L. The covariant functor $\mathrm{Id} : \mathbf{S} \to \mathbf{L}$ associates with each object S in **S** the lattice $\mathrm{Id}\ S$ of all ideals of S. If $f : S \to T$ is a **S**-morphism, then $\mathrm{Id}\ f : \mathrm{Id}\ S \to \mathrm{Id}\ T$ is the embedding as an ideal determined by setting $\mathrm{Id}\ fI = f(I)$ for each ideal I of S.

Given an object S of **S** and a congruence relation θ on S, there is a congruence relation $\Omega_S\theta$ on the lattice Id S determined by setting, for $x, y \in S$,

$$(x] \equiv (y] \, (\Omega_S\theta) \text{ if and only if } x \equiv y(\theta).$$

The following result is essentially the main result of [2] (see also Lemma II.3.19, p. 84 of [1]).

LEMMA 11. Ω *is a natural equivalence* Con $\to$ Con Id. *That is, for each object S of* **S**, *the map*

$$\Omega_S : \text{Con } S \to \text{Con Id } S$$

is an isomorphism and, given a **S**-*morphism $f : S \to T$, the diagram in* **D** *depicted in Fig. 5 commutes.*
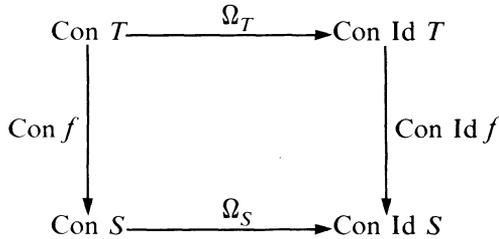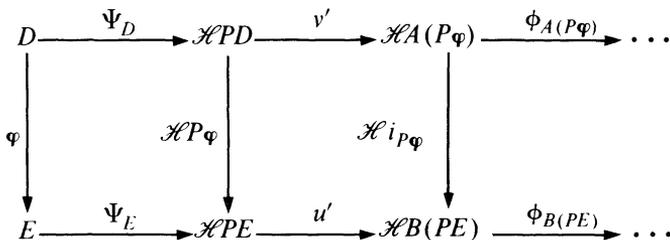

Figure 5

8. The final result. By combining the natural equivalences of Lemma 1, Theorem 1, Lemma 11, and the commutative diagram of Lemma 2 we get our main result.

THEOREM 2. *Let D and E be finite distributive lattices and let $\varphi : D \to E$ be a $0, 1$-preserving lattice homomorphism. Then there exist sectionally complemented finite lattices K, L and an embedding α of K as an ideal in L such that there are isomorphisms $\beta : D \to$ Con L, $\gamma : E \to$ Con K satisfying*

$$\gamma \circ \varphi = (\text{Con } \alpha) \circ \beta.$$
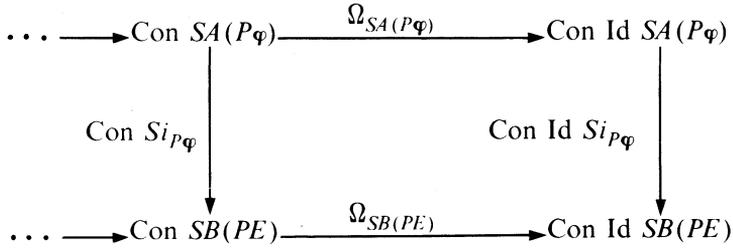
Proof. Consider the diagram in Fig. 6.

$$\cdots \longrightarrow \text{Con } SA(P\varphi) \xrightarrow{\quad\Omega_{SA(P\varphi)}\quad} \text{Con Id } SA(P\varphi)$$

$$\text{Con } Si_{P\varphi} \Bigg\downarrow \qquad\qquad\qquad \text{Con Id } Si_{P\varphi} \Bigg\downarrow$$

$$\cdots \longrightarrow \text{Con } SB(PE) \xrightarrow{\quad\Omega_{SB(PE)}\quad} \text{Con Id } SB(PE)$$

Figure 6

By Lemma 1, ψ_D and ψ_E are isomorphisms and the left-most square commutes. By Lemma 2, u' and v' are isomorphisms and the next square commutes. By Theorem 1, $\Phi_{A(P\varphi)}$ and $\Phi_{B(PE)}$ are isomorphisms and the associated square commutes. Finally, by Lemma 11, $\Omega_{SA(P\varphi)}$ and $\Omega_{SB(PE)}$ are isomorphisms and the right-most square commutes.

Set $L = \text{Id } SA(P\varphi)$, $K = \text{Id } SB(PE)$, $\alpha = \text{Id } Si_{P\varphi}$. If we set

$$\beta = \Omega_{SA(P\varphi)} \circ \Phi_{A(P\varphi)} \circ v' \circ \psi_D$$

and

$$\gamma = \Omega_{SB(PE)} \circ \Phi_{B(PE)} \circ u' \circ \psi_E$$

the theorem follows, except for the fact that K and L are sectionally complemented. This last statement can be verified using Lemma II 3.9 of [2]. (See also Exercise 33.)

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University of Manitoba,
Winnipeg, Manitoba