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GROUPS WITH AN AUTOMORPHISM CUBING MANY ELEMENTS

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1. Introduction

Let G be a group and α_n the mapping which takes every element of G to its nth power, where n is an integer. It is well known that if α_n is an automorphism then G is Abelian in the cases $n = -1, 2$, and 3. For any other integer $n (\neq 0)$ there exists a non-Abelian group which admits α_n as the identity automorphism. Indeed Miller (1929) has shown that if $n \neq 0, \pm 1, 2, 3$ then there exist non-Abelian groups which admit α_n as a non-trivial automorphism.

Confining our attention to finite groups, we consider the problem of how large a proportion of the elements of a non-Abelian group can be mapped to their nth powers by some automorphism when $n = -1, 2$ or 3. Let $\mathscr{G}_p$ denote the set of all finite groups with order divisible by the prime p but by no smaller prime. In the case $n = -1$ it is known that if G is a non-Abelian group in $\mathscr{G}_p$ then not more than $\frac{3}{4}|G|$ or $|G|/p$ of its elements can be inverted by an automorphism according as $p = 2$ or p is odd. Manning (1906) classified all groups G with an automorphism inverting $\frac{3}{4}|G|$ elements, while Liebeck and the present author (1973) classified all non-Abelian groups in $\mathscr{G}_p$ (p odd) with an automorphism inverting $|G|/p$ elements.

Liebeck (1973) has recently settled the case $n = 2$ by proving that if G is a non-Abelian group in $\mathscr{G}_p$ then no automorphism can send more than $|G|/p$ elements of G to their squares. This result includes the case $p = 2$. A complete classification of all non-Abelian groups G in $\mathscr{G}_p$ with an automorphism squaring exactly $|G|/p$ elements also appears in Liebeck (1973).

In this paper we investigate the case $n = 3$. We prove the following results:

(a) If G is a finite non-Abelian group then not more than $\frac{3}{4}|G|$ elements can be cubed by an automorphism.

(b) G is a finite group with an automorphism cubing exactly $\frac{3}{4}|G|$ elements if and only if G has central quotient group of order 4 and the centre of G has no elements of order 3.

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(c) If G is a non-Abelian group in $\mathscr{G}_p$ and p is odd then no automorphism of G can send more than $|G|/p$ elements to their cubes.

2. Notation

G	Denotes a finite group.
α	An automorphism of G.
T	$\{g \in G \mid (g)\alpha = g^3\}$.
$\mathscr{G}_p$	The set of all groups with order divisible by the prime p but by no smaller prime.
$C_G(t)$	The centralizer of the element t in the group G.
$Z(G) = Z$	The centre of G.
G'	The commutator subgroup of G.

3. Preliminary results.

LEMMA 3.1. *If* $\alpha \in \operatorname{Aut} G$ *then* $g^{-1}(g\alpha) \in C_G(T \cap g^{-1}Tg)$.

PROOF. For $g \in G, t \in T, g^{-1}tg \in T \Leftrightarrow (g^{-1}tg)^3 = (g^{-1}tg)\alpha \Leftrightarrow [g^{-1}(g\alpha), t] = 1$.

LEMMA 3.2. *If* $|G|$ *is odd and* $g\alpha = g (g \neq 1)$ *then* $T \cap Tg$ *is empty.*

PROOF. Suppose that $t \in T \cap Tg$. Then $t = t_1 g$ and applying α we get $t^3 = t_1{}^3 g$. Thus $t^2 = t_1{}^2$ and the oddness of $|G|$ gives $t = t_1$, and $g = 1$.

LEMMA 3.3. (Joseph (1969)). *If G is a non-Abelian group in $\mathscr{G}_p$ (p odd) then G has at least $|G|/p$ conjugacy classes if and only if G is nilpotent of class* 2 *with* $|G'| = p$.

PROOF. G has $(G:G')$ irreducible representations of degree 1 and hence at least $|G|/p - (G:G')$ other irreducible representations, each of degree at least p. Hence

$$|G| \geqq |G|/|G'| + p^2\left(\frac{1}{p} - \frac{1}{|G'|}\right)|G|$$

from which it follows that $|G'| \leqq p+1$. Since p is odd, $|G'| = p$, and so $G' \subseteq Z(G)$, since G belongs to $\mathscr{G}_p$. The converse is obvious.

LEMMA 3.4. *If G belongs to $\mathscr{G}_p$ and $Z(G)$ is not contained in T then* $|T| \leqq |G|/p$.

PROOF. If $Z \nsubseteq T$ then $T \cap Z$ is a proper subgroup of Z. Clearly, $|Zx \cap T| \leqq (1/p)|Z|$ for any x in G and the result follows.

4. Main Results

THEOREM 4.1. *If* $|T| > \frac{3}{4}|G|$ *then* $T = G$ *and* G *is Abelian.*

PROOF. Suppose that $|T| > \frac{3}{4}|G|$ and let t be any element of T. Then

$$|t^{-1}Tt \cap T| = |t^{-1}Tt| + |T| - |t^{-1}Tt \cup T| > \tfrac{3}{4}|G| + \tfrac{3}{4}|G| - |G| = \tfrac{1}{2}|G|.$$

By Lemma 3.1, t^2 commutes with more than half the elements of G and hence $t^2 \in \boldsymbol{Z}(G)$, for all $t \in T$.

Similarly, $|tT \cap T| > \frac{1}{2}|G|$. However, if t, s and ts belong to T then $t^3s^3 = (ts)^3$ and so $ts = st$, since t^2 is central.

Hence, $|C_G(t)| > \frac{1}{2}|G|$ for every $t \in T$ and so every element of T is central. Finally, $|\boldsymbol{Z}(G)| > \frac{3}{4}|G|$ and so $T = \boldsymbol{Z}(G) = G$.

THEOREM 4.2. *G has an automorphism for which $|T| = \frac{3}{4}|G|$ if and only if $(G:\boldsymbol{Z}(G)) = 4$ and $Z(G)$ has no elements of order 3.*

PROOF. If $(G:\boldsymbol{Z}(G)) = 4$ then $G = \boldsymbol{Z} \cup \boldsymbol{Z}a \cup \boldsymbol{Z}b \cup \boldsymbol{Z}ab$ where a^2, b^2 and $[a,b]$ all belong to $\boldsymbol{Z}$. A routine calculation shows that if $\boldsymbol{Z}$ has no elements of order 3 then the map defined by $za^ib^j \to z^3a^{3i}b^{3j}$, $0 \leqq i, j \leqq 1$ for all $z \in \boldsymbol{Z}$, defines an automorphism sending exactly $\frac{3}{4}|G|$ elements to their cubes.

Conversely, let G be a group for which $|T| = \frac{3}{4}|G|$. Clearly G is non-Abelian. Let t be any non-central element of T. We show that $C_G(t)$ is an Abelian subgroup of index 2 in G.

As in the proof of Theorem 4.1 $|C_G(t^2)| \geqq \frac{1}{2}|G|$. If $|C_G(t^2)| > \frac{1}{2}|G|$ then t^2 is central and thus $|tT \cap T| \geqq \frac{1}{2}|G|$. Thus $|C_G(t)| \geqq \frac{1}{2}|G|$ and since t central, $|C_G(t)| = \frac{1}{2}|G|$. Moreover, $C_G(t) \subset T$ and so $C_G(t)$ is Abelian.

We can now assume that $|C_G(t^2)| = \frac{1}{2}|G|$ and $C_G(t^2)$ is Abelian, since $C_G(t^2) \subset T$. Accordingly, if $gt^2 = t^2g$ then $gt = tg$ since $tt^2 = t^2t$. So $C_G(t) = C_G(t^2)$ and $C_G(t)$ is an Abelian subgroup of index 2 in G.

Finally, let a and b be a pair of non-commuting elements of T. Such a pair exists since otherwise G is Abelian. Let $A = C_G(a)$ and $B = C_G(b)$ and so A and B are distinct Abelian subgroups of index 2 in G.

Now $G = AB$ and $(G: A \cap B) = 4$. Clearly $A \cap B = \boldsymbol{Z}(G)$. Since $\boldsymbol{Z}(G) \subset T$, $\boldsymbol{Z}(G)$ has no elements of order 3 and the proof is complete.

THEOREM 4.3. *Let $G \in \mathscr{G}_p$ and let G be non-Abelian, where p is odd. Then $|T| \leqq |G|/p$, for any automorphism α of G.*

PROOF. Suppose that $G \in \mathscr{G}_p$ and $|T| > (1/p)|G|$, where G is non-Abelian. We first consider the case where α fixes a non-trivial element g of G. Now g has order at least p and by Lemma 3.2 the p sets $T, Tg, \cdots, Tg^{p-1}$ are pairwise disjoint. Then, $|G| \geqq |T \cup Tg \cup \cdots \cup Tg^{p-1}| = p|T| > |G|$, a contradiction.

We may thus assume that α is fixed-point-free. By Lemma 3.1, for $g \in G$, $t \in T$, $g^{-1}tg \in T$ if and only if $[g^{-1}(g\alpha), t] = 1$. Since α is fixed-point-free the correspondence $g^{-1}g(\alpha) \leftrightarrow g$ is one-to-one and so $g^{-1}tg \in T \Leftrightarrow [g, t] = 1$. Hence any conjugacy class contains at most one element of T. Thus G has at least $(1/p)|G|$ conjugacy classes and so by Lemma 3.3, G is nilpotent of class 2 with $|G'| = p$. Moreover, by Lemma 3.4, $Z(G) \subset T$ and so $G' \subseteq Z(G) \subset T$.

Finally, let r and s be a pair of noncommuting elements of T. Then, $[r, s]\alpha = [r^3, s^3] = [r, s]^3 = [r, s]^9$, since r, $s \in T$ and G is nilpotent of class 2. Thus $[r, s]^6 = 1$ and so $[r, s]^3 = 1$, by the oddness of $|G|$. Since T has no elements of order 3, this is a contradiction and the theorem is established.

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