LENGTH FUNCTIONS ON HYPERCENTRAL GROUPS

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In [6] the structure of any real valued length function on an abelian group G is determined. It is shown there, in Theorem 6.1., that such a length function is an extension of a non-Archimedean length function l_1 on N by an Archimedean length function l_2 on $H = G/N$. Any non-Archimedean length function is given by a chain of subgroups, as described in [5], and following from results of Nancy Harrison [2], the length l_2 is essentially the absolute value function on a subgroup of R. In the situation above if $N \neq G$ then N is a subgroup of G whose elements have bounded lengths. In this paper we show that it is an easy consequence of techniques developed in [1] that this result can be extended to hypercentral groups, thus determining the structure of any length function in this case. We point out that the result does not extend to soluble groups. The infinite dihedral group D_∞ is soluble. However if D_∞ is regarded as a free product of two cyclic groups of order 2 and is given the length function associated with a free product, as described by Lyndon [3], then N is not a subgroup of D_∞, and the lengths of its elements are unbounded.

We consider hypercentral groups, that is groups G with the property that every non-trivial epimorphic image of G has a non-trivial centre. For properties of hypercentral groups we refer to Chapter VI of [4], and for definitions and some properties of length functions used in the following theorem, we refer to [1] and [6].

Theorem. *If l is a real valued length function on a hypercentral group G, then l is an extension of a non-Archimedean length function l_1 on N, the collection of non-Archimedean elements of G, by an Archimedean length function l_2 on $H = G/N$. Moreover there exists $c \geq 0$ and an embedding $g: H \to R$ such that c is an upper bound for the lengths of the elements of N and $l_2(u) = c + |g(u)|$ for $u \in H$, $u \neq 1$.*

Proof. We consider results established in [1]. Here for l a length function on any group G the maximal l-trivializable subgroup T of G is defined by

$$T = \{a \in G; \ l(ax) = l(x) \quad \text{for all } x \notin N\}.$$

The subgroup $T \subset N$, and it is shown in Proposition 1.3. that the length function restricted to T can be changed subject only to a bound condition. Two length functions on G are then defined to be equivalent if their maximal trivializable subgroups are identical, and the length functions agree on elements outside this subgroup. By considering the cases where the centre Z of G is or is not contained in N, it is shown in Theorem 2.1 of [1] that l is an extension of a non-Archimedean length function l_1 on N

by an Archimedean length function l_2 on $H = G/N$, an abelian group, or l is equivalent to an extension of a non-Archimedean length function l_1 on C, the core of T, by a length function l_2 on $K = G/C$, a group with trivial centre. It is possible above that $C = T = N$, which is the case in (ii) of Theorem 2.1, when the equivalence is, in fact, an equality.

The factor group K above cannot be hypercentral, since it has trivial centre, and hence the group G in this case cannot by hypercentral. If G is assumed to be hypercentral it thus follows that l is an extension of a non-Archimedean length function l_1 on N by an Archimedean length function l_2 on the abelian group $H = G/N$.

In Section 3 of [2], Nancy Harrison shows in effect that if l_2 is an Archimedean length function on an abelian group H then there exists $c \geq 0$ such that $2d_2(u, u^{-1}) = 2l_2(u) - l_2(u^2) = c$ and $2d_2(u, v) = l_2(u) + l_2(v) - l_2(uv^{-1}) \geq c$ for all non-trivial $u, v \in H$. Also there is an embedding $g: H \to R$ such that $l_2(u) = c + |g(u)|$. Condition (i) of Proposition 4.1. of [6], must be satisfied by the length functions l_1 and l_2, and so c is an upper bound for the lengths of the elements of N, completing the proof of the theorem.

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