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ON IMMERSIONS OF N-MANIFOLDS IN CODIMENSION $N-1$

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Abstract

We give a simple proof, using only classical algebraic topology, of the following theorem of B. H. Li and F. P. Peterson. Any map from an N-manifold into a $(2N-1)$-manifold is homotopic to an immersion.

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Let $f\colon M \to N$ be a map between smooth manifolds of dimension n and $n+k$ respectively. In [6] Whitney proved that if $k \geq n$ then f is homotopic to an immersion. In this note we give a simple proof of the following generalization of Whitney's result due to B. H. Li and F. P. Peterson [4].

THEOREM. *Let $f\colon M \to N$ be a map between smooth manifolds of dimension n and $2n-1$, respectively. Then f is homotopic to an immersion* $(n > 1)$.

By Hirsch's theorem [2] it is enough to find an $(n-1)$-vector bundle ν such that $TM \oplus \nu \cong f^*TN$, where T denotes the tangent bundle. If ν_M denotes the normal bundle of M then this condition is equivalent to $[\nu] = [f^*TN \oplus \nu_M]$, where the brackets denote the stable class of the bundle.

Let $\xi\colon M \to BO(n+1)$ be a map representing $f^*TN \oplus \nu_M$, then the existence of the bundle ν is equivalent to the lifting problem (1) below, where p is a fibration with fiber $O(n+1)/O(n-1) = V_{n+1,2}$. The space $V_{n+1,2}$ is $(n-2)$-connected so by classical obstruction theory [5] there is only one

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obstruction $\theta_n(\xi)$ to the existence of this lifting and

$$\theta_n(\xi) \in H^n(M;\ \{\pi_{n-1}(V_{n+1,2})\})$$

(the brackets denote twisted coefficients).

$$\text{(1)} \qquad \begin{array}{ccc} & & BO(n-1) \\ & \nearrow & \big\downarrow p \\ M & \xrightarrow[\xi]{} & BO(n+1) \end{array}$$

We now have two cases:

(i) If M has boundary then $M \simeq M - \partial M$ and $M - \partial M$ is an open n-manifold so it has the homotopy type of an $(n-1)$-complex and hence

$$H^n(M;\ \{\pi_{n-1}(V_{n+1,2})\}) = 0.$$

(ii) If M has no boundary, then if M is not compact M is an open n-manifold and we proceed as in case (i). Hence we only have to consider the case when M is a closed manifold.

Suppose now that M is closed and assume without loss of generality that M is connected. We will show that $\theta_n(\xi)$ is determined by the nth Stiefel-Whitney class $w_n(\xi)$ and that $w_n(\xi) = 0$.

It is known [5] that if n is odd then $\pi_{n-1}(V_{n+1,2}) \cong \mathbf{Z}$ and that the bundle ξ twists $\mathbf{Z}$ with the homomorphism $w_1(\xi)_{\#}\colon \pi_1(M) \to \mathbf{Z}_2$ induced by the class $w_1(\xi)$. We will denote these twisted coefficients by $\mathbf{Z}_\xi$; with this notation $\theta_n(\xi) \in H^n(M;\ \mathbf{Z}_\xi)$.

LEMMA. *$H^n(M;\ \mathbf{Z}_\xi) \cong \mathbf{Z}$ or $\mathbf{Z}_2$.*

PROOF. By the Thom isomorphism with twisted coefficients [3] we have that $H^n(M;\ \mathbf{Z}_\xi) \cong \tilde{H}^{2n+1}(T\xi;\ \mathbf{Z})$. The bundle ξ is isomorphic to a smooth bundle so we can assume that the total space $E(\xi)$ is a $(2n+1)$-manifold. Furthermore, if we denote by $E(\xi)^\infty$ the one point compactification of $E(\xi)$ and by $H_c^*(-)$ the cohomology with compact supports, we have $\tilde{H}^{2n+1}(T\xi;\ \mathbf{Z}) \cong \tilde{H}^{2n+1}(E(\xi)^\infty;\ \mathbf{Z}) \cong H_c^{2n+1}(E(\xi);\ \mathbf{Z})$, and by Poincaré duality with twisted coefficients $H_c^{2n+1}(E(\xi);\ \mathbf{Z}) \cong H_0(E(\xi);\ \mathbf{Z}_{TE(\xi)})$.

Finally $H_0(E(\xi);\ \mathbf{Z}_{TE(\xi)}) \cong \mathbf{Z}/H$, where H is the subgroup generated by elements of the form $n - r \cdot n$ with $n \in \mathbf{Z}$ and $r \in \pi_1(E(\xi))$. But $r \cdot n = n$ or $-n$ so $H = 0$ or $2\mathbf{Z}$.

The obstruction class $\theta_n(\xi)$ is related to $w_n(\xi)$ as follows [5]: if n is even then $\theta_n(\xi) = w_n(\xi)$, and if n is odd consider the sequence $H^{n-1}(M;\ \mathbf{Z}_2) \xrightarrow{\delta} H^n(M;\ \mathbf{Z}_\xi) \xrightarrow{\rho} H^n(M;\ \mathbf{Z}_2)$, where δ is the twisted Bockstein and ρ is the mod 2 reduction, then $\delta(w_{n-1}(\xi)) = \theta_n(\xi)$ and $\rho(\theta_n(\xi)) = w_n(\xi)$.

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By the lemma above $H^n(M;\ \mathbf{Z}_\xi) \cong \mathbf{Z}$ or $\mathbf{Z}_2$. If this group is isomorphic to $\mathbf{Z}$ then $\delta(w_{n-1}(\xi)) = \theta_n(\xi) = 0$. If it is isomorphic to $\mathbf{Z}_2$ then ρ is an isomorphism sending $\theta_n(\xi)$ to $w_n(\xi)$.

Hence, we have proved that $\theta_n(\xi) = 0$ if and only if $w_n(\xi) = 0$.

We finish the proof of the theorem by showing that $w_n(\xi)$ is zero.

LEMMA. $w_n(\xi) = 0$.

PROOF. Let $f^!$ and $f_!$ denote the Umkehr homomorphisms in cohomology and homology, respectively, associated with the map $f\colon M \to N$. These homomorphisms have the following properties [1]: (i) $\langle f^!(a), x\rangle = \langle a, f_!(x)\rangle$; (ii) $S_q f^!(x) = f^!(w(\xi) \cup S_q(x))$, where S_q is the total Steenrod square and $w(\xi)$ is the total Stiefel-Whitney class; (iii) if [ ] denotes the fundamental class of a manifold, then $f_![N] = [M]$.

Using these properties we have

$$\langle w_n(\xi),\ [M]\rangle = \langle w_n(\xi), f_![N]\rangle = \langle f^!(w_n(\xi)), [N]\rangle = \langle S_q^n f^!(1), [N]\rangle.$$

But $f^!(1)$ is a class of dimension $n-1$ so this Kronecker product is zero, and as M is connected then $w_n(\xi) = 0$.

REMARK. It is well known that there is no immersion of the real projective space of dimension 2^r into the sphere of dimension $2(2^r) - 2$ so we do not have a similar result when the codimension is less than $n-1$.

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