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SOME REMARKS ON VARIATIONAL-LIKE AND QUASIVARIATIONAL-LIKE INEQUALITIES

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In this paper we study the variational-like inequalities, which generalise some results of Parida, Sahoo and Kumar, and we also investigate the quasivariational-like inequalities. We establish some existence theorems of a solution for the above problem.

I. FORMULATION

We denote the inner product and norm on $\mathbb{R}^n$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let C be a convex and closed set in $\mathbb{R}^n$. Given $\psi : C \to \mathbb{R}$, $\psi(x)$ is differentiable function. In recent years, many research works were published for a certain class of differentiable functions, now known as invex functions. We recall this definition given in [**2**]. Let $\psi : C \to \mathbb{R}$ be differentiable. Then ψ is η-convex on K if there exists a continuous map $\eta : C \times C \to \mathbb{R}^n$ such that

$$\psi(y) - \psi(x) \geqslant \langle \nabla\psi(x), \eta(y,x)\rangle, \text{ for all } x, y \in C,$$

where $\nabla\psi(x)$ is a gradient of ψ at x.

It is known that if $\eta(y,x) = y - x$, then ψ is convex function on C.

Suppose that f is η-convex over C for some continuous map $\eta : C \times C \to \mathbb{R}^n$. We consider the minimisation problem

$$\min f(x) \quad \text{subject to} \quad x \in C \tag{1.1}$$

where C is a convex and closed set in $\mathbb{R}^n$ and f is also continuously differentiable with $\nabla f(x) := F(x)$.

From [**12**] we know that if $\overline{x} \in K$ satisfies

$$\langle F(\overline{x}), \eta(x,\overline{x})\rangle \geqslant 0 \quad \text{for all} \quad x \in C \tag{1.2}$$

then $\overline{x}$ is the solution of the problem (1.1).

By the above fact, we study the following generalised problem :

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P_1 : Find $\overline{x} \in C$ such that

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle + \varphi(x) - \varphi(\overline{x}) \geqslant 0 \quad \text{for all} \quad x \in C,$$

where $F : C \to \mathbb{R}^n$, $\eta : C \times C \to \mathbb{R}^n$ and $\varphi : C \to R$.

We call it a generalised variational-like inequality problem.

If $\varphi(x) \equiv 0$, then P_1 is called a variational-like inequality problem in [**12**]. If $\varphi \equiv 0$, $\eta(x, \overline{x}) = x - \overline{x}$, then P_1 reduces to a variational inequality problem in [**7**]. If $\varphi \equiv 0$, $\eta(x, \overline{x}) = x - g(\overline{x})$ where $g : C \to C$, then P_1 was considered in [**10**], ...

In the formulation of the problem P_1, the underlying convex set C does not depend upon the solution. In many applications, the convex set also depends implicity on the solution $\overline{x}$ itself. In this case, for $\eta(x, \overline{x}) = x - \overline{x}$, $\varphi(x) = 0$. The problem P_1 is known as the quasi-variational inequality problem, originally studied by Bonsoussan and Lions [**3**] in impluse control. To be more specific, we introduce an extension of P_1 as follows.

Let C be a convex and closed set in $\mathbb{R}^n$, and 2^C will denote the family of all nonempty subsets of C. Given a multivalued map $E : C \to 2^C$ and two continuous maps $F : C \to \mathbb{R}^n$ and $\eta : C \times C \to \mathbb{R}^n$ and a function $\varphi : C \to \mathbb{R}^n$, we consider the problem:

P_2 : Find $\overline{x} \in C$ such that $\overline{x} \in E(\overline{x})$ and

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle + \varphi(x) - \varphi(\overline{x}) \geqslant 0 \quad \text{for all} \quad x \in E(\overline{x}).$$

We call this a generalised quasi-variational-like inequality problem. If $\varphi(x) = 0$, $\eta(x, \overline{x}) = g(x) - g(\overline{x})$ and $E(x) = m(x) + C$, then P_2 is equivalent to the general quasi complementarity problem in [**11**] and [**9**].

In this paper, we shall establish some existence theorems for the problems P_1 and P_2 under some different conditions on the subset C and the maps E, F, η and the function φ.

II. Lemma and Definition

We must use following definitions.

The map $F : C \to \mathbb{R}^n$ is said to be η-monotone on C if there exists a continuous map $\eta : C \times C \to \mathbb{R}^n$ such that

$$\text{(2.1)} \qquad \langle F(x), \eta(y, x)\rangle + \langle F(y), \eta(y, x)\rangle \leqslant 0 \quad \text{for all} \quad x, y \in C.$$

F is said to be strictly η-monotone over C if the equality holds in (2.1) only when $x = y$.

The function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semicontinuous if for every $r \in \mathbb{R}$ the set $\{x \in C : \varphi(x) \leqslant r\}$ is closed in C for every $r \in \mathbb{R}$. This is equivalent to saying that the epigraph of φ

$$epi(\varphi) = \{(x, r) \in C \times \mathbb{R} : r \geqslant \varphi(x)\}$$

is closed in $C \times \mathbb{R}$.

The following basic theorem can be found in [**6**, Theorem 1, p.1].

LEMMA 1. *Let X be a compact topological space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. Then φ is bounded below and the infimum of φ is achieved at $x_0 \in X$.*

Let $E : C \to 2^{\mathbb{R}^n}$, E is said to be upper semicontinuous, u.s.c. for short, at x_0 if for every open set $V \supset Ex_0$ there exists a neighborhood $\mathcal{U}$ of x_0 such that $Ex \subset V$ for all $x \in \mathcal{U}$.

E is said to be closed if for each $x_n \in C$, x_n converging to x, and $\{y_n\}$, with $y_n \in E(x_n)$, y_n converging to y, implies $y_0 \in E(x_0)$.

We say that E is u.s.c. (closed) at C if E is u.s.c. (closed) at every $x_0 \in C$.

We denote

$$graf\ E = \{(x, y) \in \mathbb{R}^n \times R^n : y \in E(x)\}.$$

It can be verified that [**4**], (i) E is closed if and only if graph E is also a closed set; (ii) if E is closed and $\overline{E(C)}$ is a compact set $\subset \mathbb{R}^n$, then E is u.s.c. on C; (iii) if E is u.s.c. and $E(x)$ is closed set for all $x \in C$, then E is closed; (iv) if E is u.s.c. and $E(x)$ is a compact set for each $x \in C$, then $E(C)$ is a compact set and the following theorem is proved in [**1**].

LEMMA 2. *Let $E, G : C \subset \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be such that $E(x) \cap G(x) \neq \emptyset$ for each $x \in C$. Suppose that E is u.s.c. at $x_0 \in C$, $E(x_0)$ is a compact subset and the graph of G is closed. Then the map $(E \cap G)(x) = E(x) \cap G(x)$ is also u.s.c. at x_0.*

The notion of measures of noncompactness was introduced by Kuratowski [**8**] and for applying this measure of noncompactness we can see [**5**]. We introduce the generalised measure of noncompactness as follows .

The function $\alpha : 2^{\mathbb{R}^n} \to \mathbb{R}^n = [0, \infty)$ is said to be a generalised measure of noncompactness if the following conditions are satisfied

(1) $\alpha(B) = 0$ if and only if $\overline{B}$ is compact, where $B \in 2^{\mathbb{R}^n}$.
(2) $\alpha(CoB) = \alpha(B)$, where CoB is the convex hull of B.
(3) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

Let $E : C \to 2^{\mathbb{R}^n}$, E is said to be a condensing multivalued map if and only if $\alpha(E(B)) < \alpha(B)$ whenever $\alpha(B) > 0$.

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III. Generalised Variational-like Inequalities

We shall first prove the following generalised variational inequalities.

Theorem 1. *Let C be a compact and convex set in $\mathbb{R}^n$ and let $F : C \to \mathbb{R}^n$ and $\eta : C \times C \to \mathbb{R}^n$ be two continuous maps, and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Suppose that*

$$\langle F(x), \eta((x)x)\rangle \geqslant 0 \quad \text{for each} \quad x \in C$$

and for each fixed $x \in C$, the function $\langle F(x), \eta(y,x)\rangle$ is quasi convex in $y \in C$.

Then P_1 has a solution.

Proof: As in [**12**], for each $x \in C$, define

$$S(x) = \{s \in C : \langle F(x), \eta(s,x)\rangle + \varphi(s) = \inf_{v \in C}\{\langle F(x), \eta(v,x)\rangle + \varphi(v)\}.$$

Since C is compact and $v \to \langle F(x), \eta(v,x)\rangle + \varphi(v)$ is lower semicontinuous quasi-convex in v, Lemma 1 shows that $S(x) \neq \emptyset$, closed and convex subset of C. And one can see that the multivalued map $S : C \to 2^C$ is upper semicontinuous. By Kakutani's fixed point theorem [**13**], there exists $\overline{x} \in C$ such that $\overline{x} \in S(\overline{x})$. Consequently, for all $x \in C$

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle + \varphi(x) \geqslant \langle F(\overline{x}), \eta(\overline{x}, \overline{x})\rangle + \varphi(\overline{x}).$$

We get

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle + \varphi(x) - \varphi(\overline{x}) \geqslant 0 \quad \text{for all} \quad x \in C.$$

This completes the proof of the theorem. ∎

Remark 1. If $\langle F(x), \eta(x,x)\rangle = 0$ for each $x \in C$ and $\varphi(x) \equiv 0$ then Theorem 1 is Theorem 3.1 of [**12**]. The following example shows that there exist F and η, that the above equality is not satisfied and there exists a solution to P_1. Given $C = [-1,1]$, $\eta : C \to \mathbb{R}$, $\eta : C \times C \to \mathbb{R}$ by $F(x) = x$, $\eta(y,x) = x.y^2$, then

$$\langle F(x), \eta(x,x)\rangle = x^4 = 0 \Leftrightarrow x = 0$$

and $\langle F(x), \eta(x,x)\rangle > 0$ for every $x \neq 0$; it is easy to see that $\overline{x} = 0$ is the solution to P_1.

Now, from the above fact we make the following hypothesis.

Condition 1. Let C be a convex and closed set in $\mathbb{R}^n$. Let $F : C \to \mathbb{R}^n$, $\eta : C \times C \to \mathbb{R}^n$ be two continuous maps such that

(1) $\langle F(x), \eta(x,x)\rangle \geqslant 0$ for all $x \in C$ and
(2) for each fixed $x \in C$, the function $\langle F(x), \eta(y,x)\rangle$ is convex in $y \in C$.

We are now going to study the generalised variational-like inequality problem P_1 for a noncompact set C as in [7] and [12]. For a real number $r > 0$ we shall denote $K_r = \{x : x \in C \quad \text{and} \quad \|x\| \leqslant r\}$. We always assume that there always exists an $r_0 > 0$ such that K_r is nonempty, whenever $r \geqslant r_0$. We notice that K_r is compact and convex. Let F and η be such that Condition 1 is satisfied; then by Theorem 1, there exists at least one $x_r \in K_r$ such that

$$(3.1) \qquad \langle F(x_r), \eta(x, x_r)\rangle + \varphi(x) - \varphi(x_r) \geqslant 0 \quad \text{for all} \quad x_r \in K_r$$

where φ is a lower semicontinuous function on C.

By an argument analogous to that used for the proof in [**12**], we also get some theorems and their proofs are thus omitted.

Proposition 1. *Let C, F, φ, η be a such that Condition 1 is satisfied. A necessary and sufficient condition that there exists a solution to P_1 is that there exists an $r > 0$ such that a solution $x_r \in K_r$ of (3.1) satisfies the estimate $\|x_r\| < r$.*

Proposition 2. *Let C, F, η and φ be such that Condition 1 is satisfied. Then the generalised variational-like inequality problem P_1 has a solution under each of the following conditions :*

(1) *There is a $u \in C$ and a scalar $r \geqslant \|x\|$ such that*

$$\langle F(x), \eta(u, x)\rangle + \varphi(u) - \varphi(x) \leqslant 0 \quad \text{for all} \quad x \quad \text{with} \quad \|x\| = r.$$

(2) *For some constant $r > 0$, and for each $x \in C$ with $\|x\| = r$, there is a $u \in C$ with $\|u\| r$ and*

$$\langle F(x), \eta(u, x)\rangle + \varphi(u) - \varphi(x) \leqslant 0.$$

(3) *There exists a nonempty, compact and convex subset K of C such that for every $x \in C \setminus K$, there exists a $u \in C$ such that*

$$\langle F(x), \eta(u, x)\rangle + \varphi(u) - \varphi(x) < 0.$$

We also have the following theorem for a unique solution to the generalised variational-like inequality problem P_1

Theorem 2. *Let C be a closed and convex set and $F : C \to \mathbb{R}^n$, $\eta : C \times C \to \mathbb{R}^n$, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$. If F is strictly η-monotone over C, then there exists a unique solution to P_1.*

Proof: If $\bar{x}$ and $\bar{z}$ are two distinct solutions to P_1, then we have

$$(\forall x \in C)\ \langle F(\bar{x}), \eta(x, \bar{z})\rangle + \varphi(x) - \varphi(\bar{x}) \geqslant 0.$$
$$(\forall x \in C)\ \langle F(\bar{z}), \eta(x, \bar{x})\rangle + \varphi(x) - \varphi(\bar{z}) \geqslant 0.$$

Setting $x = \overline{z}$ in the first inequality, and $x = \overline{x}$ in the second and adding the two, we obtain

$$\langle F(\overline{x}), \eta(\overline{z}, \overline{x})\rangle \ + F\langle(\overline{z}), \eta(\overline{x}, \overline{z})\rangle \ \geqslant 0.$$

Which implies that $\overline{x} = \overline{z}$ by the strict η-monotonicity of F. ∎

IV. Generalised Quasivariational-like Inequalities

Theorem 3. *Let C be a compact and convex set in $\mathbb{R}^n$, suppose that*

(1) *$E : C \to 2^C$ is u.s.c. such that for each $x \in C$, $E(x)$ is a compact convex subset of C.*

(2) *$\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous and convex function.*

(3) *$F : C \to \mathbb{R}^n$ and $\eta : C \times C \to \mathbb{R}^n$ are such that Condition 1 is satisfies.*

Then Problem P_2 has a solution

Proof: For each $x \in C$ define

$$S(x) = \{s \in E(x) : \langle F(x), \eta(s, x)\rangle + \varphi(x)\} = \inf_{v \in E(x)} \{\langle F(x), \eta(v, x)\rangle + \varphi(v)\}.$$

Since $E(x)$ is a compact convex set and we have the conditions of the theorem, by Lemma 1, this implies that $S(x)$ is a nonempty, convex and closed subset of $E(x) \subset C$. Since E is u.s.c. and the map $s \to \langle F(x), \eta(s, x)\rangle \ + \varphi(s)$ is lower semi-continuous on C, we conclude that the graph of S is closed in $C \times C$. Therefore, by Lemma 2, S is u.s.c., too. Since $S(x) \subset E(x) \subset C$ for each $x \in C$, by Kakutani's fixed point theorem [**13**], there exists $\overline{x} \in C$ such that $\overline{x} \in S(\overline{x})$, that is $\overline{x} \in E(\overline{x})$ and

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle \ + \varphi(x) \geqslant \ \langle F(\overline{x}), \eta(\overline{x}, \overline{x})\rangle \ + \varphi(\overline{x}).$$

By Condition 1, we obtain

$$\langle F(\overline{x}), \eta(x, \overline{x})\rangle \ + \varphi(x) - \varphi(\overline{x}) \geqslant 0 \quad \text{for all} \quad x \in E(\overline{x}).$$

This completes the proof. ∎

The result below is an extension of Theorem 3 to noncompact sets, but with the assumption that the multivalued map E is condensing.

Theorem 4. *Let C be a closed convex subset $\subset \mathbb{R}^n$, and α be a generalised measure of noncompactness on $2^{\mathbb{R}^n}$. Suppose*

(1) *$E : C \to 2^C$ is a condensing upper semicontinuous map such that $E(x)$ is a compact convex subset in C for each $x \in C$;*

(2) *$\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function;*

(3) *F and η are such that Condition 1 is satisfied.*

Then P_2 has a solution.

PROOF: Let v be a point in C and define

$$\mathcal{M} = \{D \subset C : D \neq \emptyset, \quad \text{D closed, convex and containing } v \text{ and } E(D) \subset D\}.$$

Since $C \in \mathcal{M}$, we have $\mathcal{M} \neq \emptyset$. For each $D \in \mathcal{M}$ set

$$g(D) = \overline{co}\,(E(D) \cup \{v\}).$$

Therefore, $g(D)$ is a closed convex subset, which contains v in D. We have

$$E(g(D)) \subset E(D) \subset \overline{co}\,(E(D) \cup \{v\}) = g(D).$$

Thus $g(D) \in \mathcal{M}$. Now we define a partial ordering ($\leqslant$) on $\mathcal{M}$ as follows : $D_1 \leqslant D_2$ if $D_1 \subseteq D_2$. Then $\mathcal{M}$ give us a partially ordered set. Let $\{D_\nu\}$ be a chain net in $\mathcal{M}$. Setting $D = \cap D_\nu$, we have $v \in D$ (since $v \in D_\nu$ for each ν). Hence it is easy to see that D is a closed convex subset and $E(D) \subset D$, that is $D \in \mathcal{M}$. Then Zorn's lemma gives us a minimal element D_0 of $\mathcal{M}$. From the above proof, $g(D_0)$ is also in $\mathcal{M}$ with $g(D_0) = \overline{co}\,(E(D_0) \cup \{v\})$. Thus, we get $g(D_0) = D_0$. In view of the definition for a measure of noncompactness, we have

$$\begin{aligned}\alpha(D_0) = \alpha(\overline{co}\,E(D) \cup \{v\}) &= \max\{\alpha(E(D_0)), \alpha\{v\}\} \\ &\leqslant \alpha(E(D_0)).\end{aligned}$$

On the other hand, by the definition of condensing map, if $\alpha(D_0) > 0$, then $\alpha(E(D_0)) < \alpha(D_0)$; we thus arrive at a contradiction. Consequently, $\alpha(D_0) = 0$ and hence this implies that D_0 is compact. We consider problem P_2 with all maps E, F, φ, η on D_0. Notice that E, F, φ, η satisfy all the conditions given in Theorem 3. Therefore, the generalised quasi-variation-like inequality problem P_2 has a solution on D_0 and it is also a solution on D. This completes the proof of the theorem. ∎

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