Fourth Meeting, February 8th, 1889.

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On a Proposition in Statics.

By Professor C. NIVEN.

It is known that a force acting along any line in space may be resolved into six components. In the most commonly employed resolution these are forces along three lines at right angles, and couples round these lines. But the six components may be taken to be forces along the six edges of a tetrahedron. It is the object of what follows to determine these forces.

Let the given force be F, and let the line in which it acts be the intersection of the planes

$$\left.\begin{array}{l} Ax + By + Cz + Du = 0 \\ A'x + B'y + C'z + D'u = 0 \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

x, y, z, u being respectively the forces OBC, OCA, OAB, ABC, (fig. 4) of the tetrahedron OABC (in quadriplanar co-ordinates).

Let the forces in OA, OB, OC, BC, CA, AB be x, y, z, P, Q, R, and let the perpendiculars on BC, CA, AB and the line of action of F be p, q, r, f; let also equal and opposite forces $\pm$P, $\pm$Q, $\pm$R, $\pm$F be applied at O parallel to their former directions. The force F at O is equivalent to the forces x, y, z, P, Q, R at O, and the couple Ff is the resultant of the three couples Pp, Qq, Rr acting in the planes OBC, OCA, OAB.

The plane in which Ff acts is given, from (1), by drawing the plane through O and the line F; its equation is

$$(AD' - A'D)x + (BD' - B'D)y + (CD' - C'D)z = 0 \quad \ldots \quad \ldots \quad (2).$$

Let us draw a parallel plane A′B′C′ to this; its equation will be

$$(AD' - A'D)x + (BD' - B'D)y + (CD' - C'D)z = E \quad \ldots \quad \ldots \quad (3).$$

Now, couples in the faces of the tetrahedron, or in planes parallel to these faces, will balance each other if their moments are proportional to the areas in which they act. This may be easily proved by going round each face in the right-handed direction, as viewed from the outside, and placing forces in each side proportional to it. All these forces mutually cancel, but those in any one face are equivalent to a couple proportional to the area of the force, thus, for example, the forces along the sides of the triangle A′B′C′ (fig. 5) make up a couple, whose moment is $2\mu.\Delta A'B'C'$.

Returning to our original figure (fig. 4), we find that

$$Pp : Ff = \Delta OB'C' : \Delta A'B'C' \ldots \quad \ldots \quad \ldots \quad (4).$$

But the areas of the faces of a tetrahedron are inversely proportional to the perpendiculars on them from the opposite vertices.

To find these, let three lines at right angles be chosen, as axes of $\xi\ \eta\ \zeta$, and let the faces of the tetrahedron OABC referred to these axes be

$$\left.\begin{aligned} x &\equiv \xi\cos\alpha_1 + \eta\cos\beta_1 + \zeta\cos\gamma_1 = 0 \\ y &\equiv \xi\cos\alpha_2 + \eta\cos\beta_2 + \zeta\cos\gamma_2 = 0 \\ z &\equiv \xi\cos\alpha_3 + \eta\cos\beta_3 + \zeta\cos\gamma_3 = 0 \\ u &\equiv \xi\cos\alpha_4 + \eta\cos\beta_4 + \zeta\cos\gamma_4 - \pi = 0 \end{aligned}\right\} \ldots \quad \ldots \quad (5).$$

Equation (3) now takes the form

$[(AD' - A'D)\cos\alpha_1 + (BD' - B'D)\cos\alpha_2 + (CD' - C'D)\cos\alpha_3]\xi$ + two similar terms $= E$,

and the perpendicular from O on $A'B'C' = E/Q$

where $Q^2 = (AD' - A'D)^2 + (BD' - B'D)^2 + (CD' - C'D)^2$

$- 2(BD' - B'D)(CD' - C'D)\cos yz$ – two similar terms $\ldots$ (6).

The perpendicular from A′ on OB′C′ is $E/(AD' - A'D)$, hence from (4) and the remark following it

$$Pp = \frac{AD' - A'D}{Q}.\ Ff \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7).$$

It remains to determine f the perpendicular from O on F. Let f_1 and f_2 be the perpendiculars on the two planes (1), the angle between these planes being θ.

From fig. 6 we have

$$f_1 = f\sin\phi, \quad f_2 = f\sin(\theta - \phi)$$

whence $\qquad f^2\sin^2\theta = f_1^2 + f_2^2 + 2f_1f_2\cos\theta$

$$\cos\theta = \frac{AA' + BB' + CC' + DD' - (AB' + A'B)\cos xy - \ldots}{\Delta_1\ .\ \Delta_2}$$

where $\qquad \Delta_1^2 = A^2 + B^2 + C^2 + D^2 - 2AB\cos xy - \ldots$

$\qquad\qquad \Delta_2^2 = A'^2 + \ldots - 2A'B'\cos xy - \ldots$

whence $\qquad \Delta_1^2\,\Delta_2^2\sin^2\theta = \Delta^2$

where $\Delta^2 = \Sigma\{T_{xy,xy}\sin^2 xy + 2T_{xy,xz}\cos xy \cos xz - 2T_{xy,zu}\cos xy \cos zu - 2T_{xy}\cos xy\}$

wherein
$$T_{xy,xy} = (AB' - A'B)^2$$
$$T_{xy,xz} = (AB' - A'B)(AC' - A'C)$$
$$T_{xy,zu} = (AC' - A'C)(BD' - B'D) + (AD' - A'D)(BC' - B'C)$$
$$T_{xy} = (AC' - A'C)(BC' - B'C) + (AD' - A'D)(BD' - B'D) \quad (8).$$

Now $\qquad f_1 = D\pi/\Delta_1$ and $f_2 = D'\pi/\Delta_2$

whence, on reduction, we find $f^2\Delta^2 = Q^2\pi^2$.

Substituting in (7), and remembering that $\sin xu = \pi/p$, we obtain finally

$$P = \frac{AD' - A'D}{\Delta}\sin xu . F \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9)$$

Δ having the form given in (8).

Kötters synthetic geometry of algebraic curves—Part II., involutions of the second and higher order.

[See Index.]

Amsler's Planimeter.

By Professor STEGGALL.

There are many proofs of the principle of this planimeter, but all that are accessible to me seem a little beyond the grasp of many students who use the instrument. It seems worth while, therefore, to notice the following proof, which, to the best of my knowledge, is new.

Let A (fig. 7) be a pivot, round which the pivoted rods AB, BC rotate, and let AB′, B′C′ be a consecutive position of the rods, when the point C has traced out the arc CC′ of the curve whose area is required; and let us first suppose that this curve does not include the point A. The element of area ABCC′B′ consists of three parts, namely :—

(1) The triangle ABB′.

(2) The parallelogram BCC″B′, where B′C″ is equal and parallel to BC.

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