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ON SOME TRACTABLE GROWTH-COLLAPSE PROCESSES WITH RENEWAL COLLAPSE EPOCHS

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ON SOME TRACTABLE GROWTH-COLLAPSE PROCESSES WITH RENEWAL COLLAPSE EPOCHS

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Abstract

In this paper we generalize existing results for the steady-state distribution of growth-collapse processes with independent exponential intercollapse times to the case where they have a general distribution on the positive real line having a finite mean. In order to compute the moments of the stationary distribution, no further assumptions are needed. However, in order to compute the stationary distribution, the price that we are required to pay is the restriction of the collapse ratio distribution from a general distribution concentrated on the unit interval to minus-log-phase-type distributions. A random variable has such a distribution if the negative of its natural logarithm has a phase-type distribution. Thus, this family of distributions is dense in the family of all distributions concentrated on the unit interval. The approach is to first study a certain Markov-modulated shot noise process from which the steady-state distribution for the related growth-collapse model can be inferred via level crossing arguments.

Keywords: Growth collapse; shot noise process; minus-log-phase-type distribution; Markov modulated

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1. Introduction

Consider a growth-collapse process that grows linearly at some given rate c. The collapses occur at renewal instants with interrenewal time distribution function F with mean μ and Laplace–Stieltjes transform (LST) G. The remaining level (e.g. funds) after a given collapse form a random fraction of the level just before the collapse occurred. It is assumed that the sequence of random proportions $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.), and independent of the underlying renewal process. Of course, since these are proportions, it is naturally assumed that $\mathrm{P}[0 \le X_1 \le 1] = 1$ and, to avoid trivialities, that $0 < \mathrm{E}\, X_1 < 1$. From, e.g. [13], it is known that this process is stable without any further conditions. Our aim is to identify a relatively broad family of distributions of X_1, which is dense in the family of all distributions on [0, 1], for which the stationary distribution of this process can be calculated.

The idea is to first consider an on/off process, where during *on* times the process increases linearly at rate c and during *off* times, whenever the process is at level x, it decreases at the rate rx for some $r > 0$. As assumed, on times have some general distribution F, while off times, denoted by $P_1, P_2, \ldots$, will be assumed to have a phase-type distribution. If we restrict the process to off times then what we obtain is a shot-noise-type process with upward jumps having distribution $F(\cdot/c)$ with mean $c\mu$, LST $G(c\alpha)$, and interarrival times which have a phase-type distribution.

Given that at the beginning of an off time the level is given by some x, as long as the period does not end, the dynamics of the process are given via

$$W(t) = x - r \int_0^t W(s)\,\mathrm{d}s, \tag{1.1}$$

where t is the time that has elapsed since the beginning of this period. Hence, as is well known, it follows that $W(t) = x\mathrm{e}^{-rt}$ for $0 \le t \le P_i$ for some index i. Thus, at the end of this period the level will be $x\mathrm{e}^{-rP_i}$. Setting $X_i = \mathrm{e}^{-rP_i}$ we see that when we restrict our process to on times, then it becomes the type of process that is described in the first paragraph. For example, when the P_i are exponential with rate r, then the X_i are Uniform$(0, 1)$. If the P_i have an Erlang distribution then the X_i are products of uniform random variables. For a general phase-type distribution, the X_i are (possibly infinite) mixtures of products of uniformly (on $(0, 1)$) distributed random variables. Since phase-type distributions are dense in the family of all distributions on $[0, \infty)$, then the family of distributions of the collapse ratio is dense in the family of all distributions on $[0, 1]$.

For the process at hand, it follows from [11] that if f_0 is the stationary density for the process restricted to on times and f_1 is the stationary density for the process restricted to off times, provided that they exist (note that when starting from a positive state, the process as described never hits 0), then $cpf_0(x) = rx(1-p)f_1(x)$, where $p = \mu/(\mu + \mathrm{E}\,P_1)$ is the fraction of on times. Thus, studying the stationary distribution of the shot-noise-type model is equivalent to studying the stationary distribution of the growth-collapse model. Also, we note that the growth-collapse model with growth rate 1 and intercollapse times with LST $G(c\alpha)$ has the same stationary distribution as the model initially proposed (with growth rate c) and, thus, we will, without loss of generality, assume that $c = 1$ from now on in order to simplify the notation.

The paper is organized as follows. Regarding shot noise, in Section 2 we actually study a more general model and then restrict to the special case of the model proposed in this introduction. In Section 3 we relate the moments of the shot noise process to the moments of the original growth-collapse process. In Section 4 we study the steady-state behavior of the growth-collapse process immediately after a collapse. Several distributions for the intercollapse times and the collapse proportions are considered.

As shown in [13], for quite general growth-collapse processes, there is a direct relationship between the time stationary distribution, the stationary distribution of the process immediately after collapses, and the stationary distribution of the process immediately before collapses. In particular, for the i.i.d. case, knowledge of one gives knowledge of the other two. For the general minus-log-phase-type collapse ratios, we found it more accessible to study the time stationary version, while, for the models of Section 4, it was more natural and easier to study the discrete-time process immediately after collapses. For some earlier studies of growth-collapse processes and their applications, see [2], [6], [8], [14], [16], [17], [18], and the references therein.

2. Shot-noise-type processes with phase-type interarrival times

In [3] a shot-noise-type process with Markov-modulated release rate was considered. Kella and Stadje [15] studied a more general model where the input is a Markov additive process (MAP) and the release rate is Markov modulated as well. In the latter paper, the MAP is not the most general possible. In particular, it did not include the additional jumps that can occur at state changes of the underlying Markov chain. This additional aspect, which we very much need here, can be included by applying a technique from [4]. We will first write some results regarding the most general setup, that is, the one-dimensional version of [15] but with the possibility of additional jumps at state change epochs. We will then specialize to the case which we need to solve the problem of this paper. Thus, let $(X, J) = \{(X(t), J(t)) \mid t \geq 0\}$ be a nondecreasing MAP (see [4]) with exponent matrix $F(\alpha) = Q \circ G(\alpha) + \mathrm{diag}(\varphi_1(\alpha), \dots, \varphi_K(\alpha))$, where $Q \circ G(\alpha) \equiv (q_{ij} G_{ij}(\alpha))$, J is an irreducible, finite state space, continuous-time Markov chain with states $1, \dots, K$, rate transition matrix $Q = (q_{ij})$, and stationary probability vector $\pi = \pi_i$, and $G_{ij}(\alpha)$ is the LST of the distribution of the (nonnegative) jump occurring when the Markov chain J changes state from i to j with $G_{ii}(\alpha) \equiv 1$ for all $\alpha \geq 0$ (LST of the constant 0). The Laplace exponent of a nondecreasing Lévy process is of the form

$$\varphi_i(\alpha) = -c_i \alpha - \int_{(0,\infty)} (1 - \mathrm{e}^{-\alpha x}) \,\mathrm{d}\nu_i(x),$$

where ν_i is a Lévy measure satisfying $\int_{(0,\infty)} \min(x, 1) \,\mathrm{d}\nu(x) < \infty$. Moreover, we assume that $\mu(i, j) \equiv -G'_{ij}(0) < \infty$ and $\rho(i) \equiv -\varphi'_i(0) = c_i + \int_{(0,\infty)} x \,\mathrm{d}\nu_i(x) < \infty$ for all i and j.

As in [4], we recall that the process X behaves like a nondecreasing Lévy process (subordinator) with exponent $\varphi_i(\cdot)$ when J is in state i and when J switches from state i to a different state j. Moreover, X jumps up by an independent amount which has a distribution with LST $G_{ij}(\cdot)$.

Now consider the following Markov-modulated linear dam process:

$$W(t) = W(0) + X(t) - \int_0^t r(J(s)) W(s) \,\mathrm{d}s. \tag{2.1}$$

Here the input is the process X and the output rate is proportional to the content of the dam, where the proportion $r(J(s))$, with $r(i) \geq 0$ for all i, is modulated by the Markov process J. Then we have the following result.

Theorem 2.1. *Suppose, in addition to the irreducibility of J and the assumptions that $\rho_i < \infty$ and $\mu(i, j) < \infty$ for all i, j (see above), that there is at least one i for which $r(i) > 0$. Then a unique stationary distribution for the joint (Markov) process (W, J) exists, and it is also the limiting distribution, which is independent of initial conditions.*

Before we prove this result, let us first show the following result concerning an alternating renewal process.

Lemma 2.1. *Let $\{(X_n, Y_n) \mid n \geq 1\}$ be independent pairs of nonnegative random variables which are identically distributed for $n \geq 2$, let $\mathrm{P}[Y_2 > 0] > 0$, and let $\mathrm{E}\, X_1, \mathrm{E}\, X_2 < \infty$. Set $S_0 = 0$, $S_n = \sum_{i=1}^n (X_i + Y_i)$ for $n \geq 1$, and*

$$I(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{n=0}^\infty [S_n, S_n + X_{n+1}), \\ 1 & \text{if } t \in \bigcup_{n=0}^\infty [S_n + X_{n+1}, S_{n+1}), \end{cases}$$

$$Z(t) = \int_0^t I(s) \,\mathrm{d}s.$$

Then, for any positive constant r, $\mathrm{E} \int_0^\infty \mathrm{e}^{-rZ(t)} \,\mathrm{d}t < \infty$.

Proof. For simplicity, we prove this for the case where (X_1, Y_1) has the same distribution as the rest of the sequence. The generalization to the case where the first pair has a different distribution is trivial. Since

$$\int_{[S_n, S_{n+1})} \mathrm{e}^{-rZ(t)} \,\mathrm{d}t = \mathrm{e}^{-rZ(S_n)} (X_{n+1} + r^{-1}(1 - \mathrm{e}^{-rY_{n+1}}))$$

and $Z(S_n) = S_n^y$, where $S_0^y = 0$ and $S_n^y = \sum_{i=1}^n Y_i$ for $n \geq 1$, it follows that

$$\int_0^\infty \mathrm{e}^{-rZ(t)}\,\mathrm{d}t = \sum_{n=0}^\infty \mathrm{e}^{-rS_n^y}(X_{n+1} + r^{-1}(1-\mathrm{e}^{-rY_{n+1}})) = r^{-1} + \sum_{n=0}^\infty \mathrm{e}^{-rS_n^y}X_{n+1}.$$

Thus, as $\mathrm{E}\,\mathrm{e}^{-rY_1} < 1$ (since $\mathrm{P}[Y_1 > 0] > 0$), it follows that

$$\mathrm{E}\int_0^\infty \mathrm{e}^{-rZ(t)}\,\mathrm{d}t = r^{-1} + \frac{\mathrm{E}\,X_1}{1-\mathrm{E}\,\mathrm{e}^{-rY_1}} < \infty,$$

as required. This completes the proof.

We note that in Lemma 2.1, the off times ($Z(t) = 0$) must have a finite mean, while, for $n \geq 2$, the on times ($Z(t) = 1$) cannot be almost surely (a.s.) 0. These are the minimal assumptions in the sense that if one of them fails to hold then $\mathrm{E}\int_0^\infty \mathrm{e}^{-rZ(t)}\,\mathrm{d}t = \infty$. We note that it is possible that X_1, X_2, or Y_1 is a.s. 0.

Proof of Theorem 2.1. From (2.1) and Theorem 1 of [16], it follows that

$$W(t) = W(0)\exp\biggl[-\int_0^t r(J(s))\,\mathrm{d}s\biggr] + \int_{(0,t]}\exp\biggl[-\int_u^t r(J(s))\,\mathrm{d}s\biggr]\mathrm{d}X(u),$$

and, thus, if we start the system with two different initial conditions $W^1(0)$ and $W^2(0)$, then

$$W^1(t) - W^2(t) = (W^1(0) - W^2(0))\exp\biggl[-\int_0^t r(J(s))\,\mathrm{d}s\biggr]. \tag{2.2}$$

Since there is at least one i for which $r(i) > 0$ and J is irreducible, it follows that, a.s., $\int_0^\infty r(J(s))\,\mathrm{d}s = \infty$, so the right-hand side of (2.2) converges a.s. to 0 as $t \to \infty$. Thus, if there is a limiting distribution for $(W(t), J(t))$, it does not depend on $W(0)$. It is standard that J can be coupled with its stationary version after an a.s. finite time. Since the value of W at this coupling time has no effect on the limiting distribution if it exists (for the same reasons as just explained for the initial conditions), we may assume without loss of generality that J is stationary. For this case, the two-dimensional process $\{(\int_0^t r(J(s))\,\mathrm{d}s, X_t) \mid t \geq 0\}$ has stationary increments in the strong sense that the distribution of $\{(\int_u^{t+u} r(J(s))\,\mathrm{d}s, X_{t+u} - X_u) \mid t \geq 0\}$ is independent of u. Thus, we can extend this process together with J to be a double-sided process having these properties. To complete the proof, it follows from Theorem 2 of [16] that it remains to show that, a.s.,

$$\int_{(-\infty,0]}\exp\biggl[-\int_u^0 r(J(s))\,\mathrm{d}s\biggr]\mathrm{d}X(u) < \infty.$$

We will in fact show that

$$\mathrm{E}\int_{(-\infty,0]}\exp\biggl[-\int_u^0 r(J(s))\,\mathrm{d}s\biggr]\mathrm{d}X(u) < \infty.$$

Defining $\bar{J}(t) = J(-t)$ and $\bar{X}(t) = -X(-t)$ for $t \geq 0$, we find that $\{(\bar{X}(t), \bar{J}(t)) \mid t \geq 0\}$ is also a MAP where $\bar{J}$ is stationary with transition rates $\bar{q}_{ij} = \pi_j q_{ji}/\pi_i$, $\bar{G}_{ij} = G_{ji}$, and $\bar{\varphi}_i = \varphi_i$. By the method of uniformization, let $\{\bar{N}(t) \mid t \geq 0\}$ be a Poisson process with some (finite) rate $\lambda \geq \max_i(-\bar{q}_{ii}) = \max_i(-q_{ii})$, in which arrival epochs we embed a (stationary) discrete-time Markov chain $\{\bar{J}_n \mid n \geq 0\}$ with transition probabilities

$$\bar{p}_{ij} = \begin{cases} \dfrac{\bar{q}_{ij}}{\lambda}, & i \neq j, \\[2ex] 1 + \dfrac{\bar{q}_{ii}}{\lambda}, & i = j. \end{cases}$$

Now, by conditioning on $\bar{J}$ we have

$$\begin{aligned} &\mathrm{E} \int_{(-\infty,0]} \exp\biggl[-\int_u^0 r(J(s))\,\mathrm{d}s\biggr]\,\mathrm{d}X(u) \\ &\quad = \mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^u r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}\bar{X}(u) \\ &\quad = \mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^u r(\bar{J}(s))\,\mathrm{d}s\biggr]\rho(J(t))\,\mathrm{d}t \\ &\qquad + \mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^t r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}\sum_{n=1}^{\bar{N}(t)} \mu(\bar{J}_{n-1}, \bar{J}_n), \end{aligned} \tag{2.3}$$

where we recall that $\rho(i) = -\varphi_i'(0) < \infty$ and $\mu(i,j) = -G_{ij}'(0) < \infty$. Defining $\bar{\rho} = \max_i \rho(i)$ and $\bar{\mu} = \max_{ij} \mu(i,j)$, we find that the right-hand side of (2.3) is bounded above by

$$\bar{\rho}\,\mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^t r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}t + \bar{\mu}\,\mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^t r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}\bar{N}(t). \tag{2.4}$$

Since $\{\bar{N}(t) - \lambda t \mid t \geq 0\}$ is a zero-mean right-continuous martingale and $\exp[-\int_0^t (\bar{J}(s))\,\mathrm{d}s]$ is adapted, continuous, and bounded, it follows that (2.4) is equal to

$$(\bar{\rho} + \lambda\bar{\mu})\,\mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^t r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}t.$$

For any i such that $r(i) > 0$,

$$\mathrm{E} \int_{[0,\infty)} \exp\biggl[-\int_0^t r(\bar{J}(s))\,\mathrm{d}s\biggr]\,\mathrm{d}t \leq \mathrm{E} \int_{[0,\infty)} \exp\biggl[-r(i)\int_0^t \mathbf{1}_{\{\bar{J}(s)=i\}}\,\mathrm{d}s\biggr]\,\mathrm{d}t, \tag{2.5}$$

and the right-hand side is finite by the irreducibility, and, hence, the positive recurrence, of $\bar{J}$ (due to that of J) and Lemma 2.1. This completes the proof.

From [4], the following is a zero-mean martingale:

$$\begin{aligned} &\int_0^t \exp[-\alpha W(s)]\,\mathbf{1}_{J(s)}\,\mathrm{d}s\,F(\alpha) + \exp[-\alpha W(0)]\,\mathbf{1}_{J(0)} - \exp[-\alpha W(t)]\,\mathbf{1}_{J(t)} \\ &\quad + \alpha \int_0^t \exp[-\alpha W(s)]\,\mathbf{1}_{J(s)}\,r(J(s))W(s)\,\mathrm{d}s. \end{aligned} \tag{2.6}$$

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Thus, if (W^*, J^*) has the stationary distribution associated with the process (W, J) then, from (2.6), it follows that

$$\mathrm{E}\, \mathrm{e}^{-\alpha W^*} \mathbf{1}_{J^*} F(\alpha) = \alpha \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathrm{E}\, \mathrm{e}^{-\alpha W^*} \mathbf{1}_{J^*} r(J^*).$$

Thus, with $w_i(\alpha) = \mathrm{E}\, \mathrm{e}^{-\alpha W^*} \mathbf{1}_{\{J^*=i\}}$, $w(\alpha) = (w_i(\alpha))$, and $D_r = \mathrm{diag}(r(1), \dots, r(K))$, we have

$$w(\alpha)^\top F(\alpha) = \alpha w'(\alpha)^\top D_r, \tag{2.7}$$

where $\pi_i = w_i(0)$ is the stationary distribution for the Markov chain J, summing to 1 and satisfying $\pi^\top Q = 0$. We do not expect to be able to solve (2.7) analytically. Nevertheless, it can immediately be deduced from this equation by differentiation that

$$\sum_{k=0}^{n} \binom{n}{k} w^{(k)}(0)^\top F^{(n-k)}(0) = n w^{(n)}(0)^\top D_r,$$

and since $F^{(0)}(0) = F(0) = Q$, we have

$$\sum_{k=0}^{n-1} \binom{n}{k} w^{(k)}(0)^\top F^{(n-k)}(0) = w^{(n)}(0)^\top (n D_r - Q).$$

Thus, when $nD_r - Q$ is invertible, we have a recursion formula that computes the moments $\mathrm{E}(W^*)^n \mathbf{1}_{\{J^*=i\}}$.

Specializing to the problem we set out to solve, assume that instead of K states for the modulating Markov chain there are $K + 1$ states, indexed by $0, \dots, K$. We will first consider the modulated process with $r(0) = 0$, $r(1) = \cdots = r(K) = 1$, $G_{i0}(\alpha) = G(\alpha)$, and, for all other i, j, $G_{ij}(\alpha) = 1$. Finally, we assume that, other than the jump that occurs when entering state 0 and the specified rates, nothing happens. That is, $\varphi_0(\alpha) = \cdots = \varphi_K(\alpha) = 0$. Thus, we see that if we restrict the process to the intervals where the modulating Markov chain is in states $1, \dots, K$ then we have a shot noise process with phase-type interarrival times and general i.i.d. jumps.

It is easy to check that (2.7) becomes

$$\sum_{i=0}^{K} w_i(\alpha) q_{ij} = \alpha w_j'(\alpha)$$

for $j \neq 0$ and, for $j = 0$, we have

$$-q_0 w_0(\alpha) + G(\alpha) \sum_{i=1}^{K} w_i(\alpha) q_{i0} = 0,$$

where $q_0 = -q_{00}$. By substitution we thus have, for $j \neq 0$,

$$\sum_{i=1}^{K} w_i(\alpha) \left(q_{ij} + \frac{q_{i0} q_{0j}}{q_0} G(\alpha) \right) = \alpha w_j'(\alpha) \tag{2.8}$$

with initial conditions $w_i(0) = \pi_i$, where π is the stationary distribution for the modulating Markov chain. Denote by $\mathbf{1}$ a K-vector of 1s, let $S = (s_{ij})$ with $s_{ij} = q_{ij}$ for $1 \le i, j \le K$, let

$\beta_j = q_{0j}/q_0$, and note that $q_{i0} = -\sum_{j=1}^K s_{ij}$. Then the underlying phase-type distribution of P_n defined earlier is

$$\mathrm{P}[P_n \le x] = 1 - \beta^\top \mathrm{e}^{-Sx}\mathbf{1}.$$

Thus, in matrix notation (2.8) becomes, with $\hat{w}(\alpha) = (w_i(\alpha))_{1\le i\le K}$ and I the identity matrix,

$$(I + \beta\mathbf{1}^\top G(\alpha))S^\top \hat{w}(\alpha) = \alpha\hat{w}'(\alpha).$$

Therefore, the stationary LST for the shot noise process with phase-type interarrival times and jumps with distribution having LST G is given by

$$w(\alpha) = \frac{\sum_{i=1}^K w_i(\alpha)}{1-\pi_0} = \frac{\mathbf{1}^\top\hat{w}(\alpha)}{1-\pi_0}.$$

It is easy to check that

$$\sum_{j=1}^K\left(q_{ij} + \frac{q_{i0}q_{0j}}{q_0}\right) = 0 \tag{2.9}$$

and that

$$\sum_{j=1}^K \pi_i\left(q_{ij} + \frac{q_{i0}q_{0j}}{q_0}\right) = 0; \tag{2.10}$$

thus, (2.8) can also be written as

$$\mu\frac{q_{0j}}{q_0}G_{\mathrm{e}}(\alpha)\sum_{i=1}^K w_i(\alpha)q_{i0} = -\sum_{i=1}^K \frac{w_i(0)-w_i(\alpha)}{\alpha}\left(q_{ij} + \frac{q_{i0}q_{0j}}{q_0}\right) - w_j'(\alpha),$$

where $G_{\mathrm{e}}(\alpha) = (1-G(\alpha))/\alpha\mu$ is the stationary residual lifetime LST associated with G. If we similarly define

$$w_{\mathrm{e},i}(\alpha) = \frac{w_i(0)-w_i(\alpha)}{-w_i'(0)\alpha} = \frac{1-\mathrm{E}[\mathrm{e}^{-\alpha W^*} \mid J^* = i]}{\mathrm{E}[W^* \mid J^* = i]\alpha}$$

and let $\mu_{n,i}^w = \mathrm{E}[(W^*)^n \mid J^* = i]$, then

$$\mu\frac{q_{0j}}{q_0}G_{\mathrm{e}}(\alpha)\sum_{i=1}^K w_i(\alpha)q_{i0} = -\sum_{i=1}^K \pi_i\mu_{1,i}^w w_{\mathrm{e},i}(\alpha)\left(q_{ij} + \frac{q_{i0}q_{0j}}{q_0}\right) - w_j'(\alpha).$$

In particular, letting $\alpha \downarrow 0$,

$$\begin{aligned}
\mu\pi_0 q_{0j} &= \mu\frac{q_{0j}}{q_0}\sum_{i=1}^K \pi_i q_{i0} \\
&= -\sum_{i=1}^K \pi_i\mu_{1,i}^w\left(q_{ij} + \frac{q_{i0}q_{0j}}{q_0}\right) + \pi_j\mu_{1,j}^w \\
&= \sum_{i=1}^K \pi_i\mu_{1,i}^w(\delta_{ij} - \tilde{q}_{ij}),
\end{aligned} \tag{2.11}$$

where $\tilde{q}_{ij} = q_{ij} + q_{i0}q_{0j}/q_0$. It follows from (2.9) and (2.10) that $\tilde{Q} = (\tilde{q}_{ij})_{1\le i,j\le K}$ is a rate transition matrix with stationary distribution $\pi_i/(1-\pi_0)$ for $i = 1, \ldots, K$.

The following is a straightforward exercise, but we include it for ease of reference.

Lemma 2.2. *If P is a stochastic matrix, D_1 and D_2 are nonnegative diagonal matrices, and $D_1 + D_2$ has a strictly positive diagonal, then $D_1 - (D_2(P - I))$ is nonsingular.*

Proof. Note that $D_1 - (D_2(P - I)) = (D_1 + D_2)(I - (D_1 + D_2)^{-1} D_2 P)$ and, thus, it suffices to show that, with $A = ((D_1 + D_2)^{-1} D_2 P)$, $A^n \to 0$ as $n \to \infty$, since then $(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$. To show this, we note that $(D_1 + D_2)^{-1} D_2$ is a nonnegative diagonal matrix where the diagonal entries are all strictly less than 1. Thus, if we let d denote the maximum of these entries then $A^n \le d^n P^n$ and, since the entries of P^n are bounded, the result immediately follows.

Since a rate transition matrix of a finite state space, continuous-time Markov chain is of the form $D(P - I)$ for some nonnegative diagonal matrix D and some stochastic matrix P, it follows from Lemma 2.2 that $I - \tilde{Q}$ is nonsingular and, thus, (2.11) has a unique solution for the unknowns $\mu_{1,i}^w$. Denoting by μ_n the nth moment with respect to the jump distribution F (with LST G), then, since $\mu G_{\mathrm{e}}^{(n-1)}(0) = (-1)^{n-1} \mu_n / n$ and, similarly,

$$\mu_{1,i}^w w_{\mathrm{e},i}^{(n-1)}(0) = (-1)^{n-1} \frac{\mu_{n,i}^w}{n},$$

it is easy to check that, after differentiating $n-1$ times and letting $\alpha \downarrow 0$, the following recursion holds:

$$\frac{q_{0j}}{q_0} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\mu_{k+1}}{k+1} \sum_{i=1}^{K} \pi_i \mu_{n-1-k,i}^w q_{i0} = \sum_{i=1}^{K} \pi_i \mu_{n,i}^w \left(\delta_{ij} - \frac{\tilde{q}_{ij}}{n} \right).$$

Upon multiplying by n, observing that

$$\binom{n-1}{k} \frac{n}{k+1} = \binom{n}{k+1},$$

and making an obvious change of variables in the first sum, we obtain

$$\frac{q_{0j}}{q_0} \sum_{k=1}^{n} \binom{n}{k} \mu_k \sum_{i=1}^{K} \pi_i \mu_{n-k,i}^w q_{i0} = \sum_{i=1}^{K} \pi_i \mu_{n,i}^w (n\delta_{ij} - \tilde{q}_{ij}).$$

Finally, denoting by $\tilde{p}_0$ a vector with coordinates $p_{0j} = q_{0j}/q_0$, we have, with $\tilde{a}^\top = \tilde{p}_0^\top (I - \tilde{Q})^{-1}$,

$$\pi_j \mu_{n,j}^w = \tilde{a}_j \sum_{k=1}^{n} \binom{n}{k} \mu_k \sum_{i=1}^{K} \pi_i \mu_{n-k,i}^w q_{i0}.$$

Thus, setting $m_n^w = \sum_{i=1}^{K} \pi_i \mu_{n,i}^w q_{i0}$ and

$$\tilde{a} = \sum_{i=1}^{K} \tilde{a}_i q_{i0} = \frac{1}{q_0} \sum_{i=1}^{K} \sum_{j=1}^{K} q_{0i} (I - \tilde{Q})_{ij}^{-1} q_{j0},$$

we have, for $n \ge 1$,

$$m_n^w = \tilde{a} \sum_{k=1}^{n} \binom{n}{k} \mu_k m_{n-k}^w, \tag{2.12}$$

so that

$$\mu_{n,j}^w = \frac{\tilde{a}_j}{\pi_j \tilde{a}} m_n^w$$

and the unconditional moment is

$$\frac{1}{1-\pi_0}\sum_{j=1}^{K}\pi_j \mu_{n,j}^w = \frac{m_n^w}{(1-\pi_0)\tilde{a}}\sum_{j=1}^{K}\tilde{a}_j. \tag{2.13}$$

From (2.12), it follows that

$$m_n^w = \frac{\tilde{a}}{1+\tilde{a}}\sum_{k=0}^{n}\binom{n}{k}\mu_k m_{n-k}^w + \frac{1}{1+\tilde{a}}\delta_{0n},$$

and, upon multiplying by $(-\alpha)^n$, dividing by $n!$, and summing, and noting that if G is uniquely defined by its moments then $G(\alpha) = \sum_{k=0}^{\infty}(-1)^k \mu_k \alpha^k / k!$, we obtain

$$m^w(\alpha) = \sum_{n=0}^{\infty}\frac{(-1)^n m_n^w}{n!}\alpha^n = \frac{1/(1+\tilde{a})}{1-\tilde{a}G(\alpha)/(1+\tilde{a})} = \sum_{k=0}^{\infty}\frac{1}{1+\tilde{a}}\left(\frac{\tilde{a}}{1+\tilde{a}}\right)^k G^k(\alpha).$$

This implies that if we let

$$H(x) = \sum_{k=0}^{\infty}\frac{1}{1+\tilde{a}}\left(\frac{\tilde{a}}{1+\tilde{a}}\right)^k F^{*k}(x)$$

then m_n^w is the nth moment with respect to H. If S_n is a sum of n i.i.d. random variables distributed like F ($S_0 = 0$) and N is an independent geometric random variable with probability of success $(1+\tilde{a})^{-1}$, counting only the number of failures, then H is the distribution of S_N. Although this seems nice, we should point out that the analysis is not complete as we have not shown that these moments define a unique distribution.

2.1. The $K = 2$ case

With $a_{ij} = q_{i0}q_{0j}/q_0$, (2.8) reduces to

$$\alpha w_1'(\alpha) = w_1(\alpha)(q_{11} + a_{11}G(\alpha)) + w_2(\alpha)(q_{21} + a_{21}G(\alpha)), \tag{2.14}$$
$$\alpha w_2'(\alpha) = w_1(\alpha)(q_{12} + a_{12}G(\alpha)) + w_2(\alpha)(q_{22} + a_{22}G(\alpha)). \tag{2.15}$$

Differentiating the second of these equations, using (2.14) for $w_1'(\alpha)$ and, finally, (2.15) to eliminate $w_1(\alpha)$, we obtain the following second-order differential equation in $w_2(\alpha)$:

$$\begin{aligned}\alpha w_2''(\alpha) = {}& w_2'(\alpha)\left(-1 + q_{22} + a_{22}G(\alpha) + q_{11} + a_{11}G(\alpha) + \frac{\alpha a_{12}G'(\alpha)}{q_{12} + a_{12}G(\alpha)}\right)\\ &+ w_2(\alpha)\left(\frac{1}{\alpha}(q_{21} + a_{21}G(\alpha))(q_{12} + a_{12}G(\alpha)) + a_{22}G'(\alpha)\right.\\ &\left. - \frac{1}{\alpha}(q_{22} + a_{22}G(\alpha))(q_{11} + a_{11}G(\alpha)) + \frac{q_{22} + a_{22}G(\alpha)}{q_{12} + a_{12}G(\alpha)}a_{12}G'(\alpha)\right).\end{aligned} \tag{2.16}$$

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Now consider the special case of $\exp(\mu)$ jumps, i.e. $G(\alpha) = \mu/(\mu+\alpha)$, and Markov transition rates $q_{12} = -q_{11} = \nu_1$, $q_{20} = -q_{22} = \nu_2$, and $q_{01} = -q_{00} = \nu_0$, i.e. the times in state i are $\exp(\nu_i)$-distributed, for $i = 0, 1, 2$. Then (2.16) reduces to

$$\alpha(\mu+\alpha)w_2''(\alpha) + (\nu_1+\nu_2+1)(\mu+\alpha)w_2'(\alpha) + \nu_1\nu_2 w_2(\alpha) = 0. \tag{2.17}$$

Setting $z = \mu + \alpha$, $a = \nu_1$, $b = \nu_2$, and $c = 0$ in Equation (15.5.1) of [1, p. 562] reduces that differential equation to (2.17). Its solution is given by the hypergeometric functions in Equations (15.5.20) and (15.5.21) of [1, p. 564].

3. Moments for the growth-collapse model

In this section we relate the nth moment of the stationary distribution of the growth-collapse model to the $(n+1)$th moment of the stationary distribution of the shot noise model. Consider the growth-collapse model with collapse ratio distribution of minus-log-phase-type. Let P denote a generic interarrival time of the corresponding shot noise process; P is phase type (cf. Section 1). An expression for $\mathrm{E}\,P$ is given via the system of equations for the t_i, which are mean interarrival times when the first phase is i, i.e.

$$t_i = \frac{1}{q_i} + \sum_{j\neq 0,i} \frac{q_{ij}}{q_i} t_j,$$

or, equivalently,

$$\sum_{j\neq 0} q_{ij} t_j = -1,$$

which has a unique solution. Then $\mathrm{E}\,P$ is a weighted average of $t_1, \ldots, t_K$, where the weights are the initial distribution of the phase-type distribution, which in our case is chosen to be q_{0i}/q_0.

From [11] we recall that, for the on/off model of Section 1, the relationship between the stationary density during on times $(f_0(\cdot))$ and that during off times $(f_1(\cdot))$ is given by $pf_0(x) = (1-p)xf_1(x)$, where $p = \mu/(\mu + \mathrm{E}\,P)$. Hence,

$$\int_0^\infty \mathrm{e}^{-\alpha x} f_0(x)\,\mathrm{d}x = \frac{\mathrm{E}\,P}{\mu}\int_0^\infty \mathrm{e}^{-\alpha x} x f_1(x)\,\mathrm{d}x = -\frac{\mathrm{E}\,P}{\mu}\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^\infty \mathrm{e}^{-\alpha x} f_1(x)\,\mathrm{d}x,$$

so that the nth moment of the stationary distribution for the growth-collapse model is given by $\mathrm{E}\,P/\mu$ times the $(n+1)$th moment (see (2.13)) for the shot noise model with phase-type interarrival times.

We also observe that in general there is a more direct way of computing the moments, as pointed out in [13]. That is, if V has the stationary distribution of the process immediately after collapses, Y has the intercollapse time distribution, and X has the collapse ratio distribution, then $V \stackrel{\mathrm{D}}{=} (V+Y)X$, where X, Y, and V are independent, and the time stationary distribution is $T \stackrel{\mathrm{D}}{=} V + Y_\mathrm{e}$, where V and Y_e are independent, and Y_e has the stationary excess time distribution of Y. From this, as in Equation (34) of [13],

$$\mathrm{E}\,V^n = \mathrm{E}\,X^n \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\,V^k\,\mathrm{E}\,Y^{n-k}.$$

It will be shown in the next section that, whenever $\mathrm{E}\,Y^n < \infty$, $\mathrm{E}\,V^n < \infty$ and, thus, recursively (with $\mathrm{E}\,V^0 = \mathrm{E}\,Y^0 = 1$),

$$\mathrm{E}\,V^n = \frac{\mathrm{E}\,X^n}{1 - \mathrm{E}\,X^n} \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{E}\,V^k\,\mathrm{E}\,Y^{n-k}, \tag{3.1}$$

and in particular $\mathrm{E}\,V = \mathrm{E}\,X\,\mathrm{E}\,Y/(1-\mathrm{E}\,X)$. Hence, if $\mathrm{E}\,Y^{n+1} < \infty$ then

$$\begin{aligned}
\mathrm{E}\,T^n &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\,V^k\,\mathrm{E}\,Y_{\mathrm{e}}^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathrm{E}\,V^k \frac{\mathrm{E}\,Y^{n-k+1}}{(n-k+1)\,\mathrm{E}\,Y} \\
&= \frac{1}{(n+1)\,\mathrm{E}\,Y} \sum_{k=0}^{n} \binom{n+1}{k} \mathrm{E}\,V^k\,\mathrm{E}\,Y^{n+1-k} \\
&= \frac{(1-\mathrm{E}\,X^{n+1})\,\mathrm{E}\,V^{n+1}}{(n+1)\,\mathrm{E}\,X^{n+1}\,\mathrm{E}\,Y} \\
&= \frac{(1-\mathrm{E}\,X^{n+1})\,\mathrm{E}\,X\,\mathrm{E}\,V^{n+1}}{\mathrm{E}\,X^{n+1}(1-\mathrm{E}\,X)(n+1)\,\mathrm{E}\,V} \\
&= \frac{(1-\mathrm{E}\,X^{n+1})\,\mathrm{E}\,X}{\mathrm{E}\,X^{n+1}(1-\mathrm{E}\,X)}\,\mathrm{E}\,V_{\mathrm{e}}^n,
\end{aligned} \tag{3.2}$$

where V_{e} has the stationary excess time distribution of V. In this particular case, since $X = \mathrm{e}^{-P}$ with $\mathrm{P}[P > x] = \beta^\top \mathrm{e}^{-Sx}\mathbf{1}$, then $\mathrm{E}\,X^\alpha$ is the LST of a phase-type distribution, which is well known and can be written as $\beta^\top(S(S+\alpha I)^{-1})\mathbf{1}$. Thus, as $I - S(S+\alpha I)^{-1} = (S+\alpha I - S)(S+\alpha I)^{-1} = \alpha(S+\alpha I)^{-1}$, we have

$$1 - \mathrm{E}\,X^n = 1 - \beta^\top(S(S+nI)^{-1})\mathbf{1} = \beta^\top(I - S(S+nI)^{-1})\mathbf{1} = n\beta^\top(S+nI)^{-1}\mathbf{1}.$$

Thus, we can use these equations together with (3.1) and (3.2) to compute the moments. Algorithmically, the complexity of doing it this way or via the related shot noise process is more or less the same.

4. The discrete-time process embedded after collapse epochs

In this section we study the steady-state behavior of the growth-collapse process immediately after the nth collapse, V_n, defined by

$$V_n = (V_{n-1} + Y_n)X_n, \tag{4.1}$$

where V_0 is the initial state.

As observed, for example, in Equation (2) of [13],

$$V_n = V_0 \prod_{j=1}^{n} X_j + \sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j.$$

We assume in the sequel that $\{X_i \mid i \geq 1\}$ and $\{Y_i \mid i \geq 1\}$ are independent sequences of i.i.d. random variables distributed like X and Y, where X and Y are independent and nonnegative.

We will initially assume that X has support $[0, 1]$. From [13], it follows that, when $\mathrm{E}\, X < 1$ and $\mathrm{E}\, Y < \infty$, a limiting distribution for the process $\{V_n \mid n \geq 0\}$ exists, which is independent of the initial condition V_0, and it has a unique stationary version. It is easy to check that this continues to hold when X is nonnegative but not necessarily restricted to $[0, 1]$, as, when $\mathrm{E}\, X_1 < 1$, $\prod_{j=1}^{n} X_j \to 0$ a.s. as $n \to \infty$, and $\sum_{i=1}^{n} Y_i \prod_{j=i}^{n} X_j$ is stochastically increasing (as it is distributed like $\sum_{i=1}^{n} Y_i \prod_{j=1}^{i} X_j$) and its mean is bounded above by $\mathrm{E}\, Y\, \mathrm{E}\, X/(1 - \mathrm{E}\, X) < \infty$. Thus, throughout this section, it is assumed that $\mathrm{E}\, X < 1$ and $\mathrm{E}\, Y < \infty$.

The fact that, when starting from $V_0 = 0$, V_n is stochastically increasing can also be used to justify the fact that the limiting distribution of V_n has a finite mth moment if and only if $\mathrm{E}\, Y^m < \infty$ and $\mathrm{E}\, X^m < 1$, as in this case

$$(1 - \mathrm{E}\, X^m)\, \mathrm{E}\, V_n^m \leq \mathrm{E}\, V_n^m - \mathrm{E}\, X^m\, \mathrm{E}\, V_{n-1}^m = \mathrm{E}\, X^m \sum_{k=0}^{m-1} \binom{m}{k} \mathrm{E}\, V_{n-1}^k\, \mathrm{E}\, Y^{m-k}, \tag{4.2}$$

where, by induction, $\mathrm{E}\, V_n^k < \infty$ and converges to the kth moment of the limiting distribution by monotone convergence (finite or infinite). Thus, if the first $m - 1$ moments of the limiting distribution of V_n are finite, $\mathrm{E}\, Y^m < \infty$, and $\mathrm{E}\, X^m < 1$, then the mth moment is finite as well. If either $\mathrm{E}\, Y^m = \infty$ or $\mathrm{E}\, X^m \geq 1$, then (4.2) also implies that the mth moment of the limiting distribution of V_n is necessarily infinite.

Let V denote a random variable having this distribution, such that X, Y, and V are independent. Then

$$V \stackrel{\mathrm{D}}{=} (V + Y)X. \tag{4.3}$$

We might also focus on $Z := V + Y$, which has the limiting distribution of the state of the system immediately before collapses. This leads to

$$Z \stackrel{\mathrm{D}}{=} ZX + Y.$$

Much of the literature on (4.1) has concentrated on the existence of a limiting distribution, and on the tail behavior of that limiting distribution. In the present section we know that this limiting distribution exists provided that $\mathrm{E}\, X < 1$. Our goal is to determine it, for a number of choices of the distributions of X and Y. We start with the following formula for the LST $\psi(\alpha)$ of V:

$$\psi(\alpha) = \mathrm{E}\, \psi(\alpha X)\eta(\alpha X) = \int_{[0,1]} \psi(\alpha x)\eta(\alpha x)\, \mathrm{d}F_X(x). \tag{4.4}$$

Here $\eta(\alpha)$ denotes the LST of Y, and $F_X(x) = \mathrm{P}[X \leq x]$. We will also study $\mathrm{E}\, V^n$ in a number of cases, comparing it with (3.1). In the sequel, we assume that all required moments of Y are finite, with the exception of an example of regular variation at the end of the section, and note that, since X has support $[0, 1]$ and $\mathrm{E}\, X < 1$, then $\mathrm{E}\, X^n < 1$ for all $n \geq 1$ also.

We start with a case that has already been treated in [20]. We review it here, as it is the basis for extensions later in this section.

4.1. Case 1: $X \sim \mathrm{Beta}(D, 1)$

In this case, X has distribution $F(x) = \mathrm{P}[X \leq x] = x^D$, $0 \leq x \leq 1$. When D is a positive integer, X is distributed like the maximum of D independent $U[0, 1]$-distributed random variables. From (4.4) we have

$$\psi(\alpha) = \int_0^{\alpha} \psi(u)\eta(u) D \left(\frac{u}{\alpha}\right)^{D-1} \frac{\mathrm{d}u}{\alpha},$$

or

$$\alpha^D \psi(\alpha) = D \int_0^\alpha \psi(u)\eta(u)u^{D-1}\,\mathrm{d}u.$$

Differentiation yields, after some rearrangement,

$$\psi'(\alpha) = -D\psi(\alpha)\frac{1-\eta(\alpha)}{\alpha}, \tag{4.5}$$

so, since $\psi(0) = 1$,

$$\psi(\alpha) = \exp\bigg[-D\int_{v=0}^\alpha \frac{1-\eta(v)}{v}\,\mathrm{d}v\bigg] = \exp\bigg[-D\,\mathrm{E}\,Y\int_0^\alpha \eta_\mathrm{e}(v)\,\mathrm{d}v\bigg], \tag{4.6}$$

where $\eta_\mathrm{e}(v) = (1-\eta(v))/v\,\mathrm{E}\,Y$ is the LST of the stationary residual lifetime distribution of Y.

Remark 4.1. For $D = 1$, $\psi(\alpha)$ is the LST of the classical shot noise process; see [12]. For an integer $D > 1$, V is apparently the sum of D independent shot noise processes, each having $D = 1$. This is not a coincidence. It follows from the fact that if

$$W_i(t) = X_i(t) - r\int_0^t W_i(s)\,\mathrm{d}s$$

for $i = 1, \dots, D$ then

$$\sum_{i=1}^D W_i(t) = \sum_{i=1}^D X_i(t) - r\int_0^t \sum_{i=1}^D W_i(s)\,\mathrm{d}s,$$

so that if the $X_i(\cdot)$ are independent processes then the $W_i(\cdot)$ are also independent shot noise processes and $\sum_{i=1}^D W_i(\cdot)$ is itself a shot noise process with driving process $\sum_{i=1}^D X_i(\cdot)$. In this particular case we may observe from the relationship discussed earlier between the shot noise and growth-collapse processes that a uniformly distributed jump ratio for the growth-collapse process corresponds to exponentially distributed interjump times for the shot noise process. Thus, in this case, for $D = 1$, the $X_i(\cdot)$ are independent compound Poisson processes with arrival rate $\lambda = 1$ and jumps distributed like Y, so that $\sum_{i=1}^D X_i(\cdot)$ is also a compound Poisson process with arrival rate D and jumps distributed like Y.

4.1.1. *Moments.* It follows from (4.5), after $k-1$ differentiations and denoting by Y_e a random variable with stationary residual lifetime distribution associated with Y, that

$$\mathrm{E}\,V^n = D\,\mathrm{E}\,Y\sum_{k=0}^{n-1}\binom{n-1}{k}\mathrm{E}\,V^k\,\mathrm{E}\,Y_\mathrm{e}^{n-1-k},$$

from which recursion all moments of V can be obtained (starting with $\mathrm{E}\,V = D\,\mathrm{E}\,Y$). The equivalence with (3.1) follows by observing that $\mathrm{E}\,X^n = D/(D+n)$ and that $\mathrm{E}\,Y_\mathrm{e}^n = \mathrm{E}\,Y^{n+1}/(n+1)\,\mathrm{E}\,Y$.

4.2. Case 2: $X \sim \mathrm{Beta}(\zeta_1, \zeta_2)$ and $Y \sim \mathrm{Gamma}(\zeta_2, \beta)$

In this case there is the following shortcut. It is well known (usually given as a standard exercise in a first-year probability course when discussing multidimensional transformations and Jacobians) that if V is $\mathrm{Gamma}(\zeta_1, \beta)$-distributed and Y is $\mathrm{Gamma}(\zeta_2, \beta)$-distributed, and

V and Y are independent, then $V/(V+Y)$ is distributed like $\mathrm{Beta}(\zeta_1, \zeta_2)$ and is independent of $V+Y$, which is distributed like $\mathrm{Gamma}(\zeta_1+\zeta_2, \beta)$. Thus, the joint distribution of $(X, V+Y)$ is the same as that of $(V/(V+Y), V+Y)$, which implies that $X(V+Y)$ is distributed like V, so that (4.3) is satisfied. As there is a unique limiting distribution for recursion (4.1), it must be $\mathrm{Gamma}(\zeta_1, \beta)$.

Remark 4.2. It is easily verified that, in the case of Y being exponentially distributed, i.e. $\mathrm{Gamma}(1, \beta)$, and X being $\mathrm{Beta}(D, 1)$-distributed as in Case 1, (4.6) yields $\psi(\alpha) = (1/(1+\beta\alpha))^D$; so, indeed, V has a $\mathrm{Gamma}(D, \beta)$ distribution. This particular case is mentioned in [20, p. 765].

4.2.1. *Moments.* Since V has a $\mathrm{Gamma}(\zeta_1, \beta)$ distribution, it immediately follows that

$$\mathrm{E}\, V^n = \beta^n \frac{\Gamma(\zeta_1 + n)}{\Gamma(\zeta_1)}.$$

4.3. Case 3: *X* has an atom at 0

Suppose that $\mathrm{P}[X = 0] = p > 0$ and that X assumes, with probability $1-p$, values on $(0, \infty)$ (so we do not necessarily restrict X to $[0, 1]$). It is easy to see that the limiting distribution of $\{V_n \mid n \geq 0\}$ always exists, as 0 is a regenerative state with geometrically distributed (hence, aperiodic finite-mean) regeneration epochs. We will study several subcases.

4.4. Case 3(a): *X* has atoms at 0 and *c*

Assume that $\mathrm{P}[X = 0] = 1 - \mathrm{P}[X = c] = p$, with $p > 0$ and $c > 0$ (also allowing $c > 1$). From (4.1),

$$\psi(\alpha) = p + (1-p)\psi(c\alpha)\eta(c\alpha),$$

of which repeated iterations yield

$$\psi(\alpha) = \sum_{j=0}^{\infty} p(1-p)^j \prod_{i=1}^{j} \eta(c^i\alpha).$$

The sum obviously converges for all $0 < p \leq 1$. Inversion of the LST reveals that

$$V \stackrel{\mathrm{D}}{=} \sum_{i=1}^{N} c^i Y_i, \tag{4.7}$$

where N is geometrically distributed (counting only failures) with probability of success p and is independent of $\{Y_i \mid i \geq 1\}$. Indeed, this also follows directly by applying (4.1), in the form $V_n = c(V_{n-1} + Y_n)$, N times, with $V_0 = 0$. See also [20, p. 762] (where $c = 1$).

4.4.1. *Moments.* From (4.7),

$$\mathrm{E}\, V^n = \sum_{j=0}^{\infty} p(1-p)^j \sum_{\sum_{i=1}^{j} n_i = n} \frac{n!}{\prod_{i=1}^{j} n_i!} c^{\sum_{i=1}^{j} i n_i} \prod_{i=1}^{j} \mathrm{E}\, Y^{n_i}.$$

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4.5. Case 3(b): $X \sim$ mixture of an atom at 0 and Beta(D,1)

Assume that $\mathrm{P}[X \le x] = p + (1-p)x^D$, $0 \le x \le 1$, $p > 0$. In this case, (4.4) reduces to

$$\psi(\alpha) = p + (1-p)\int_0^1 \psi(\alpha x)\eta(\alpha x)\,\mathrm{d}x,$$

yielding, after manipulations similar to those in Case 1,

$$\psi'(\alpha) = \frac{pD}{\alpha} + \psi(\alpha)\frac{(1-p)D\eta(\alpha) - D}{\alpha}. \tag{4.8}$$

The solution of this first-order inhomogeneous differential equation is

$$\begin{aligned}\psi(\alpha) = C\exp\biggl[D\int_0^\alpha \frac{(1-p)D\eta(v) - D}{v}\,\mathrm{d}v\biggr] \\ + \int_0^\alpha \frac{pD}{z}\exp\biggl[\int_z^\alpha \frac{(1-p)D\eta(v) - D}{v}\,\mathrm{d}v\biggr]\mathrm{d}z.\end{aligned} \tag{4.9}$$

It is easily seen that the first integral on the right-hand side of (4.9) diverges, so we have to take $C = 0$. By observing that $((1-p)D\eta(v) - D)/v$ is bounded between $-D/v$ and $-pD/v$, and, hence, that the expression in the last line of (4.9) is bounded between

$$\int_0^\alpha \frac{pD}{z}\left(\frac{z}{\alpha}\right)^{pD}\mathrm{d}z \quad \text{and} \quad \int_0^\alpha \frac{pD}{z}\left(\frac{z}{\alpha}\right)^{D}\mathrm{d}z,$$

it follows that the expression on the last line of (4.9) has a value between p and 1. When $\alpha \downarrow 0$, the above bound $-pD/v$ becomes tight and the expression on the last line of (4.9) approaches 1.

4.5.1. *Moments.* The most suitable approach to obtain $\mathrm{E}\,V^n$ via the LST here seems to be to multiply both sides of (4.8) with α and differentiate $k-1$ times. However, the resulting calculation is not much easier than when starting from (3.1), and, hence, we omit it.

4.5.2. *Tail behavior.* Suppose that the distribution of Y is regularly varying at ∞ with index $-\nu$. Then application of Lemma 8.1.6 of [5] to (4.9) readily shows that V is also regularly varying, with the same index. We do not provide details because considerably more general tail results can be obtained for (4.1); see [19] for regularly varying Y and [7] for light tailed Y.

4.6. Case 3(c): $X \sim$ mixture of an atom at 0 and a product of two i.i.d. U[0,1]

The density of the product of two i.i.d. random variables which are uniformly distributed on $[0,1]$ is $-\ln x$, $0 < x < 1$. Formula (4.4) now reduces to

$$\psi(\alpha) = p - (1-p)\int_0^1 \psi(\alpha x)\eta(\alpha x)\ln x\,\mathrm{d}x,$$

so

$$\alpha\psi(\alpha) = p\alpha - (1-p)\int_0^\alpha \psi(u)\eta(u)\ln\left(\frac{u}{\alpha}\right)\mathrm{d}u,$$

which, after two differentiations, leads to

$$\alpha^2\psi''(\alpha) + 3\alpha\psi'(\alpha) + (1 - (1-p)\eta(\alpha))\psi(\alpha) = p. \tag{4.10}$$

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In the special case that $Y \sim \exp(\mu)$ (hence, $\eta(\alpha) = \mu/(\mu+\alpha)$) this equation simplifies to

$$\alpha^2(\mu+\alpha)\psi''(\alpha) + 3\alpha(\mu+\alpha)\psi'(\alpha) + (\mu+\alpha-(1-p)\mu)\psi(\alpha) = p.$$

For $p = 0$, we obtain a known case:

$$\alpha(\mu+\alpha)\psi''(\alpha) + 3(\mu+\alpha)\psi'(\alpha) + \psi(\alpha) = 0.$$

Note that this is differential equation (2.17) for the case of $\nu_1 = \nu_2 = 1$, which makes sense: Y being exponential and X being a product of two independent $U[0, 1]$ random variables corresponds to having an exponential on-time distribution and an Erlang-2 off-time distribution in the on/off model of Section 1 (which was directly related to the growth-collapse model and the shot noise model). Slightly more generally, if $X = U_1^{1/\nu_1} U_2^{1/\nu_2}$, with U_1 and U_2 independent and $U[0, 1]$-distributed, we obtain (2.17) with ν_1 and ν_2.

Remark 4.3. We note that the density of the product of $k \geq 2$ i.i.d. random variables which are uniformly distributed on $[0, 1]$ is $(-\ln(x))^{k-1}/(k-1)!$, $0 < x < 1$; thus, in a similar manner we may derive a kth-order differential equation for $\psi(\alpha)$.

Remark 4.4. When $p = 0$ and $\eta(\alpha) = (\mu/(\mu+\alpha))^2$, i.e. Y is Erlang-2, then (4.10) becomes

$$\psi''(\alpha) + \frac{3}{\alpha}\psi'(\alpha) + \frac{\alpha+2\mu}{\alpha(\mu+\alpha)^2}\psi(\alpha) = 0. \tag{4.11}$$

When $p = 0$ and $\eta(\alpha) = b\mu_1/(\mu_1+\alpha) + (1-b)\mu_2/(\mu_2+\alpha)$ with $0 < b < 1$, i.e. Y is hyperexponentially distributed, then (4.10) becomes

$$\psi''(\alpha) + \frac{3}{\alpha}\psi'(\alpha) + \frac{b(\mu_2+\alpha) + (1-b)(\mu_1+\alpha)}{\alpha(\mu_1+\alpha)(\mu_2+\alpha)}\psi(\alpha) = 0. \tag{4.12}$$

Both (4.11) and (4.12) are special cases of Heun's differential equation; cf. [9].

4.7. Case 4: $X \sim U[0,a]$

We are interested in studying the case in which X is not restricted to $[0, 1]$. We assume that X is $U[0, a]$-distributed, with $\mathrm{E}\,X = a/2 < 1$. As noted in the first paragraph of this section, together with $\mathrm{E}\,Y < \infty$, this implies stability. Formula (4.4) now becomes

$$\psi(\alpha) = \frac{1}{a}\int_0^a \psi(\alpha x)\eta(\alpha x)\,\mathrm{d}x = \frac{1}{a\alpha}\int_0^{a\alpha} \psi(u)\eta(u)\,\mathrm{d}u,$$

and differentiation gives (see (4.5))

$$\psi'(\alpha) = \psi(a\alpha)\frac{\eta(a\alpha)}{\alpha} - \frac{\psi(\alpha)}{\alpha}. \tag{4.13}$$

By introducing $\zeta(\alpha) \equiv \psi(\mathrm{e}^{\alpha})$ we can rewrite (4.13) as the differential-difference equation

$$\zeta'(\alpha) = \zeta(\alpha+c)\xi(\alpha+c) - \zeta(\alpha),$$

with $c = \ln a < 0$ and $\xi(\alpha) := \eta(\mathrm{e}^{\alpha})$. There is an extensive literature on differential-difference equations; see, for example, [10]. However, solutions of such equations are only known in special cases, such as when $\xi(\alpha)$ is a constant. Below we consider (4.13) in the special case that $Y \sim \exp(\mu)$. Equation (4.13) then reduces to

$$(\mu + a\alpha)\alpha\psi'(\alpha) = \mu\psi(a\alpha) - (\mu + a\alpha)\psi(\alpha).$$

We might solve this equation by introducing the Taylor series expansion $\psi(\alpha) \equiv \sum_{n=0}^{\infty} f_n\alpha^n$, with $f_0 = \psi(0) = 1$, and solving the resulting recursion for f_n (which is $(-1)^n \mathrm{E}\, V^n/n!$).

We prefer an alternative approach, starting from (3.1):

$$\mathrm{E}\, V^n = \frac{\mathrm{E}\, X^n}{1 - \mathrm{E}\, X^n} \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{E}\, V^k\, \mathrm{E}\, Y^{n-k} = n!\, \frac{a^n}{n + 1 - a^n} \sum_{k=0}^{n-1} \frac{\mathrm{E}\, V^k}{k!} \frac{1}{\mu^{n-k}}.$$

If we define $B_n = \sum_{k=0}^{n} \mu^k\, \mathrm{E}\, V^k/k!$ then $B_n - B_{n-1} = a^n B_{n-1}/(n + 1 - a^n)$ so that $B_n = (n + 1)B_{n-1}/(n + 1 - a^n)$, which implies that $B_n = (n + 1)!/\prod_{i=1}^{n}(i + 1 - a^i)$. Hence,

$$\frac{\mu^n\, \mathrm{E}\, V^n}{n!} = B_n - B_{n-1} = \frac{a^n n!}{\prod_{i=1}^{n}(i + 1 - a^i)},$$

and, thus, when $a < (1 + n)^{1/n}$,

$$\mathrm{E}\, V^n = \frac{(a/\mu)^n (n!)^2}{\prod_{i=1}^{n}(i + 1 - a^i)}.$$

Remark 4.5. For $a = 1$, we obtain $\mathrm{E}\, V^n = n!/(\mu)^n$, corresponding to V being exponentially distributed; see Remark 4.2. We further recall that $\mathrm{E}\, V^n < \infty$ if and only if $\mathrm{E}\, X^n = a^n/(n + 1) < 1$ (see (4.2) and the discussion there). Note that $(1+n)^{1/n}$ equals 2 for $n = 1$ and decreases to 1 as $n \to \infty$, so that, for $0 < a \le 1$, all the moments exist, but not for $1 < a < 2$.

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