BENDING IN THE SPACE OF QUASI-FUCHSIAN STRUCTURES

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0. Introduction. In [**2**] I described the deformation of bending a hyperbolic manifold along an embedded totally geodesic hypersurface. As I remarked there, the deformation is particularly interesting in the case of a surface, because a surface contains many embedded totally geodesic hypersurfaces, namely simple closed curves, along which it is possible to bend. Furthermore, for a surface it is possible to extend the definition of bending to the case of a geodesic lamination, by using the fact that the set of simple closed geodesics is dense in the space of geodesic laminations. This direction has been developed by Epstein and Marden in [**1**].

In the present article I want to look into another generalization of the idea of bending, which again is possible only in dimension 2, namely that of bending a quasi-Fuchsian structure.

It is possible to define bending with reference only to the boundary of the universal cover of the n-manifold, and in particular to the $(n-2)$-spheres which form the boundary of the components of the lift of the hypersurface. In general, a quasi-conformal homeomorphism would deform such a sphere into a non spherical Jordan hypersurface in $\mathbb{R}^{n-1}$, which would not be invariant by any one-parameter group of Möbius transformations of $\mathbb{R}^{n-1}$. However, if $n=2$ the deformed 0-spheres still define geodesics. Hence it is possible to define bending starting from any quasi-Fuchsian structure. In this case we do not obtain an explicit bending embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$; because of this I follow [**1**] in using cocycles to define the deformation. There is however a naturally defined bending map of a pleated surface representing the bending lamination, which is a local isometry. There is also an embedding of the spheres at infinity.

In a subsequent article this deformation will be used to study the geometry of the space of quasi-Fuchsian structures.

1. Preliminaries. A. I shall use the upper half-plane $\mathbb{H}^2=\{z\in\mathbb{C},\ \operatorname{Im} z>0\}$ and the upper half-space $\mathbb{H}^3=\{(x,y,z)\in\mathbb{R}^3,\ z>0\}$, with their respective Poincaré metrics; c will denote the point $i\in\mathbb{H}^2$ or the point $(0,0,1)\in\mathbb{H}^3$ and γ the geodesic $\{ti,\ t>0\}\subset\mathbb{H}^2$ or $\{(0,0,t),\ t>0\}\subset\mathbb{H}^3$. $\bar{\mathbb{H}}^2$ and $\bar{\mathbb{H}}^3$ will denote the closure of $\mathbb{H}^2$ and $\mathbb{H}^3$ inside $\mathbb{C}\cup\{\infty\}$ and $\mathbb{R}^3\cup\{\infty\}$ respectively. Elements of $\mathrm{PSL}(2,\mathbb{C})$ will be identified with the corresponding isometries of $\mathbb{H}^3$.

Let S be a closed surface. The space $Q(S)$ of *quasi-Fuchsian structures* on S is a quotient of the space of injective homomorphisms $\rho:\pi_1(S)\to PSL(2,\mathbb{C})$ such that $\Gamma=\operatorname{im}\rho$ is discrete and $S\times I\cong\mathbb{H}^3/\Gamma$. Two homomorphisms $\rho_1,\ \rho_2$ define the same point in $Q(S)$ if there is an inner automorphism $\operatorname{ad} A$ of $PSL(2,\mathbb{C})$ such that $\rho_1=\operatorname{ad} A\circ\rho_2$.

A *Fuchsian point* in $Q(S)$ is the class of a homomorphism with image in $PSL(2,\mathbb{R})$. The Fuchsian points of $Q(S)$ form the Teichmüller space $T(S)$. A point $\rho\in T(S)$ determines a hyperbolic structure on S: there is a homeomorphism $S\to\mathbb{H}^2/\Gamma$ inducing $\rho:\pi_1(S)\to\Gamma$.

We shall fix a Fuchsian point represented by $\rho:\pi_1(S)\to\Gamma\subset PSL(2,\mathbb{R})$ and shall identify $\pi_1(S)$ with Γ under ρ. We shall consider S with the hyperbolic structure defined by ρ.

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For every $\rho_0 \in Q(S)$, there is an embedding $\varphi: \mathbb{H}^2 \to \mathbb{H}^3$, extending to the boundary by an embedding $\partial\varphi: \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$, such that $\rho_0: \Gamma \to \Gamma_0$ is given by

$$\rho_0(A)|_{\varphi(\mathbb{H}^2)} = \varphi \circ A \circ \varphi^{-1}|_{\varphi(\mathbb{H}^2)}.$$

B. Let S be a surface with a hyperbolic structure. A *geodesic lamination* Λ on S is a foliation of a closed subset of S such that the leaves are geodesics. The leaves of Λ together with the components of $S - \Lambda$ will be called *strata* of Λ. A geodesic lamination is *finite* if it has only finitely many leaves; it is *discrete* if every compact subset of S intersects finitely many leaves of Λ. A geodesic lamination is *maximal* if every component of $S - \Lambda$ is an ideal triangle.

PROPOSITION 1.1 [5]. *Any geodesic lamination on a closed hyperbolic surface is contained in a maximal lamination.*

If $\varphi: S_1 \to S_2$ is a homeomorphism of closed hyperbolic surfaces and Λ is a geodesic lamination on S_1, define $\varphi_*(\Lambda)$ to be the unique geodesic lamination whose leaves are in the isotopy class of the curves $\varphi(f)$, for f a leaf of Λ. In particular, if S_1 and S_2 are distinct hyperbolic structures on a closed surface S, a geodesic lamination on S_1 determines a unique geodesic lamination on S_2.

If $\varphi: \bar{\mathbb{H}}^2 \to \bar{\mathbb{H}}^3$ is an embedding and Λ a geodesic lamination on $\mathbb{H}^2$, $\varphi_*(\Lambda)$ is the set of geodesics in $\mathbb{H}^3$ asymptotic with the curves $\varphi(f)$ for f a leaf of Λ.

A *pleated surface* in a hyperbolic 3-manifold N is a complete hyperbolic surface S together with a continuous map $\psi: S \to N$ which satisfies:

(a) ψ is isometric, in the sense that every geodesic segment in S is taken to a rectifiable arc in N which has the same length;

(b) for each point $x \in S$, there is at least one open geodesic segment α_x through x which is mapped to a geodesic segment in N.

The set of points with a unique segment α_x satisfying (b) is the *pleating locus* of ψ. The pleating locus is a geodesic lamination on S.

The *hyperbolic convex hull* $H(B)$ of a closed set $B \subset \bar{\mathbb{H}}^3$ is the intersection of all closed hyperbolic half spaces containing B. For a quasi-Fuchsian point $\rho_0 \in Q(S)$, the convex hull $H_0 = H(\partial\varphi(\partial\mathbb{H}^2))$ is invariant by the group Γ_0, and the *convex hull manifold* of Γ_0 is the quotient space H_0/Γ_0.

PROPOSITION 1.2. [5]. *Let S be a closed hyperbolic surface, Γ a quasi-Fuchsian group isomorphic to $\pi_1(S)$, and M the convex hull manifold of Γ. For every maximal geodesic lamination Λ on S there is a unique hyperbolic structure S_Λ, and a pleated surface $\psi: S_\Lambda \to M$ such that the pleating locus of ψ is contained in Λ.*

Let Λ_S be a lamination in a hyperbolic surface S. Then Λ_S lifts to a lamination Λ in $\mathbb{H}^2$.

A *transverse measure* μ for Λ is a finite complex valued Borel measure μ_α on each compact non trivial geodesic segment α in $\mathbb{H}^2$, such that:

(a) if α is contained entirely in a stratum of Λ, then $\mu_\alpha(\alpha) = 0$;

(b) if α and β are geodesic segments and their respective endpoints lie in the same strata of Λ, then $\mu_\alpha(\alpha) = \mu_\beta(\beta)$.

If α and β are geodesic segments, U is a set with more than one point and $U \subset \alpha \cap \beta$, then $\mu_\alpha(U) = \mu_\beta(U)$ and we shall denote it by $\mu(U)$. If x is contained in a leaf f of Λ,

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$\mu(\{x\})$ will denote the μ_α-measure of the set $\{x\}$, where α is a geodesic segment containing x and not contained in f.

A *measured lamination* (Λ, μ) is a geodesic lamination with a transverse measure. If Λ is finite with leaves $f_1, \ldots, f_k$ and $x_i \in f_i$, $\mu(\{x_i\}) = z_i$, we shall denote the measured lamination (Λ, μ) by $\{f_i, z_i\}$.

If Λ is the lift to $\mathbb{H}^2$ of a geodesic lamination Λ_S on S, and the transverse measure μ of Λ is invariant by the action of the fundamental group of S on $\mathbb{H}^2$, it defines a transverse measure μ_S for Λ_S.

C. The following definition generalizes the definition of a cocycle in [**1**, Definition 3.5.2]. Let $\rho \in T(S)$, $\rho_0 \in Q(S)$, be as in subsection A.

A (Γ, ρ_0)*-cocycle* on $\mathbb{H}^2$ is a mapping $Z : \mathbb{H}^2 \times \mathbb{H}^2 \to PSL(2, \mathbb{C})$ such that

(a) $Z(x, x) = I$ for all $x \in \mathbb{H}^2$;

(b) $Z(x, z) = Z(x, y)Z(y, z)$ for $x, y, z \in \mathbb{H}^2$;

(c) $Z(Ax, Ay) = \rho_0(A)Z(x, y)\rho_0(A^{-1})$ for $x, y \in \mathbb{H}^2$, $A \in \Gamma$.

A (Γ, ρ_0)-cocycle determines a cocycle for Γ in the usual sense (i.e. a crossed homomorphism). Fix a point $x \in \mathbb{H}^2$ and define

$$Z_A = Z(x, Ax) \quad \text{for} \quad A \in \Gamma.$$

Then $Z_{AB} = Z(x, ABx) = Z(x, Ax)Z(Ax, ABx)$, and using (c) we have

$$Z_{AB} = Z_A \rho_0(A) Z_B \rho_0(A^{-1}). \tag{1}$$

This in turn determines a representation of Γ in $PSL(2, \mathbb{C})$ and hence a quasi-Fuchsian structure

$$\rho_Z(A) = Z_A \rho_0(A).$$

By (1), ρ_Z is a homomorphism.

2. Construction of the cocycle. A. For every measured geodesic lamination (Λ_S, μ_S) on S we shall define a (Γ, ρ_0)-cocycle on $\mathbb{H}^2$ which will give rise to a deformation of ρ_0. To begin with assume that Λ_S is a finite lamination. Then Λ is a discrete lamination in $\mathbb{H}^2$.

Let $x, y \in \mathbb{H}^2$. If x and y lie in the same stratum of Λ define $Z(x, y) = I$. Otherwise let $[x, y]$ be the closed geodesic segment from x to y, oriented from x to y. Number the leaves of Λ which intersect $[x, y]$ starting from $x : f_0, f_1, \ldots, f_k$. Give f_i the orientation so that it crosses the segment $[x, y]$ from right to left, and let $z_i = \mu(f_i \cap [x, y])$.

If f is a geodesic in $\mathbb{H}^2$, with endpoints u, v and orientation from u to v, and z a complex number, define $C(f, z)$ to be the element of $PSL(2, \mathbb{C})$ determined as follows. $C(f, z)$ is conjugate to

$$C(\gamma, z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}$$

and has repulsive fixed point $u' = \partial\varphi(u)$ and attractive fixed point $v' = \partial\varphi(v)$, where $\varphi : \mathbb{H}^2 \to \mathbb{H}^3$ is the embedding defined in section **1**(A). $C(f, z)$ is given by the matrix

$$(v' - u')^{-1} \begin{pmatrix} v'e^{z/2} - u'e^{-z/2} & u'v'(e^{-z/2} - e^{z/2}) \\ e^{z/2} - e^{-z/2} & v'e^{-z/2} - u'e^{z/2} \end{pmatrix}.$$

Define $C_i = C(f_i, z_i)$ for $i = 0, \ldots, k-1$ by

$$C_0 = \begin{cases} C(f_0, z_0) & \text{if } x \notin f_0 \\ C(f_0, \tfrac{1}{2}z_0 & \text{if } x \in f_0 \end{cases} \quad \text{and} \quad C_k = \begin{cases} C(f_k, z_k) & \text{if } y \notin f_k \\ C(f_k, \tfrac{1}{2}z_k) & \text{if } y \in f_k. \end{cases}$$

Then define $Z(x, y) = C_0 \ldots C_k$.

LEMMA 2.1. *Z is a (Γ, ρ_0)-cocycle on $\mathbb{H}^2$.*

Proof. (a) By definition, $Z(x, x) = I$.

(b) First assume that y lies on the geodesic segment $[x, z]$ and not on a leaf of Λ. Let $f_0, \ldots, f_m$ be the leaves of Λ which intersect $[x, y]$, $f_{m+1}, \ldots, f_k$ those which intersect $[y, z]$. Then

$$Z(x, y)Z(y, z) = C_0 \ldots C_m C_{m+1} \ldots C_k = Z(x, z).$$

If $y \in f_m \cap [x, z]$, then

$$\begin{aligned} Z(x, y)Z(y, z) &= (C_0 \ldots C_{m-1}C(f_m, \tfrac{1}{2}z_m))(C(f_m, \tfrac{1}{2}z_m)C_{m+1} \ldots C_k) \\ &= C_0 \ldots C_k = Z(x, z). \end{aligned}$$

The same calculation holds if the stratum containing y intersects the segment $[x, z]$. Otherwise there are points t_1, t_2, t_3 on $[y, z]$, $[x, z]$, $[x, y]$ such that t_1 separates the leaves which intersect $[x, y]$ and $[y, z]$ from those which intersect $[y, z]$ and $[x, z]$, and similarly for t_2, t_3. Then $Z(x, t_2) = Z(x, t_3)$, $Z(t_3, y) = Z(t_1, y)$ and $Z(t_2, z) = Z(t_1, z)$. Hence

$$\begin{aligned} Z(x, z) &= Z(x, t_2)Z(t_2, z) = Z(x, t_3)Z(t_1, z) \\ &= Z(x, t_3)Z(t_3, y)Z(y, t_1)Z(t_1, z) \\ &= Z(x, y)Z(y, z). \end{aligned}$$

(c) (Λ, μ) is invariant by Γ, hence if $A \in \Gamma$ the leaves which intersect $[Ax, Ay]$ are $A(f_0), \ldots, A(f_k)$, and for $x_i \in f_i$, $\mu(\{A(x_i)\}) = \mu(\{x_i\}) = z_i$. If u, v are the end-points of f_i, $C(A(f_i), z_i)$ has fixed points $\partial\varphi(Au)$, $\partial\varphi(Av)$. Let $B \in PSL(2, \mathbb{C})$ be such that $B(\partial\varphi(u)) = 0$ and $B(\partial\varphi(v)) = \infty$. Then $B\rho_0(A^{-1})(\partial\varphi(Au)) = 0$ and $B\rho_0(A^{-1})(\partial\varphi(Av)) = \infty$. Therefore

$$C(A(f_i), z_i) = \rho_0(A)C(f_i, z_i)\rho_0(A^{-1}).$$

This completes the construction of the (Γ, ρ_0)-cocycle for a finite lamination.

B. If the lamination Λ_S is not finite, we shall approximate the value of the cocycle at (x, y) by grouping together nearby leaves of Λ_S, to obtain a finite lamination.

A solid *cylinder over a disc D* in $\mathbb{H}^3$ is the union of all geodesics orthogonal to a two dimensional hyperbolic disc in $\mathbb{H}^3$. The endpoints of these geodesics lie in two discs D_1 and D_2 in $\mathbb{H}^3$; we shall say that the cylinder is *supported* by D_1 and D_2. The cylinder is uniquely determined by the disc D. The *radius* of the cylinder is the hyperbolic radius of the disc D.

LEMMA 2.2. *Given ε and η positive numbers, there is $\delta > 0$ such that if D_1 and D_2 are two discs in $\mathbb{R}^2 \subset \partial\mathbb{H}^3$, with euclidean diameter $<\delta$ and the distance between D_1 and D_2 is $>\eta$, then the cylinder supported by D_1 and D_2 has radius $r < \varepsilon$.*

Proof. It is clear that the radius of the cylinder increases with an increase in the diameter of the supporting discs and decreases with an increase in the distance between the discs. It is therefore enough to calculate the radius r of the cylinder supported by two discs of diameter δ and distance η apart. This is given by

$$2r = \log \frac{\frac{1}{2}\eta + \delta}{\frac{1}{2}\eta} = \log(1 + 2\delta/\eta).$$

We can therefore take $\delta = \frac{1}{2}\eta(e^{2\varepsilon} - 1)$.

Let (Λ_S, μ_S) be an arbitrary measured lamination on S, and (Λ, μ) its lift to $\mathbb{H}^2$. Let $[x, y]$ be a closed oriented geodesic segment in $\mathbb{H}^2$. There are two disjoint closed intervals I and J on $\partial\mathbb{H}^2$ such that each leaf of Λ which intersects $[x, y]$ has one endpoint in I and one in J. Assume that $\partial\varphi : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$ maps I and J into $\mathbb{R}^2$. Let $\eta = \inf\{|\partial\varphi(u) - \partial\varphi(v)|, u \in I, v \in J\}$, $\varepsilon = 1/n$ and δ_n the number given by Lemma 2.2.

Since I and J are compact there exist partitions $\Pi = \{u_0, \ldots, u_q\}$ of I and $\Sigma = \{v_0, \ldots, v_s\}$ of J such that $\partial\varphi([u_{i-1}, u_i])$, $i = 1, \ldots, q$ and $\partial\varphi([v_{j-1}, v_j])$, $j = 1, \ldots, s$ are each contained in a disc of diameter δ_n. Using Π and Σ, we can construct a partition $P_n = \{x_0, \ldots, x_p\}$, $x = x_0$, $y = x_p$, such that $\mu(\{x_i\}) = 0$ for $0 < i < p$ and all the leaves of Λ which intersect $[x_{i-1}, x_i]$ have their endpoints in the same subinterval of I or J. This condition implies that the set of geodesics in $\varphi_*(\Lambda)$ which correspond to leaves of Λ which intersect $[x_{i-1}, x_i]$ all lie in a hyperbolic cylinder of radius $1/n$ in $\mathbb{H}^3$.

Given any partition $P = \{x_0, \ldots, x_p\}$ of $[x, y]$ with $\mu(\{x_i\}) = 0$ for $0 < i < p$, define a *finite approximation of* (Λ, μ) *subordinate to* P to be a finite lamination $\{f_i, z_i\}$ constructed in the following way. If there are leaves of Λ intersecting $[x_{i-1}, x_i]$ choose one of them and label it f_{i-1}. If there is a leaf of Λ through x_0 or through x_p choose that leaf for f_0 or f_{p-1}. Relabel the geodesics $f_0, \ldots, f_r$ to avoid repetitions and subintervals with empty intersection with Λ. Now define a finite measured lamination $\{f_i, z_i\}$ where z_i is the sum of the measures under μ of the (one or two) subintervals of P associated with f_i. We have $\sum_{i=0}^{r} z_i = \mu([x, y])$. Define

$$Z_P = C(f_0, z_0) \ldots C(f_r, z_r)$$

where z_0 is replaced by $\frac{1}{2}z_0$ if x lies on Λ and z_r is replaced by $\frac{1}{2}z_r$ if y lies on Λ.

C. Let $Z_n = Z_{P_n}$ for the partition P_n constructed above. In the remainder of this section I shall show that $\{Z_n\}$ converges to a limit in $PSL(2, \mathbb{C})$ and that the limit is independent of the choices made in the definition of Z_n.

Because we shall have to act by elements of $PSL(2, \mathbb{C})$ on the right as well as on the left, it is not convenient to use the geometrically defined left invariant metric on $PSL(2, \mathbb{C})$. Instead we shall work with a norm in the space of 2×2 complex matrices.

In the vector space $\mathbb{C}^2$ introduce the norm $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$. A complex matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{C}^2$ and has norm

$$\|A\| = \max\{|a| + |b|, |c| + |d|\}.$$

LEMMA 2.3 [**1**, Lemma 3.3.1]. *Let X be a set of matrices in $SL(2, \mathbb{C})$ and $c = (0, 0, 1) \in \mathbb{H}^3$. Then the following are equivalent.*

(a) *The closure of X is compact.*
(b) *There is a constant M such that if $A \in X$ then $\|A\| \leq M$.*
(c) *There is a constant M such that if $A \in X$ then $\|A\| \leq M$ and $\|A^{-1}\| \leq M$.*
(d) *There is a constant R such that if $A \in X$ then $d(c, A(c)) \leq R$.*

The following elementary lemma is proved by analyticity. Recall that

$$C(\gamma, z) = \begin{pmatrix} e^{z/2} & 0 \\ 0 & e^{-z/2} \end{pmatrix}.$$

LEMMA 2.4 [**1**, Lemma 3.4.1]. *If K is a compact subset of $SL(2, \mathbb{C})$ and $M > 0$, there is a constant N such that for any A, $B \in K$ and $z \in \mathbb{C}$ with $|z| \leq M$, we have*

$$\|A\, C(\gamma, z)\, A^{-1} - B\, C(\gamma, z)\, B^{-1}\| \leq N \,\|A - B\|\, |z|.$$

We need to know that the approximations Z_n all remain within a compact subset of $SL(2, \mathbb{C})$. To obtain this I look at the pleated surface $\psi: S \to \mathbb{H}^3/\Gamma_0$ representing a maximal lamination containing Λ_S. Let $\tilde{\psi}: \mathbb{H}^2 \to \mathbb{H}^3$ be the lift of ψ.

LEMMA 2.5. *Let K be a compact disc of radius R about c in $\mathbb{H}^3$, and $M > 0$. There is a number $N > 0$ with the following property. If $[x, y]$ is a geodesic segment in $\mathbb{H}^2$ such that $\tilde{\psi}([x, y]) \subset K$ and $\{f_i, z_i\}$, $i = 0, \ldots, k$, is a finite measured lamination with support contained in the support of Λ, whose leaves all intersect $[x, y]$ and are numbered in order from x to y, and such that $\sum_{i=0}^{k} |\mathrm{Re}\, z_i| < M$, then*

$$\|C(f_0, z_0) \ldots C(f_k, z_k)\| \leq N.$$

Proof. Since $\{f_i\}$ is a sublamination of Λ, $\tilde{\psi}(f_i)$ are geodesics in $\mathbb{H}^3$. Let $x_i = f_i \cap [x, y]$. Then $\tilde{\psi}([x_{i-1}, x_i])$ is a rectifiable path in $\mathbb{H}^3$ of length $d(x_{i-1}, x_i)$. For $i = 1, \ldots, k$ define $\tau_i: [x_{i-1}, x_i] \to \mathbb{H}^3$ by $\tau_i = C(f_0, z_0) \ldots C(f_{i-1}, z_{i-1}) \circ \tilde{\psi}|_{[x_{i-1}, x_i]}$. Each $\tau_i([x_{i-1}, x_i])$ is a rectifiable path of length $d(x_{i-1}, x_i)$.

Define a rectifiable path from $\tilde{\psi}(x_0)$ to $C(f_0, z_0) \ldots C(f_k, z_k) \circ \tilde{\psi}(x_k)$ by joining the geodesic segments and the rectifiable paths $[\tilde{\psi}(x_0), \tau_1(x_0)]$, $\tau_1([x_0, x_1])$, $[\tau_1(x_0), \tau_2(x_1)], \ldots, \tau_k([x_{k-1}, x_k])$, $[\tau_k(x_k), C(f_0, z_0) \ldots C(f_k, z_k) \circ \tilde{\psi}(x_k)]$. This path has length less than $d(x_0, x_k) + M \leq 2R + M$. Therefore $d(\tilde{\psi}(x_0), C(f_0, z_0) \ldots C(f_k, z_k) \circ \tilde{\psi}(x_k)) \leq 2R + M$. But $\tilde{\psi}(x_0)$ and $\tilde{\psi}(x_k)$ both lie in a disc of radius R from c, hence

$$d(c, C(f_0, z_0) \ldots C(f_k, z_k)(c)) \leq 4R + M.$$

By Lemma 2.3, there is N such that $\|C(f_0, z_0) \ldots C(f_k, z_k)\| \leq N$.

In the following two lemmas we obtain the basic estimates that will be used in the proof of the convergence.

LEMMA 2.6. *Let K be a compact set in $\mathbb{H}^3$. There exists a constant M with the following property. If D is a disc of radius r contained in K, and α, β two geodesics contained in the cylinder over D, then there is an element $A \in SL(2, \mathbb{C})$ such that $A(\alpha) = \beta$ and $\|A - I\| \leq Mr$.*

Proof. It is enough to prove the result for α the geodesic orthogonal to D through its centre because

$$\|AB - I\| \leq \|A - I\|\, \|B\| + \|B - I\|.$$

Since K is compact, we can further assume that $\alpha = \gamma$, $D \cap \alpha = c$ and $b = \beta \cap D$ is the point $(a, 0, \sqrt{(1-a^2)})$, because

$$\|A - I\| \leq \|B^{-1}\| \, \|BAB^{-1} - I\| \, \|B\|.$$

If $\delta = d(c, b)$, ϑ is the angle between β and the perpendicular to D through b and φ the angle between the projection of β onto the plane of D and the geodesic $(t, 0, \sqrt{(1-t^2)})$ we can take A to be

$$\begin{pmatrix} \cosh \frac{1}{2}\delta & \sinh \frac{1}{2}\delta \\ \sinh \frac{1}{2}\delta & \cosh \frac{1}{2}\delta \end{pmatrix} C(\gamma, i\varphi) \begin{pmatrix} \cos \frac{1}{2}\vartheta & \sin \frac{1}{2}\vartheta \\ -\sin \frac{1}{2}\vartheta & \cos \frac{1}{2}\vartheta \end{pmatrix} C(\gamma, -i\varphi).$$

Then

$$\begin{aligned} \|A - I\| \leq |1 - \cosh \tfrac{1}{2}\delta \cos \tfrac{1}{2}\vartheta| + |\cosh \tfrac{1}{2}\delta \sin \tfrac{1}{2}\vartheta| \\ + |\sinh \tfrac{1}{2}\delta \cos \tfrac{1}{2}\vartheta| + |\sinh \tfrac{1}{2}\delta \sin \tfrac{1}{2}\vartheta)|. \end{aligned}$$

Since β is contained in the cylinder, we have $\delta \leq r$ and $|\sin \vartheta| \leq \tanh r$. For $r < \operatorname{diam}(K)$, there is M such that $\|A - I\| \leq Mr$.

LEMMA 2.7. *Let K be a compact subset of $\mathbb{H}^3$ and $M > 0$. Then there is a constant N with the following property. Let D be a disc in K of radius r and F the hyperbolic cylinder over D. Let $f_0, \ldots, f_k$ be geodesics in F and $z_0, \ldots, z_k$ complex numbers with $\sum_{i=0}^{k} |\operatorname{Re} z_i| < M$. If f is any geodesic in F then*

$$\left\| C(f_0, z_0) \ldots C(f_k, z_k) - C\left(f, \sum_{i=0}^{k} z_i\right) \right\| \leq Nr \sum_{i=0}^{k} |z_i|.$$

Proof. It is clearly sufficient to prove the result for f the geodesic orthogonal to D through its centre, and since K is compact we can assume that $f = \gamma$ and $\gamma \cap D = c$. Let A_i be the matrix given by Lemma 2.5, mapping f to f_i. Then

$$\|C(f_i, z_i) - C(f, z_i)\| = \|A_i C(f, z_i) A_i^{-1} - C(f, z_i)\|.$$

By Lemma 2.4, there is N_1 depending only on K and M such that

$$\|C(f_i, z_i) - C(f, z_i)\| \leq N_1 \, \|A_i - I\| \, |z_i|$$

and, by Lemma 2.6, there exists a constant N_2 depending only on K such that

$$\|C(f_i, z_i) - C(f, z_i)\| \leq N_1 N_2 r \, |z_i|.$$

Write C_i for $C(f_i, z_i)$. We have

$$\begin{aligned} \left\| C_0 \ldots C_k - C\left(f, \sum_{i=0}^{k} z_i\right) \right\| &\leq \|C_0 \ldots C_k - C_0 \ldots C_{k-1} C(f, z_k)\| \\ &\quad + \|C_0 \ldots C_{k-1} C(f, z_k) - C_0 \ldots C_{k-2} C(f, z_{k-1} + z_k)\| \\ &\quad + \ldots + \left\| C_0 C\left(f, \sum_{i=1}^{k} z_i\right) - C\left(f, \sum_{i=0}^{k} z_i\right) \right\| \\ &\leq \|C_0 \ldots C_{k-1}\| \, \|C_k - C(f, z_k)\| \\ &\quad + \|C_0 \ldots C_{k-2}\| \, \|C_{k-1} - C(f, z_{k-1})\| \, \|C(f, z_k)\| \\ &\quad + \ldots + \|C_0 - C(f, z_0)\| \left\| C\left(f, \sum_{i=1}^{k} z_i\right) \right\|. \end{aligned}$$

By Lemma 2.5 there is N_3 such that

$$\left\| C_0 \dots C_k - C\left(f, \sum_{i=0}^{k} z_i\right) \right\| \leq N_3 \sum_{i=0}^{k} \|C_i - C(f, z_i)\| \leq N_1 N_2 N_3 r \sum_{i=0}^{k} |z_i|.$$

The following lemma implies that $\{Z_n\}$ converges and that the limit is independent of the choice of the sequence of finite approximations.

LEMMA 2.8. *Let $\varepsilon > 0$. There exists $N > 0$ with the following property. For $n > N$, let P_n be a partition of $[x, y]$ defined as in Section 2.B, $\{f_i, z_i\}$, $i = 0, \dots, p$, a finite approximation to (Λ, μ) subordinate to P_n, and $Z_n = C(f_0, z_0) \dots C(f_p, z_p)$. If Q is a refinement of P_n with no interior points of $[x, y]$ of positive measure, $\{h_i, v_i\}$, $i = 0, \dots, q$, a finite approximation of (Λ, μ) subordinate to Q, and $Z_Q = C(h_0, v_0) \dots C(h_q, v_q)$, then*

$$\|Z_n - Z_Q\| < \varepsilon.$$

Proof. Let $I_i \subset [x, y]$ be the subsegment associated with the leaf f_i (I_i consists of one or two of the subsegments of P_n), and similarly $J_j \subset [x, y]$ the subsegment associated with the leaf h_j. Let $h_{i_1}, \dots h_{i_m}$ be the leaves of $\{h_j, v_i\}$ which intersect I_i and put $\bar{v}_{i_k} = \mu(J_{i_k} \cap I_i)$. Define $B_i = C(h_{i_1}, \bar{v}_{i_1}) \dots C(h_{i_m}, \bar{v}_{i_m})$. The only cases where $\bar{v}_{i_k} \neq v_{i_k}$ occur when $h_{i_k} \cap [x, y] \subset P_n$. Since no interior points of P_n have positive measure we have $Z_Q = B_0 \dots B_p$.

$$\begin{aligned}
\|Z_n - Z_Q\| &= \|C_0 \dots C_p - B_0 \dots B_p\| \\
&\leq \|C_0 \dots C_p - C_0 \dots C_{p-1} B_p\| \\
&\quad + \dots + \|C_0 \dots C_k B_{k+1} \dots B_p - C_0 \dots C_{k-1} B_k \dots B_p\| \\
&\quad + \dots + \|C_0 B_1 \dots B_p - B_0 \dots B_p\| \\
&\leq \|C_0 \dots C_{p-1}\| \, \|C_p - B_p\| \\
&\quad + \dots + \|C_0 \dots C_{k-1}\| \, \|C_k - B_k\| \, \|B_{k+1} \dots B_p\| \\
&\quad + \dots + \|C_0 - B_0\| \, \|B_1 \dots B_p\|.
\end{aligned}$$

By Lemma 2.5 there is a constant N_1, and by Lemma 2.7 there is a constant N_2 such that

$$\|Z_n - Z_Q\| \leq N_1^2 N_2 \frac{2}{n} \sum_{i,j} |\bar{v}_{i_j}|.$$

It is therefore enough to take

$$N > \frac{N_1^2 N_2}{\varepsilon} |\mu| ([x, y]).$$

THEOREM 2.9. *Given a closed hyperbolic surface $S \to \mathbb{H}^2/\Gamma$, a measured lamination (Λ_S, μ_S) on S and a quasi-Fuchsian structure $\rho_0 : \Gamma \to \Gamma_0$, for any pair of points $x, y \in \mathbb{H}^2$, the sequence $\{Z_n\}$ constructed in Section 2.B, converges in $PSL(2, \mathbb{C})$ to $Z(x, y)$, the limit is independent of the choice of the sequence and defines a (Γ, ρ_0)-cocycle Z on $\mathbb{H}^2$.*

Proof. The convergence of Z_n and the independence of $Z(x, y)$ from the choice of the sequence of finite approximations are immediate consequences of Lemma 2.8. Condition (a) for a cocycle holds obviously. Condition (b) is proved by constructing finite approximations for the sublamination of Λ consisting of the leaves intersecting $[x, y]$ and $[y, z]$ and then applying Lemma 2.1. Condition (c) is proved similarly.

The (Γ, ρ_0)-cocycle Z determined by the measured lamination (Λ_S, μ_S) defines a section homomorphism as in 1.C.,

$$\rho_\mu(A) = Z(c, Ac)\rho_0(A).$$

Let $\tilde{\psi}: \mathbb{H}^2 \to \mathbb{H}^3$ be the lift of the pleated surface representing a maximal lamination containing Λ_S, as in Lemma 2.5. Then the (Γ, ρ_0)-cocycle Z determines a mapping $\tilde{\psi}_\mu: \mathbb{H}^2 \to \mathbb{H}^3$ by

$$\tilde{\psi}_\mu(x) = Z(c, x)\tilde{\psi}(x).$$

The restriction of $\tilde{\psi}_\mu$ to each stratum of Λ is an isometry. The mapping $\tilde{\psi}_\mu$ is continuous except at points $x \in \mathbb{H}^2$ for which $\operatorname{Re} \mu(\{x\}) > 0$.

For $t \in \mathbb{C}$, let $t\mu$ denote the measure μ multiplied by t, and define a deformation of ρ_0 by $t \mapsto \rho_{t\mu}$. The space of quasi-Fuchsian structures is open in the space of discrete faithful representations of Γ (see [**4**]). Hence there is an open neighbourhood U of 0 such that if $t \in U$, $\rho_{t\mu}$ gives a quasi-Fuchsian structure in $Q(S)$. In [**2**], I determined a lower bound for the size of U when Λ_S is a finite lamination and ρ_0 is Fuchsian: if $(\Lambda_S, \mu_S) = \{f_i, z_i\}$, and a is the injectivity radius of the lamination $\{f_i\}$ in $\mathbb{H}^2/\Gamma_0$, then $\rho_{t\mu}$ is quasi-Fuchsian provided that $|\operatorname{Im}(tz_i)| < \pi$ and $\cos \frac{1}{2}(\operatorname{Im}(tz_i)) > 1/\cosh a$. In [**2, 3**] I give examples where this estimate is the best possible.

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