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A DUALITY THEOREM FOR NONDIFFERENTIABLE CONVEX PROGRAMMING WITH OPERATORIAL CONSTRAINTS

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A duality theorem of Wolfe for non-linear differentiable programming is now extended to minimization of a non-differentiable, convex, objective function defined on a general locally convex topological linear space with a non-differentiable operatorial constraint, which is regularly subdifferentiable. The gradients are replaced by subgradients. This extended duality theorem is then applied to a programming problem where the objective function is the sum of a positively homogeneous, lower semi continuous, convex function and a subdifferentiable, convex function. We obtain another duality theorem which generalizes a result of Schechter.

1. Introduction

The following pair of programming problems has been studied by Wolfe [9]:

(P) minimize $f(x)$

subject to $h_i(x) \geq 0$, $i = 1, \ldots, m$;

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$$\text{(D)} \quad \text{maximize} \quad f(x) - \sum_{i=1}^{m} u_i h_i(x)$$

$$\text{subject to} \quad u \geq 0 \quad \text{and} \quad \nabla f(x) = \sum_{i=1}^{m} u_i \nabla h_i(x) \; .$$

Here f is a convex function on R^n and the h_i's are concave functions. f and h_i are assumed differentiable. Furthermore a constraint qualification is assumed satisfied. Then Wolfe has proved the duality theorem that if x_0 is optimal for (P), there exists a vector u_0 such that (x_0, u_0) is optimal for (D) and furthermore the two problems have the same extremal value. Geoffrian [3] and Rockafellar [7] have studied duality theory without differentiability in a direction different from that of Wolfe's. On the other hand Mond and Schechter [5] have studied some particular problems very much in the spirit of Wolfe.

In this paper we derive a duality theorem in Section 3, very much like Wolfe's in a general locally convex topological linear space. Here we do not assume differentiability, and we replace functional constraints by operatorial constraints and gradients by subgradients. Finally in Section 4, by applying this duality theorem to a programming problem where the objective function is the sum of a positively homogeneous, lower semi continuous, convex function and a subdifferentiable, convex function, we get another duality theorem which generalizes a result of Schechter [8].

2. Preliminaries

In this paper V and V^*, as well as Y and Y^*, shall be pairs of real vector spaces in duality, with their respective weak topologies. Thus all the spaces will be locally convex spaces. We let $C \subset Y$ be a closed convex cone defining a partial order in Y- for $x, y \in Y$; $x \leq y$ if $y-x \in C$. (When Y is R, it is understood that the cone C is $[0, \infty)$.) C^* shall stand for the polar-cone namely,

$$C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \text{ for every } y \in C\} \; .$$

Let A be a non-empty closed convex subset of V, and let $G : A \to Y$. G is said to be *convex* if $G(tx+(1-t)y) \leq tG(x) + (1-t)G(y)$ for all $x, y \in A$ and $0 \leq t \leq 1$.

A continuous linear map $T : V \to Y$ is said to be a *subgradient of* G *at a point* $u_0 \in A$ if $T(u-u_0) \leq G(u) - G(u_0)$ for every $u \in A$. The set of all subgradients of G at u_0 is called the *subdifferential of* G *at* u_0 and is denoted by $\partial G(u_0)$.

G is said to be *regularly subdifferentiable at* u_0 if $\partial(y^* \circ G)(u_0) = y^* \circ \partial G(u_0)$ for every $y^* \in C^*$ [1]. If G is regularly subdifferentiable at every point of A, then G is said to be regularly subdifferentiable on A.

3. The duality theorem

Let $J : A \to R$ be a lower semi continuous, convex function, and let $G : A \to Y$ be a convex operator, which is regularly subdifferentiable on A.

Let $U = \{u \in A : G(u) \leq 0\}$ be non-empty.

The primal problem (P) is

$$(P) \qquad \inf_{u \in U} J(u) .$$

The proof of the following theorem can be found in [1], [2].

THEOREM 1. *Let* $\inf_{u \in U} J(u)$ *be finite, and assume that there is a* $u_0 \in A$ *such that* $G(u_0) < 0$ *(that is,* $-G(u_0)$ *is an interior point of* C *). Then* $\bar{u} \in A$ *is a solution of* (P) *if and only if there is* $\bar{p}^* \in C^*$ *such that* $(\bar{u}, \bar{p}^*)$ *satisfies*

$$(1) \quad J(\bar{u}) + \langle p^*, G(\bar{u})\rangle \leq J(\bar{u}) + \langle \bar{p}^*, G(\bar{u})\rangle \leq J(u) + \langle \bar{p}^*, G(u)\rangle$$

for every $u \in A$, $p^* \in C^*$. *Further, in this case,* $\langle \bar{p}^*, G(\bar{u})\rangle = 0$.

NOTE. From the second inequality in (1), it follows that $\bar{u}$ is a minimum point for the function $(J+\bar{p}^* \circ G)(u)$, and hence $0 \in \partial(J+\bar{p}^* \circ G)(\bar{u})$ ([4], page 81).

Consequently, we have the following generalized Kuhn-Tucker theorem for operatorial constraints.

THEOREM 2. *If we further assume that* G *is continuous at some point* *in* A, *then* $\bar{u} \in A$ *is a solution of* (P) *if, and only if, there is* $\bar{p}^* \in C^*$ *such that* $\langle \bar{p}^*, G(\bar{u})\rangle = 0$ *and* $0 \in \partial J(\bar{u}) + \bar{p}^* \circ \partial G(\bar{u})$.

This is so because, if G is continuous at some point in A, then, by the Morean-Rockafellar theorem [6],

$$\begin{aligned}\partial(J+\bar{p}^* \circ G)(\bar{u}) &= \partial J(\bar{u}) + \partial(\bar{p}^* \circ G)(\bar{u}) \\ &= \partial J(\bar{u}) + \bar{p}^* \circ \partial G(\bar{u}) ,\end{aligned}$$

since G is regularly subdifferentiable on A .

Based on Theorem 2, we define the following dual problem $(\mathcal{D})$:

$$(\mathcal{D}): \quad \text{maximize} \quad J(u) + \langle y^*, G(u)\rangle$$
$$\text{subject to} \quad y^* \in C^* , \text{ and } \quad 0 \in \partial J(u) + y^* \circ \partial G(u) .$$

Now we have the following analogue of Wolfe's duality theorem [9] in the case of operatorial constraints.

THEOREM 3. *Assume the hypotheses of Theorems* 1 *and* 2. *If* u_0 *is a solution for problem* (P), *then there exists* $y_0^* \in Y^*$ *such that* (u_0, y_0^*) *is a solution for problem* $(\mathcal{D})$. *Furthermore, the two problems have the same extremal value.*

Proof. By Theorem 2, feasible solutions exist for $(\mathcal{D})$.

Let (x, y^*) be a feasible solution for problem $(\mathcal{D})$. Then $y^* \geq 0$, and there exist $v \in \partial J(u)$ and $T \in \partial G(u)$ such that $0 = v + y^* \circ T$.

Now

$$\begin{aligned}J(u_0) - [J(u)+\langle y^*, G(u)\rangle] \\ &\geq \langle v, u_0-u\rangle - \langle y^*, G(u)\rangle = -\langle y^* \circ T, u_0-u\rangle - \langle y^*, G(u)\rangle \\ &\geq \langle y^*, G(u)-G(u_0)\rangle - \langle y^*, G(u)\rangle = -\langle y^*, G(u_0)\rangle \\ &\geq 0 ,\end{aligned}$$

since $y^* \geq 0$, and $G(u_0) \leq 0$. Thus

$$(2) \qquad J(u_0) \geq J(u) + \langle y^*, G(u)\rangle$$

for any feasible solution (u, y^*) for problem $(\mathcal{D})$. Since u_0 is an optimal solution of problem (P), we have from Theorem 2, that there exists $y_0^* \in C^*$ such that $\langle y_0^*, G(u_0)\rangle = 0$ and $0 \in \partial J(u_0) + y_0^* \circ \partial G(u_0)$. In other words, (u_0, y_0^*) is a feasible solution for $(\mathcal{D})$. Hence

(3) $$J(u_0) = J(u_0) + \langle y_0^*, G(u_0)\rangle .$$

This shows that from (2) and (3), (u_0, y_0^*) is an optimal solution for $(\mathcal{D})$, and that the two problems have the same extremal value.

4. Applications

We next apply the above theorem to the case where the objective function is the sum of a positively homogeneous, lower semi continuous, convex function and a subdifferentiable convex function.

We shall need the following definition and propositions.

DEFINITION. Let A be a subset of a locally convex space V^*. Then the *support function of* A, denoted by $S(\cdot/A)$ is defined by

$$S(u/A) = \sup\{\langle u, u^*\rangle : u^* \in A\} .$$

NOTE. Let F be a positively homogeneous, lower semi continuous, convex function, defined on a locally convex space V. Then

$$\partial F(0) = \{u^* \in V^* : F(u) \geq \langle u, u^*\rangle \text{ for all } u \in V\} ,$$

since $F(0) = 0$.

The following proposition is proved in ([4], page 192):

PROPOSITION 1. *Let* F *be a positively homogeneous, lower semi continuous, convex function defined on a locally convex space* V. *Then* F *is the support function of* $\partial F(0)$.

REMARK. Note that $\partial F(0)$ is a non-empty, convex, compact subset of V^*. In fact, there is a one to one correspondence between compact convex subsets of V^* and positively homogeneous, lower semi continuous, convex functions on V.

PROPOSITION 2. *Let* F *be a positively homogeneous, lower semi continuous, convex function defined on a locally convex space* V; *and let* $u \neq 0$. *Then*

(4) $$\partial F(u) = \{u^* \in \partial F(0) : F(u) = \langle u, u^*\rangle\} .$$

This follows from Proposition 1, and the result that $u^* \in \partial F(u)$ if, and only if, $F(u) + F^*(u^*) = \langle u, u^*\rangle$ ([4], page 198), where F^* denotes the conjugate function of F.

Let the objective function $J : A \to \mathbb{R}$ be of the form $J = F_1 + F_2$, where F_1 is a positively homogeneous, lower semi continuous, convex function and F_2 is a convex function and let F_2 be continuous at some point of A. Also, let $G : A \to Y$ be regularly subdifferentiable on A.

The primal problem (P_1) is

(P_1): minimize $J(u)$ subject to $G(u) \le 0$.

Let $(\mathcal{D}_1)$ and $(\mathcal{D}_2)$ denote the following dual problems:

$(\mathcal{D}_1)$: maximize $F_2(u) + \langle w^*, u\rangle + \langle y^*, G(u)\rangle$

subject to $y^* \in C^*$, $w^* \in \partial F_1(0)$, $\langle w^*, u\rangle = F_1(u)$

and $0 \in \partial F_2(u) + w^* + y^* \circ \partial G(u)$;

$(\mathcal{D}_2)$: maximize $F_2(u) + \langle w^*, u\rangle + \langle y^*, G(u)\rangle$

subject to $y^* \in C^*$, $w^* \in \partial F_1(0)$

and $0 \in \partial F_2(u) + w^* + y^* \circ \partial G(u)$.

THEOREM 4. *If u_0 is optimal for (P_1), then there exist y_0^* and w_0^* such that (u_0, y_0^*, w_0^*) is optimal for $(\mathcal{D}_2)$. Further, the two problems, have the same extremal value.*

Proof. Since u_0 is optimal for (P_1), by Theorem 2, there exists $y^* \in C^*$ such that $\langle y^*, G(u_0)\rangle = 0$ and $0 \in \partial J(u_0) + y^* \circ \partial G(u_0)$. But $\partial J(u_0) = \partial F_1(u_0) + \partial F_2(u_0)$ by the Moreau-Rockafellar theorem [6]. Also $\partial F_1(u_0) = \{u^* \in \partial F_1(0) : F_1(u_0) = \langle u_0, u^*\rangle\}$ by (4). Therefore, $0 \in \partial F_2(u_0) + \{u^* \in \partial F_1(0) : F_1(u_0) = \langle u_0, u^*\rangle\} + y^* \circ \partial G(u_0)$. Hence there is $w^* \in \partial F_1(0)$ satisfying $F_1(u_0) = \langle u_0, w^*\rangle$ such that $0 \in \partial F_2(u_0) + w^* + y^* \circ \partial G(u_0)$. Thus feasible solutions for $(\mathcal{D}_2)$ exist.

Let (u, y^*, w^*) be any feasible solution for $(\mathcal{D}_2)$. Then $y^* \in C^*$, $w^* \in \partial F_1(0)$ and there exist $v \in \partial F_2(u)$ and $T \in \partial G(u)$ such that $0 = v + w^* + y^* \circ T$.

Now

$$F_1(u_0) + F_2(u_0) - [\langle w^*, u\rangle + F_2(u) + \langle y^*, G(u)\rangle]$$
$$\geq [F_2(u_0) - F_2(u)] + [\langle w^*, u_0\rangle - \langle w^*, u\rangle] - \langle y^*, G(u)\rangle \quad (\text{since } w^* \in \partial F_1(0))$$
$$\geq \langle v, u_0 - u\rangle + \langle w^*, u_0 - u\rangle - \langle y^*, G(u)\rangle$$
$$= \langle v + w^*, u_0 - u\rangle - \langle y^*, G(u)\rangle$$
$$= -\langle y^* \circ T, u_0 - u\rangle - \langle y^*, G(u)\rangle$$
$$= -\langle y^*, T(u_0 - u)\rangle - \langle y^*, G(u)\rangle$$
$$\geq -\langle y^*, G(u_0) - G(u)\rangle - \langle y^*, G(u)\rangle \quad (\text{since } T \in \partial G(u))$$
$$= -\langle y^*, G(u_0)\rangle \geq 0 \quad (\text{since } y^* \in C^*, \ -G(u_0) \in C) .$$

Thus $F_1(u_0) + F_2(u_0) \geq \langle w^*, u\rangle + F_2(u) + \langle y^*, G(u)\rangle$ for every feasible solution (u, y^*, w^*) of $(\mathcal{D}_2)$.

Now, since u_0 is optimal for (P_1), there are $y_0^* \in C^*$, $w_0^* \in \partial F_1(0)$ satisfying $F_1(u_0) = \langle u_0, w_0^*\rangle$ such that $0 \in \partial F_2(u_0) + w_0^* + y_0^* \circ \partial G(u_0)$ and such that $\langle y_0^*, G(u_0)\rangle = 0$.

Hence

$$F_1(u_0) + F_2(u_0) + \langle y_0^*, G(u_0)\rangle \geq \langle w^*, u\rangle + F_2(u) + \langle y^*, G(u)\rangle$$

for every feasible solution (u, y^*, w^*) of $(\mathcal{D}_2)$. That is, (u_0, y_0^*, w_0^*) is optimal for $(\mathcal{D}_2)$.

Clearly, the extremal values of the two problems are the same.

REMARKS. (1) In Theorem 4, if u_0 is optimal for (P_1), then the (u_0, y_0^*, w_0^*) which has been obtained optimizing $(\mathcal{D}_2)$, in fact, also optimizes $(\mathcal{D}_1)$.

(2) Theorem 4 generalizes a result of Schechter [8].

References

[1] V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces* (Sijthoff & Noordhoff, The Netherlands, 1978).

[2] E. Ivar Ekeland, Roger Teman, *Convex analysis and variational problems* (Studies in Mathematics and its Applications, 1. North-Holland, Amsterdam, Oxford; American Elsevier, New York; 1976).

[3] A.M. Geoffrian, "Duality in nonlinear programming: a simplified applications-oriented development", *SIAM Rev.* 13 (1971), 1-37.

[4] A.D. Ioffe, V.M. Tihomirov, *Theory of extremal problems* (Studies in Mathematics and its Applications, 6. North-Holland, Amsterdam, New York, Oxford, 1979).

[5] Bertram Mond and Murray Schechter, "A programming problem with an L_p norm in the objective function", *J. Austral. Math. Soc. Ser. B* 19 (1976), 333-342.

[6] R.T. Rockafellar, "Extension of Fenchel's duality theorem for convex functions", *Duke Math. J.* 33 (1966), 81-89.

[7] R. Tyrrell Rockafellar, *Conjugate duality and optimization* (Regional Conference Series in Applied Mathematics, 16. Society for Industrial and Applied Mathematics, Philadelphia, 1974).

[8] Murray Schechter, "A subgradient duality theorem", *J. Math. Anal. Appl.* 61 (1977), 850-855.

[9] Philip Wolfe, "A duality theorem for non-linear programming", *Quart. Appl. Math.* 19 (1961), 239-244.

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