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COMPARISON THEOREMS OF LIAPUNOV-RAZUMIKHIN TYPE FOR NFDE S WITH INFINITE DELAY

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ABSTRACT. Some comparison theorems of Liapunov-Razumikhin type are provided for uniform (asymptotic) stability and uniform (ultimate) boundedness of solutions to neutral functional differential equations with infinite delay with respect to a given phase space pair. Examples are given to illustrate how the comparison theorems and stability and boundedness of solutions depend on the choice(s) of phase space(s) and are related to asymptotic behavior of solutions to some difference and integral equations.

1. **Introduction.** In [19] and [20], Seifert provided informative examples to illustrate that Liapunov-Razumikhin type results related to (uniform) asymptotic stability for functional differential equations (FDEs) with finite delay do not always carry over readily to an infinite delay setting. Nevertheless, there have been some successful efforts to extend Liapunov-Razumikhin theory to retarded FDEs with infinite delay. Among these efforts are [4–6], [8], [11–13], [15], [18] and [21].

For neutral functional differential equations (NFDEs)

$$\frac{d}{dt}D(t,x_t) = f(t,x_t), \tag{1.1}$$

the generalization of Liapunov-Razumikhin theorems becomes even more difficult, since the behavior of solutions depends heavily on the property of the so-called generalized difference equation

$$D(t,x_t) = h(t). \tag{1.2}$$

In the case where the delay is finite, under the restriction that the above D-operator is linear in the second argument and stable, Lopes [16] and [29] obtained some stability and boundedness results of Liapunov-Razumikhin type for finite delay NFDEs. In [25], one of the authors established some comparison principles of Liapunov-Razumikhin type for NFDEs with infinite delay on the space BC of bounded continuous functions with supremum norm.

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As indicated in [7] among other places, there are certain drawbacks in restricting the setting to the space BC mentioned above. Along these lines one of the purpose of this paper is to establish stability and boundedness comparison theorems of Liapunov-Razumikhin type in a unified way so that the results are available both for general admissible phase spaces and the space BC of bounded continuous functions.

We obtain several comparison theorems of Liapunov-Razumikhin type for (uniform, asymptotic) stability and (uniform, ultimate) boundedness with respect to a given phase space pair (X, Y) with the intention that various choices of phase space pairs allow us to apply the obtained comparison theorems to study the existence of periodic solutions (see, *cf.* [28]) as well as the precompactness of positive orbits (see, *cf.* [7]) for NFDEs with infinite delay. The comparison theorems given here are reminiscent of those provided by Lakshmikantham and Leela [14] and Lopes [16] for finite delay retarded and neutral FDEs. However, it should be pointed out that the technicalities involved in dealing with neutral FDEs with infinite delay and the associated D-operator are by nature more complicated than those for retarded FDEs. Similarly, the difficulties are compound from the fact that we must be concerned also with the choice(s) of phase space(s) for infinite delay equations.

The rest of this paper is organized as follows. In Section 2, we introduce the definition and examples of a fundamental phase space which will be used throughout this paper. Section 3 is devoted to a precise description of stability and boundedness with respect to a given phase space pair. Several examples are also given to show that there is often a natural way to follow in connection with the choices involved in phase space selection. In Section 4, we prove several comparison theorems for uniform stability and uniform boundedness, and we show by examples how sufficient conditions of stability and boundedness depend on the choices of phase spaces. In Section 5, we use an argument of [26] to investigate asymptotic behaviors of the D-operator associated with a class of neutral integrodifferential and difference equations. Finally, in Section 6 we prove several comparison results for uniformly asymptotic stability and uniformly ultimate boundedness which include the classical Liapunov-Razumikhin theorems and a theorem in [23] as special cases.

2. **Phase spaces and** NFDEs **with infinite delay.** Let $\tilde{B}$ be a linear space of R^n-valued functions on $(-\infty, 0]$ with a semi-norm $p(\cdot)$ so that the quotient space $B = \tilde{B}/p(\cdot)$ of the equivalent classes of elements of $\tilde{B}$ under the norm $|\cdot|_B$ induced by $p(\cdot)$ is a Banach space.

The space $(B, |\cdot|_B)$ is called a *fundamental phase space*, if it satisfies the following conditions:

(B1) there exists a constant $K > 0$ such that $|\tilde{\phi}(0)| \leq Kp(\tilde{\phi})$ for all $\tilde{\phi} \in \tilde{B}$ and

(B2) for any $t_0 \in R, \delta > 0$ and any function $x: (-\infty, t_0 + \delta) \longrightarrow R^n$ with $x_{t_0} \in \tilde{B}$ and $x: [t_0, t_0 + \delta] \longrightarrow R^n$ continuous, we have $x_t \in \tilde{B}$ for all $t \in [t_0, t_0 + \delta]$, where x_t is defined by $x_t(s) = x(t + s)$ for $s \in (-\infty, 0]$.

Among the typical and important fundamental phase spaces are the spaces R^n, C_r, C_g and BC, which are defined below.

(a) R^n is the usual n-dimensional Euclidean space with the norm $|\cdot|$. It can be regarded as a fundamental phase space where $\tilde{B}$ is the linear space of all R^n-valued functions on $(-\infty,0]$ with the semi-norm $p(\cdot)$ defined by $p(\tilde{\phi}) = |\tilde{\phi}(0)|$.

(b) For any positive constant $r > 0$, $C_r = C([-r,0],R^n)$ is defined as the space of continuous R^n-valued functions with the sup-norm $|\cdot|_r$ defined by

$$|\phi|_r = \sup_{-r\le s\le 0} |\phi(s)|,$$

where $\tilde{B}$ is the linear space of all R^n-valued continuous functions on $(-\infty,0]$ with the semi-norm $p(\cdot)$ defined by $p(\tilde{\phi}) = |\tilde{\phi}_{[-r,0]}|_r$.

(c) For any function $g\colon(-\infty,0] \longrightarrow [1,\infty)$ satisfying the following condition:

(g1) $g\colon(-\infty,0] \longrightarrow [1,\infty)$ is a continuous nonincreasing function such that $g(0) = 1$.

Let C_g denote the space of continuous functions which map $(-\infty,0]$ into R^n such that

$$\sup_{s\le 0} \frac{|\phi(s)|}{g(s)} < \infty.$$

Then C_g equipped with the norm

$$|\phi|_{C_g} = \sup_{s\le 0} \frac{|\phi(s)|}{g(s)}$$

is a fundamental phase space.

(d) BC is a special case of C_g with $g(s) = 1$ for all $s \in (-\infty,0]$. For BC, the C_g norm is denoted by $|\cdot|_\infty$, that is

$$|\phi|_\infty = \sup_{s\le 0} |\phi(s)|.$$

Throughout this paper, we consider the following neutral functional differential equations with infinite delay

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{2.1}$$

subject to the following Cauchy initial condition

$$x_{t_0} = \phi \tag{2.2}$$

in which $t \ge t_0 \ge 0$, B is a fundamental phase space, and $D,f\colon[0,\infty)\times B \longrightarrow R^n$. A *solution* of (2.1)–(2.2) (denoted by $x(t_0,\phi)(\cdot)$) is defined as a function $x\colon(-\infty,t_0+\delta) \longrightarrow R^n$, $\delta > 0$, so that $x_{t_0} = \phi \in B$, $x\colon[t_0,t_0+\delta) \longrightarrow R^n$ is continuous, $D(t,x_t)$ is differentiable on $[t_0,t_0+\delta)$ and (2.1) is satisfied on $[t_0,t_0+\delta)$.

For fundamental results on existence, uniqueness, continuation and continuous dependence, we refer to [22] (for admissible phase spaces), [24] (for bounded continuous function spaces) and [27] (for general fundamental phase spaces).

We conclude this section by recalling some notation.

(e) For a continuous function $V: [0,\infty) \times R^n \longrightarrow [0,\infty)$, the derivative of V along solutions of (2.1) is defined as

$$\dot V_{(2.1)}\big(t, D(t,x_t)\big) = \limsup_{h\to 0^+} \frac{1}{h}\Big[V\big(t+h, D\big(t+h, x_{t+h}(t,x_t)\big)\big) - V\big(t, D(t,x_t)\big)\Big],$$

where $x: R \longrightarrow R^n$ is a solution of (2.1).

(f) A *wedge* Q is a continuous strictly increasing function $Q: [0,\infty) \longrightarrow [0,\infty)$ with $Q(0) = 0$.

(g) An *unbounded pseudo wedge* S is an unbounded strictly increasing continuous function $S: [0,\infty) \longrightarrow [0,\infty)$.

3. **Uniform stability and uniform boundedness: definitions and examples.** Throughout the remainder of this paper, let X, Y and B be fundamental phase spaces with $X \underset{=}{\subset} B$ and $Y \underset{=}{\subset} B$. We assume that $D(t,0) = f(t,0) = 0$ for all $t > 0$ so that equation (2.1) possesses the *zero solution*.

In this section, we introduce the concepts of stability and boundedness of neutral equation (2.1) and its associated D-operator with respect to a phase space pair (X, Y). Examples are given to illustrate the dependence of these concepts on the choice of phase space.

DEFINITION 3.1. The zero solution of (2.1) is (X, Y)-*uniformly stable* if, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $[\phi \in X, |\phi|_X < \delta$ and $t \geq t_0 \geq 0]$ imply $|x_t(t_0,\phi)|_Y < \varepsilon$.

DEFINITION 3.2. An operator $D: [0,\infty) \times B \longrightarrow R^n$ is (X, Y)-*uniformly stable* if there exists a wedge Q such that for any $t_0 \in [0,\infty)$, $x: R \longrightarrow R^n$ and $\varepsilon > 0$ so that $x_{t_0} \in X$, $x: [t_0,\infty) \longrightarrow R^n$ being continuous, $|x_{t_0}|_X < Q(\varepsilon)$ and $\sup_{t\geq t_0} |D(t,x_t)| < Q(\varepsilon)$, we have $|x_t|_Y < \varepsilon$ for all $t \geq t_0$.

DEFINITION 3.3. The solutions of (2.1) are (X, Y)-*uniformly bounded* if, for any $\alpha > 0$ there exists a $\beta > 0$ such that $[\phi \in X, |\phi|_X \leq \alpha$ and $t \geq t_0 \geq 0]$ imply $|x_t(t_0,\phi)|_Y < \beta$.

Here, and in what follows, when discussing the boundedness of solutions to equation (2.1), we do not require that $D(t,0) = 0$ and $f(t,0) = 0$ for all $t > 0$.

DEFINITION 3.4. An operator $D: [0,\infty) \times B \longrightarrow R^n$ is (X, Y)-*uniformly bounded* if there exists an unbounded pseudo wedge S such that for any $t_0 \in [0,\infty)$, $x: R \longrightarrow R^n$ and $H > 0$ so that $x_{t_0} \in X$, $x: [t_0,\infty) \longrightarrow R^n$ being continuous, $|x_{t_0}|_X < H$ and $\sup_{t\geq t_0} |D(t,x_t)| < H$, we have $|x_t|_Y < S(H)$ for $t \geq t_0$.

REMARK 3.5. Usually it will be the case that either $X = Y = B$ or $X = B$ and $Y = R^n$, where R^n is regarded as an embedded subspace of constant functions in B. That is, if $c \in R^n$, then the corresponding element in B is the constant function ϕ_c defined by $\phi_c(s) = c, s \leq 0$. In this case $|x_t|_Y$ is merely $|x(t)|$.

Now we consider some examples for which the left-hand side of (2.1) has the form

$$\frac{d}{dt}\Big[x(t) - \sum_{i=1}^{\infty} B_i x(t-r_i) - \int_{-\infty}^{0} G(-s)x(t+s)\,ds - e(t)\Big],$$

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where $e\colon [0,\infty) \longrightarrow R^n$ is bounded, continuous and

$$M = \sup_{0 \le t < \infty} |e(t)|.$$

In this case the operator D reads

$$D(t,\phi) = \phi(0) - \sum_{i=1}^{\infty} B_i\phi(-r_i) - \int_{-\infty}^{0} G(-s)\phi(s)\,ds - e(t). \tag{3.1}$$

A special case, of course, is the "finite delay" left-hand side $\frac{d}{dt}[x(t) - cx(t-r)]$, which often is found in the literature.

PROPOSITION 3.6. *Suppose that B_i $(i = 1,2,\ldots)$ are $n \times n$ constant matrices, $G\colon [0,\infty) \longrightarrow R^{n\times n}$ is continuous, $\{r_i\}$ is an increasing unbounded sequence of positive numbers, and*

$$\sum_{i=1}^{\infty} |B_i| + \int_{-\infty}^{0} |G(-s)|\,ds = m < 1. \tag{3.2}$$

Then for any $m^* \in (m,1)$, we can find a function $g\colon (-\infty,0] \longrightarrow [1,\infty)$ satisfying **(g1)***,*

(g2) *$\frac{g(s+u)}{g(s)} \longrightarrow 1$ as $u \longrightarrow 0^+$ uniformly for $s \in (-\infty,0]$, and*

(g3) *$g(s) \longrightarrow \infty$ as $s \longrightarrow -\infty$*

so that the D-operator defined by (3.1) is (C_g, R^n)-uniformly bounded with $S(H) = \frac{H+M}{1-m^*}$. Moreover, if $e \equiv 0$, then the D-operator defined by (3.1) is (C_g, R^n)-uniformly stable with $Q(\varepsilon) = (1-m^*)\varepsilon$.

PROOF. In [7], it is shown that (3.2) implies the existence of a function $g\colon (-\infty,0] \longrightarrow [1,\infty)$ satisfying **(g1)**, **(g2)** and **(g3)** such that

$$\sum_{i=1}^{\infty} |B_i| g(-r_i) + \int_{-\infty}^{0} |G(-s)| g(s)\,ds \le m^*. \tag{3.3}$$

For any $t_0 \in [0,\infty)$, $x\colon R \longrightarrow R^n$ and $H > 0$ with $x_{t_0} \in C_g$, $x\colon [t_0,\infty) \longrightarrow R^n$ being continuous, $|x_{t_0}|_{C_g} < H$ and $\sup_{t\ge t_0} |D(t,x_t)| < H$, let

$$h(t) = x(t) - \sum_{i=1}^{\infty} B_i x(t-r_i) - \int_{-\infty}^{t} G(t-s)x(s)\,ds - e(t).$$

If there exists a $\tau > t_0$ such that $|x(\tau)| = \max_{t_0\le s\le \tau} |x(s)|$, then

$$|x(\tau)| = \left|\sum_{i=1}^{\infty} B_i x(\tau - r_i) + \int_{-\infty}^{\tau} G(-s)x(s)\,ds + e(\tau) + h(\tau)\right|.$$

Let K be an integer so that $r_K \le \tau - t_0 < r_{K+1}$. We have

$$
\begin{aligned}
|x(\tau)| &\le \sum_{i=1}^{K} |B_i x(\tau - r_i)| + \sum_{i=K+1}^{\infty} |B_i x(t_0 + \tau - t_0 - r_i)| \\
&\quad + \int_{-\infty}^{t_0} |G(\tau - s)| x(s)\, ds + \int_{t_0}^{\tau} |G(\tau - s)| x(s)\, ds + H + M \\
&\le \Big[\sum_{i=1}^{K} |B_i| + \int_{t_0}^{\tau} |G(\tau - s)|\, ds\Big] |x(\tau)| \\
&\quad + \sum_{i=K+1}^{\infty} |B_i| g(-r_i) \frac{g(\tau - t_0 - r_i)}{g(-r_i)} \frac{|x_{t_0}(\tau - t_0 - r_i)|}{g(\tau - t_0 - r_i)} \\
&\quad + \int_{-\infty}^{0} |G(\tau - t_0 - u)| g(t_0 + u - \tau) \frac{g(u)}{g(t_0 + u - \tau)} \frac{|x_{t_0}(u)|}{g(u)} du + H + M \\
&\le \Big[\sum_{i=1}^{K} |B_i| + \int_{t_0 - \tau}^{0} |G(-s)|\, ds\Big] |x(\tau)| \\
&\quad + \Big[\sum_{i=K+1}^{\infty} |B_i| g(-r_i) + \int_{-\infty}^{t_0 - \tau} |G(-s)| g(s)\, ds\Big] |x_{t_0}|_{C_g} + H + M \\
&\le \Big[\sum_{i=1}^{\infty} |B_i| g(-r_i) + \int_{-\infty}^{0} |G(-s)| g(s)\, ds\Big] \max\{|x(\tau)|, |x_{t_0}|_{C_g}\} + H + M \\
&\le m^* \max\{|x(\tau)|, |x_{t_0}|_{C_g}\} + H + M
\end{aligned}
$$

by (3.3), since $g(s) \ge 1$ for all $s \in (-\infty, 0]$ by **(g1)**. Now, if $|x(\tau)| > |x_{t_0}|_{C_g}$, it follows from the above inequality that

$$|x(\tau)| \le m^* |x(\tau)| + H + M,$$

which implies

$$|x(\tau)| \le \frac{H + M}{1 - m^*}.$$

Therefore

$$|x(\tau)| \le \max\Big(|x_{t_0}|_{C_g}, \frac{H + M}{1 - m^*}\Big) = \frac{H + M}{1 - m^*}.$$

This proves the (C_g, R^n)-uniform boundedness of D. Similarly, we can prove the (C_g, R^n)-uniform stability of D when $e = 0$. ∎

Following the proof of Proposition 3.6, one can show the following

PROPOSITION 3.7. *Under the assumptions of Proposition 3.6, the D-operator defined by (3.1) is (BC, R^n)-uniformly bounded with $S(H) = \frac{H+M}{1-m}$ and is (BC, R^n)-uniformly stable when $e = 0$ with $Q(\varepsilon) = (1 - m)\varepsilon$.*

The next result allows more flexible B_i's and G and includes the previous propositions. For the sake of brevity, the proof is omitted.

PROPOSITION 3.8. *Suppose that* $B_i\colon [0,\infty)\times R^n \longrightarrow R^n$, $G\colon [0,\infty)\times R\times R^n \longrightarrow R^n$ *and* $e\colon [0,\infty)\longrightarrow R^n$ *are continuous. Suppose also that there exist constants* $K>0$ *and* $k_1\in[0,1)$ *such that*

(i) *for any* $t\geq t_0\geq 0$ *and* $\phi\in X$, *we have*

$$\sum_{i=1}^{\infty}\sup_{-r_i\leq s\leq 0}\left|B_i\big(t,\phi(s)\big)\right| + \int_{-\infty}^{0}\left|G\big(t,t_0+s,\phi(s)\big)\right| ds \leq K|\phi|_X;$$

(ii) *for any continuous* $x\colon [t_0,\infty)\longrightarrow R^n$, *we have*

$$\sum_{i=1}^{N}\left|B_i\big(t,x(t-r_i)\big)\right| + \int_{t_0}^{t}\left|G\big(t,s,x(s)\big)\right| ds \leq k_1\sup_{t_0\leq s\leq t}|x(s)|,$$

where N is an integer so that $r_N\leq t-t_0\leq r_{N+1}$.

Then the D-operator

$$D(t,\phi)=\phi(0)-\sum_{i=1}^{\infty}B_i\big(t,\phi(-r_i)\big)-\int_{-\infty}^{0}G\big(t,t+s,\phi(s)\big)\,ds-e(t)$$

is (X,R^n)*-uniformly bounded with* $S(H)=\frac{(1+K)H+M}{1-k_1}$ *if* $\sup_{t\geq 0}|e(t)|\leq M<\infty$, *and is* (X,R^n)*-uniformly stable with* $Q(\varepsilon)=\frac{1-k_1}{1+K}\varepsilon$ *if* $e=0$.

We conclude this section with a simple result to contrast with the finite delay equations.

PROPOSITION 3.9. *If there exist constants* $K_1,K_2\geq 0$ *such that for* $t_0\in[0,\infty)$, $x\colon R\longrightarrow R^n$ *with* $x_{t_0}\in X$, $x\colon [t_0,\infty)\longrightarrow R^n$ *being continuous and*

$$|x_t|_Y\leq K_1|x_{t_0}|_X+K_2\sup_{t_0\leq s\leq t}|D(t,x_t)|.$$

Then the operator D is (X,Y)*-uniformly stable and* (X,Y)*-uniformly bounded with* $Q(\varepsilon)=\frac{\varepsilon}{K_1+K_2}$ *and* $S(H)=(K_1+K_2)H$, *respectively.*

It follows that if $D\colon [0,\infty)\times C_r\longrightarrow R^n$ is stable in the sense of Lopes [16] (or [9]), then this D-operator is (C_r,R^n) (or (C_r,C_r))-uniformly bounded and (C_r,R^n) (or (C_r,C_r))-uniformly stable.

4. **Uniform stability and uniform boundedness: comparison theorems.**

In this section, we are concerned with comparison theorems and their applications to uniform stability and uniform boundedness. We start with a uniform boundedness theorem.

THEOREM 4.1. *Suppose the operator D is* (X,R^n)*-uniformly bounded and there exist unbounded pseudo wedges* $W_i (i=1,2,3)$, *a constant* $M>0$ *and continuous functions* $V\colon [0,\infty)\times R^n\longrightarrow[0,\infty)$, $W\colon [0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ *such that*

(i) $|D(t,\phi)|\leq W_3(|\phi|_X)$;

(ii) $W_1(|x|)\leq V(t,x)\leq W_2(|x|)$;

(iii) for any $(t_0,\phi)\in[0,\infty)\times X$ *and any* $x\colon R\longrightarrow R^n$ *with* $x_{t_0}=\phi$ *and* $x\colon[t_0,\infty)\longrightarrow R^n$ *continuous, we have*

$$\dot V_{(2.1)}\Big(t,D(t,x_t)\Big)\le W\Big(t,V\big(t,D(t,x_t)\big)\Big) \tag{4.1}$$

at $t\ge t_0$ *where* $\max\{|x_{t_0}|_X,\sup_{t_0\le s\le t}|x(s)|\}\le S\circ W_1^{-1}\circ V\big(t,D(t,x_t)\big)$ *and* $V\big(t,D(t,x_t)\big)\ge M$*, here* S *is given in Definition 3.4;*

(iv) the solutions of $\dot z=W(t,z)$ *are uniformly bounded.*

Then the solutions of (2.1) are (X,R^n)*-uniformly bounded.*

PROOF. For any $(t_0,\phi)\in[0,\infty)\times X$ and $\alpha\ge 0$, if $|\phi|_X\le\alpha$, then

$$|D(t_0,\phi)|\le W_3(|\phi|_X)\le W_3(\alpha)$$

and

$$V\big(t_0,D(t_0,\phi)\big)\le W_2\big(|D(t_0,\phi)|\big)\le W_2\circ W_3(\alpha).$$

Choose $\gamma=\max\{\alpha,M,W_1(\alpha),W_2\circ W_3(\alpha),W_1\circ S^{-1}(\alpha)\}$, by assumption (iv), there exists a $\beta_1>0$ such that for any $t_0\ge 0$, if $z(t)$ is the maximal solution of

$$\begin{cases}\dot z(t)=W\big(t,z(t)\big)\\ z(t_0)=\gamma,\end{cases}$$

then $|z(t)|\le\beta_1$ for all $t\ge t_0$.

We claim that

$$V\big(t,D(t,x_t)\big)\le z(t) \tag{4.2}$$

for all $t\ge t_0$. Suppose it is not true, then we can find a positive integer m and a real number $t_1>t_0$ such that

$$V\big(s,D(s,x_s)\big)<z_m(s)$$

for $s\in[t_0,t_1)$, and

$$V\big(t_1,D(t_1,x_{t_1})\big)=z_m(t_1),$$

and there exists a sequence $\tau_n\longrightarrow t_1^+$ such that

$$V\big(\tau_n,D(\tau_n,x_{\tau_n})\big)>z_m(\tau_n)$$

for $n=1,2,\ldots$, where $z_m(t)$ is a solution of the initial value problem

$$\begin{cases}\dot z_m(t)=W\big(t,z_m(t)\big)+\frac{1}{m}\\ z_m(t_0)=\gamma\end{cases}$$

(see, *cf.* [14]). Therefore

$$\begin{aligned}\dot V\big(t_1,D(t_1,x_{t_1})\big)&\ge\lim_{n\to\infty}\frac{V\big(\tau_n,D(\tau_n,x_{\tau_n})\big)-V\big(t_1,D(t_1,x_{t_1})\big)}{\tau_n-t_1}\\&\ge\lim_{n\to\infty}\frac{z_m(\tau_n)-z_m(t_1)}{\tau_n-t_1}\\&=\dot z_m(t_1)\\&=W\big(t_1,z_m(t_1)\big)+\frac{1}{m}\\&=W\Big(t_1,V\big(t_1,D(t_1,x_{t_1})\big)\Big)+\frac{1}{m}.\end{aligned} \tag{4.3}$$

On the other hand, since $z_m(t)$ is increasing, it follows that

$$W_1\Big(|D(s,x_s)|\Big) \le V\Big(s, D(s,x_s)\Big) \le z_m(s) \le z_m(t_1)$$

for $t_0 \le s \le t_1$, that is

$$|D(s,x_s)| \le W_1^{-1}\big(z_m(t_1)\big)$$

for $t_0 \le s \le t_1$. Since $\gamma \ge W_1(\alpha)$, we have

$$|\phi|_X \le \alpha \le W_1^{-1}(\gamma) = W_1^{-1}\big(z_m(t_0)\big) \le W_1^{-1}\big(z_m(t_1)\big).$$

By the Definition 3.4, we have

$$|x(s)| \le S \circ W_1^{-1}\big(z_m(t_1)\big)$$

for $t_0 \le s \le t_1$. Moreover, the choice of γ gives

$$|\phi|_X \le \alpha \le S \circ W_1^{-1}(\gamma) \le S \circ W_1^{-1}\big(z_m(t_1)\big).$$

Hence (4.1) implies that

$$\dot V\big(t_1, D(t_1, x_{t_1})\big) \le W\Big(t_1, V\big(t_1, D(t_1, x_{t_1})\big)\Big),$$

which contradicts (4.3). Therefore, (4.2) must hold. Thus we have that for all $t \ge t_0$

$$V\big(t, D(t,x_t)\big) \le z(t) \le \beta_1,$$

which implies

$$|D(t,x_t)| \le W_1^{-1}(\beta_1)$$

for all $t \ge t_0$. Since $\gamma \le \beta_1$, and hence $|\phi|_X \le \alpha \le W_1^{-1}(\beta_1)$, by the (X,R^n)-uniform boundedness of the operator D, we have

$$|x(t)| \le \beta = S \circ W_1^{-1}(\beta_1).$$

This completes the proof. ∎

THEOREM 4.2. *Suppose the operator D is (X,R^n)-uniformly stable and there exist wedges $W_i (i = 1,2,3)$, a constant $M > 0$ and continuous functions $V: [0,\infty) \times R^n \longrightarrow [0,\infty)$, $W: [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ such that*

(i) $|D(t,\phi)| \le W_3(|\phi|_X)$;

(ii) $W_1(|x|) \le V(t,x) \le W_2(|x|)$;

(iii) for any $(t_0,\phi) \in [0,\infty) \times X$ and any $x: R \longrightarrow R^n$ with $x_{t_0} = \phi$ and $x: [t_0,\infty) \longrightarrow R^n$ continuous, we have

$$\dot V_{(2.1)}\big(t, D(t,x_t)\big) \le W\Big(t, V\big(t, D(t,x_t)\big)\Big)$$

at $t \ge t_0$ where $\max\{|x_{t_0}|_X, \sup_{t_0 \le s \le t} |x(s)|\} \le Q^{-1} \circ W_1^{-1} \circ V\big(t, D(t,x_t)\big)$, here Q is given in Definition 3.2;

(iv) the zero solution of $\dot{z} = W(t,z)$ is uniformly stable.
Then the zero solution of (2.1) is (X, R^n)-uniformly stable.

The above theorems allow us to determine the uniform boundedness and uniform stability for NFDEs with infinite delay. In the following example, we examine (BC, R^n)- and (C_g, R^n)-uniform boundedness (stability) of certain integrodifferential equations. In the C_g case we see how the choice of underlying phase space is involved in determining the uniform boundedness and uniform stability.

EXAMPLE 4.3. Consider the following linear nonhomogeneous Volterra integrodifferential equation

$$\begin{aligned}\frac{d}{dt}\Big[x(t) - \sum_{i=1}^{\infty} B_i x(t-r_i) - \int_{-\infty}^{t} G(t-s)x(s)\,ds - f(t)\Big]& \\ = Ax(t) + \sum_{i=1}^{\infty} A_i x(t-r_i) + \int_{-\infty}^{t} H(t-s)x(s)\,ds + k(t).&\end{aligned} \tag{4.4}$$

We assume that

(1) A is a stable $n \times n$ constant matrix. That is, there exist an $n \times n$ positive definite matrix P and constants $\beta \geq \alpha > 0$ such that

$$A^T P + PA = -I, \quad \alpha^2 x^T x \leq x^T P x \leq \beta^2 x^T x;$$

(2) $f, k \colon (-\infty, +\infty) \longrightarrow R^n$ are continuous and $|f(t)| \leq M_1$, $|k(t)| \leq M_2$ for some constants $M_1, M_2 > 0$;

(3) $\sum_{i=1}^{\infty} |B_i| + \int_{-\infty}^{0} |G(-s)|\,ds = m < 1$;

(4) $\sum_{i=1}^{\infty} |A_i| + \int_{-\infty}^{0} |H(-s)|\,ds < \infty$.

Let $V(x) = x^T P x$. Then for the D-operator defined by

$$D(t,\phi) = \phi(0) - \sum_{i=1}^{\infty} B_i \phi(-r_i) - \int_{-\infty}^{0} G(t-s)\phi(s)\,ds - f(t),$$

we have

$$\begin{aligned}\dot{V}_{(4.4)}\big(t, D(t,x_t)\big) &= D^T(t,x_t)P\Big\{AD(t,x_t) + \sum_{i=1}^{\infty}(A_i + AB_i)x(t-r_i) \\ &\quad + \int_{-\infty}^{t}[H(t-s) + AG(t-s)]x(s)\,ds + k(t) + Af(t)\Big\} \\ &\quad + \Big\{AD(t,x_t) + \sum_{i=1}^{\infty}(A_i + AB_i)x(t-r_i) \\ &\quad + \int_{-\infty}^{t}[H(t-s) + AG(t-s)]x(s)\,ds + k(t) + Af(t)\Big\}^T PD(t,x_t) \\ &= -D^T(t,x_t)D(t,x_t) + 2D^T(t,x_t)P\Big\{\sum_{i=1}^{\infty}(A_i + AB_i)x(t-r_i) \\ &\quad + \int_{-\infty}^{t}[H(t-s) + AG(t-s)]x(s)\,ds + k(t) + Af(t)\Big\}.\end{aligned}$$

By Proposition 3.7, D is (BC, R^n)-uniformly bounded with $S(H) = \frac{H+M_1}{1-m}$. If

$$|x(s)| \leq S([D^T(t,x_t)PD(t,x_t)]^{\frac{1}{2}}/\alpha)$$

for $s \leq t$, then

$$|x(s)| \leq \frac{\frac{\beta}{\alpha}|D(t,x_t)| + M_1}{1-m}$$

for $s \leq t$, and therefore

$$\begin{aligned}
\dot{V}_{(4.4)}\big(t, D(t,x_t)\big) \leq & -D^T(t,x_t)D(t,x_t) + 2|D(t,x_t)|\,|P|\Big[\sum_{i=1}^{\infty}|A_i + AB_i| \\
& + \int_{-\infty}^{t} |H(t-s) + AG(t-s)|\,ds\Big] \frac{\frac{\beta}{\alpha}|D(t,x_t)| + M_1}{1-m} \\
& + 2|D(t,x_t)|\,|P|\,|k(t) + Af(t)| \\
\leq & -\left(1 - \frac{2\beta|P|u_1}{(1-m)\alpha}\right)|D(t,x_t)|^2 \\
& + 2|P|\left(\frac{u_1M_1}{1-m} + M_2 + |A|M_1\right)|D(t,x_t)|,
\end{aligned}$$

where

$$u_1 = \sum_{i=1}^{\infty}|A_i + AB_i| + \int_{-\infty}^{0}|H(-s) + AG(-s)|\,ds.$$

Thus, by Theorem 4.1 we have

PROPOSITION 4.4. *Under the conditions (1)–(4), if*
(i) $u_1 < \frac{(1-m)\alpha}{2\beta|P|}$, *or*
(ii) $u_1 \leq \frac{(1-m)\alpha}{2\beta|P|}$ *and* $f(t) = k(t) = 0$ *for* $t \geq 0$,
then the solutions of (4.4) are (BC, R^n)*-uniformly bounded.*

REMARK 4.5. Using Theorem 4.2, we can prove that if assumption (ii) in Proposition 4.4 holds, then the zero solution of (4.4) is (BC, R^n)-uniformly stable.

We now consider the (C_g, R^n)-uniform boundedness and uniform stability of (4.4). We will see how the choice of phase space enters into the considerations. Let $m < m^* < 1$. By a result in [7], there exists a continuous function $g: (-\infty, 0] \longrightarrow [1, \infty)$ satisfying (**g1**), (**g2**) and (**g3**) such that

$$\sum_{i=1}^{\infty}|B_i|g(-r_i) + \int_{-\infty}^{0}|G(-s)|g(s)\,ds \leq m^*$$

and

$$\sum_{i=1}^{\infty}|A_i|g(-r_i) + \int_{-\infty}^{0}|H(-s)|g(s)\,ds < \infty.$$

By Proposition 3.6, D is (C_g, R^n)-uniformly bounded with

$$S(H) = \frac{H + M_1}{1 - m^*}.$$

If

$$|x_{t_0}|_{C_g} \le \frac{1}{1-m^*}\left(\frac{[D^T(t,x_t)PD(t,x_t)]^{\frac{1}{2}}}{\alpha} + M_1\right)$$

and

$$|x(s)| \le \frac{1}{1-m^*}\left(\frac{[D^T(t,x_t)PD(t,x_t)]^{\frac{1}{2}}}{\alpha} + M_1\right)$$

for $t_0 \le s \le t$, then

$$\begin{aligned}
\dot{V}_{(4.4)}\big(t, D(t,x_t)\big) &\le -D^T(t,x_t)D(t,x_t) + 2|D(t,x_t)|\,|P|\Big[\sum_{i=1}^{\infty}|A_i + AB_i|g(-r_i) \\
&\quad + \int_{-\infty}^{0}|H(-s) + AG(-s)|g(s)\,ds\Big]\frac{1}{1-m^*}\Big[\frac{\beta}{\alpha}|D(t,x_t)| + M_1\Big] \\
&\quad + 2|D(t,x_t)|\,|P|(M_2 + |A|M_1) \\
&= -\Big(1 - \frac{2\beta|P|u_2}{(1-m^*)\alpha}\Big)|D(t,x_t)|^2 \\
&\quad + 2|P|\Big(\frac{u_2M_1}{1-m^*} + M_2 + |A|M_1\Big)|D(t,x_t)|,
\end{aligned}$$

where

$$u_2 = \sum_{i=1}^{\infty}|A_i + AB_i|g(-r_i) + \int_{-\infty}^{0}|H(-s) + AG(-s)|g(s)\,ds.$$

Therefore, by Theorem 4.1 we have

PROPOSITION 4.6. *If*

(i) $u_2 < \frac{(1-m^*)\alpha}{2\beta|P|}$, *or*

(ii) $u_2 \le \frac{(1-m^*)\alpha}{2\beta|P|}$ *and* $f(t) = k(t) = 0$ *for* $t \ge 0$,

then the solutions of (4.4) are (C_g, R^n)*-uniformly bounded.*

REMARK 4.7. Using Theorem 4.2, we can prove that if assumption (ii) in Proposition 4.6 holds, then the zero solution of (4.4) is (C_g, R^n)-uniformly stable.

It is interesting to note that the (C_g, R^n)-uniform boundedness (stability) under the conditions in Proposition 4.6 implies the (BC, R^n)-uniform boundedness (stability) by Proposition 4.5.

5. Uniform asymptotic stability and uniform ultimate boundedness: definitions and examples. In this section, we introduce the concepts and sufficient conditions of asymptotic stability and ultimate boundedness of neutral equation (2.1) and its associated D-operator with respect to a given phase space pair (X, Y).

DEFINITION 5.1. The origin (X, Y)*-attracts the solutions of (2.1) uniformly* if, for any $M > 0$ and $\eta > 0$ there exists a $T(\eta, M) > 0$ such that for any solution $x(t)$ of (2.1)–(2.2) defined for $t \ge t_0$ with $\max\{|x_{t_0}|_X, \sup_{s \ge t_0}|x(s)|\} \le M$, we have $|x_t(t_0,\phi)|_Y < \eta$ for $t \ge t_0 + T(\eta, M)$.

DEFINITION 5.2. The solutions of (2.1) are *(X, Y)-weakly uniformly ultimately bounded* for bound $B > 0$ if, for any $\beta > 0$ there exists a $T(\beta) > 0$ such that for any solution $x(t)$ of (2.1)–(2.2) defined for $t \geq t_0$ with $\max\{|x_{t_0}|_X, \sup_{s\geq t_0} |x(s)|\} \leq \beta$ we have $|x_t(t_0, \phi)|_Y < B$ for $t \geq t_0 + T(\beta)$.

DEFINITION 5.3. Suppose that $D(t, 0) = f(t, 0) = 0$. The zero solution of (2.1) is *(X, Y)-uniformly asymptotically stable* if, it is (X, Y)-uniformly stable and there exists a constant $\delta_0 > 0$ such that for any $\varepsilon > 0$ there is a $T(\varepsilon) > 0$ so that for any solution $x(t)$ of (2.1)–(2.2) defined for $t \geq t_0$ with $|x_{t_0}|_X < \delta_0$, we have $|x_t(t_0, \phi)|_Y < \varepsilon$ for $t \geq t_0 + T(\varepsilon)$.

DEFINITION 5.4. The solutions of (2.1) are *(X, Y)-uniformly ultimately bounded* for bound $B > 0$, if for any $\alpha > 0$ there exists a $T(\alpha) > 0$ such that for any solution $x(t)$ of (2.1)–(2.2) defined for $t \geq t_0$ with $|x_{t_0}|_X \leq \alpha$, we have $|x_t(t_0, \phi)|_Y < B$ for all $t \geq t_0 + T(\alpha)$.

Obviously, if $D(t, 0) = f(t, 0) = 0$, then the (X, R^n)-uniform stability of the zero solution of (2.1) and the (X, R^n)-uniformly attractivity of the origin imply the (X, R^n)-uniform asymptotic stability of the zero solution of (2.1). Similarly, if the solutions of (2.1) are (X, R^n)-uniformly bounded and (X, R^n)-weakly uniformly ultimately bounded, then the solutions of (2.1) are (X, R^n)-uniformly ultimately bounded.

DEFINITION 5.5. An operator D is called *(X, Y)-pseudo uniformly asymptotically stable* if there exists a wedge P such that for any $\varepsilon, M > 0$ there is a $T_1(\varepsilon, M) > 0$ such that for any $t_0 \in [0, \infty)$, $x\colon R \longrightarrow R^n$ with $x_{t_0} \in X$, $x\colon [t_0, \infty) \longrightarrow R^n$ being continuous, $\max\{|x_{t_0}|_X, \sup_{s\geq t_0} |x(s)|\} \leq M$ and $\sup_{s\geq t_0} |D(s, x_s)| < P(\varepsilon)$, we have $|x_t|_Y < \varepsilon$ for $t \geq t_0 + T_1(\varepsilon, M)$.

An operator D which is both (X, Y)-uniformly stable and (X, Y)-pseudo uniformly asymptotically stable is called *(X, Y)-uniformly asymptotically stable*.

DEFINITION 5.6. An operator D is called *(X, Y)-pseudo uniformly ultimately bounded* if there exists an unbounded pseudo wedge B such that for any $M_1, M_2 > 0$ there is a $T_2(M_1, M_2) > 0$ such that for any $t_0 \in [0, \infty)$, $x\colon R \longrightarrow R^n$ with $x_{t_0} \in X$, $x\colon [t_0, \infty) \longrightarrow R^n$ being continuous, $\max\{|x_{t_0}|_X, \sup_{s\geq t_0} |x(s)|\} \leq M_1$ and $\sup_{s\geq t_0} |D(s, x_s)| \leq M_2$, we have $|x_t|_Y \leq B(M_2)$ for $t \geq T_0 + T_2(M_1, M_2)$.

An operator D which is both (X, Y)-uniformly bounded and (X, Y)-pseudo uniformly ultimately bounded is called *(X, Y)-uniformly ultimately bounded*.

To illustrate the above concepts, we consider the D-operator defined by

$$D(t, \phi) = \phi(0) - \sum_{i=1}^{\infty} B_i\big(t, \phi(-r_i)\big) - \int_{-\infty}^{0} G\big(t, t + u, \phi(u)\big)\, du, \tag{5.1}$$

where $B_i (i = 1, 2, \ldots)\colon [0, \infty) \times R^n \longrightarrow R^n$ and $G\colon [0, \infty) \times R \times R^n \longrightarrow R^n$ are continuous, $\{r_i\}$ is an unbounded increasing sequence of positive real numbers. By a similar argument to that in [23], we obtain the following

PROPOSITION 5.7. *Suppose that there exists a nonnegative constant $l < 1$, and for any $\varepsilon > 0, M > 0$ there exists an integer $K = K(\varepsilon, M) > 0$ such that for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$, $x: [t_0, \infty) \longrightarrow R^n$ being continuous and* $\max\{|x_{t_0}|_X, \sup_{s \geq t_0} |x(s)|\} \leq M$, *we have*

$$\int_{-\infty}^{-r_K} \left|G\big(t, t+s, x_t(s)\big)\right| ds + \sum_{i=K+1}^{\infty} \left|B_i\big(t, x_t(-r_i)\big)\right| < \varepsilon$$

and

$$\int_{-r_K}^{0} \left|G\big(t, t+s, x_t(s)\big)\right| ds + \sum_{i=1}^{K} \left|B_i\big(t, x_t(-r_i)\big)\right| \leq l \max_{t-r_K \leq s \leq t} |x(s)|$$

for $t \geq t_0 + r_K$. Then the D-operator defined by (5.1) is (X, R^n)-pseudo uniformly asymptotically stable with $P(r) = qr$, where q is any given constant in $(0, 1-l)$.

PROOF. Choose $\delta > 0$ so that $l + q + \delta < 1$. Let $K = K(\delta\varepsilon, M)$, $x: R \longrightarrow R^n$ be given with $x_{t_0} \in X$, $x: [t_0, \infty) \longrightarrow R^n$ being continuous and satisfy the following inequalities

$$\max\Big\{|x_{t_0}|_X, \sup_{s \geq t_0} |x(t)|\Big\} \leq M, \quad \sup_{s \geq t_0} |h(s)| < q\varepsilon,$$

where

$$h(t) = x(t) - \sum_{i=1}^{\infty} B_i\big(t, x(t-r_i)\big) - \int_{-\infty}^{t} G\big(t, s, x(s)\big)\, ds.$$

Then for any $t \geq t_0 + r_K$, we have

$$\int_{-\infty}^{t-r_K} \left|G\big(t, s, x(s)\big)\right| ds + \sum_{i=K+1}^{\infty} \left|B_i\big(t, x(t-r_i)\big)\right| < \delta\varepsilon$$

and

$$\int_{t-r_K}^{t} \left|G\big(t, s, x(s)\big)\right| + \sum_{i=1}^{K} \left|B_i\big(t, x(t-r_i)\big)\right| \leq l \max_{t-r_K \leq s \leq t} |x(s)|.$$

Consider now the consecutive intervals $I_n = [t_0 + nr_K, t_0 + (n+1)r_K]$ for $n \geq 1$ and find $t_n \in [t_0 + nr_K, t_0 + (n+1)r_K]$ so that $|x(t_n)| = \max_{s \in I_n} |x(s)|$. Then

$$\begin{aligned} |x(t_n)| &\leq \int_{-\infty}^{t_n - r_K} \left|G\big(t_n, s, x(s)\big)\right| ds + \sum_{i=K+1}^{\infty} \left|B_i\big(t_n, x(t_n - r_i)\big)\right| \\ &\quad + \int_{t_n - r_K}^{t_n} \left|G\big(t_n, s, x(s)\big)\right| + \sum_{i=1}^{K} \left|B_i\big(t_n, x(t_n - r_i)\big)\right| + |h(t_n)| \\ &\leq \delta\varepsilon + q\varepsilon + l \max_{t_n - r_K \leq s \leq t_n} |x(s)|. \end{aligned}$$

Therefore either

$$|x(t_n)| \leq (q+\delta)\varepsilon + l|x(t_{n-1})|$$

if there exists $t^* \in [t_n - r_K, t_0 + nr_K]$ so that $|x(t^*)| = \max_{t_n - r_K \leq s \leq t_n} |x(s)|$, or

$$|x(t_n)| \leq (q+\delta)\varepsilon + l|x(t_n)|$$

if no such t^* exists. So we assert that either

(1) $|x(t_k)| \leq \frac{q+\delta}{1-l}\varepsilon \leq \varepsilon$, for $k \geq N$, where N is some integer, or
(2) $|x(t_n)| \leq (q+\delta)\varepsilon + l|x(t_{n-1})|$, for $n = 2, 3, \ldots$.

In the second case, we have

$$\begin{aligned} |x(t_n)| &\leq (q+\delta)\varepsilon(1 + l + l^2 + \cdots + l^{n-3}) + l^{n-2}|x(t_2)| \\ &\leq \frac{q+\delta}{1-l}\varepsilon + l^{n-2}M \end{aligned}$$

for $n = 2, 3, \ldots$. Choose N^* so that

$$N^* > 2 + \Big[\ln(1 - \frac{q+\delta}{1-l})\varepsilon - \ln M\Big] / \ln l.$$

Then for $n \geq N^*$, we have

$$|x(t_n)| < \frac{q+\delta}{1-l}\varepsilon + \Big(1 - \frac{q+\delta}{1-l}\Big)\varepsilon = \varepsilon.$$

This shows that $|x(t)| < \varepsilon$ for all $t > t_0 + N^* r_K$. This completes the proof. ■

Similarly, we have the following

PROPOSITION 5.8. *Suppose that there exist constants* $C_1, C_2 \geq 0$ *and* $l \in [0, 1)$ *such that for any* $M_1 > 0$ *there exists an integer* $K = K(M_1)$ *so that for any* $x: R \longrightarrow R^n$ *with* $x_{t_0} \in X$, $x: [t_0, \infty) \longrightarrow R^n$ *being continuous and* $\max\{|x_{t_0}|_X, \sup_{s \geq t_0} |x(s)|\} \leq M_1$, *we have*

$$\int_{-\infty}^{-r_K} \Big|G\big(t, t+s, x_t(s)\big)\Big|\, ds + \sum_{i=K+1}^{\infty} \Big|B_i\big(t, x_t(-r_i)\big)\Big| \leq C_1$$

and

$$\int_{-r_K}^{0} \Big|G\big(t, t+s, x_t(s)\big)\Big|\, ds + \sum_{i=1}^{K} \Big|B_i\big(t, x(t - r_i)\big)\Big| \leq l \max_{t - r_K \leq s \leq t} |x(s)| + C_2$$

for $t \geq t_0 + r_K$, *then for any bounded continuous* $f: R \longrightarrow R^n$ *the D- operator*

$$D(t, \phi) = \phi(0) - \sum_{i=1}^{\infty} B_i\big(t, \phi(-r_i)\big) - \int_{-\infty}^{0} G\big(t, t+u, \phi(u)\big)\, du - f(t)$$

is (X, R^n)*-pseudo uniformly ultimately bounded with*

$$B(M_2) = \frac{C_1 + C_2 + M_2 + \sup_{t \in R} |f(t)|}{q},$$

where q is any given constant in $(0, 1 - l)$.

We complete this section by presenting a simple result to contrast with the finite delay case (*cf.* [3]).

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PROPOSITION 5.9. *If there exist constants K_1, K_2 and $a > 0$ such that for any $t_0 \in [0,\infty)$, $x: R \longrightarrow R^n$ with $x_{t_0} \in X$, $x: [t_0,\infty) \longrightarrow R^n$ being continuous, we have*

$$|x_t|_Y \leq K_1 e^{-a(t-t_0)}|x_{t_0}|_X + K_2 \sup_{t_0 \leq s \leq t} |D(s,x_s)|$$

for $t \geq t_0$, then the D-operator is (X,Y)-uniformly asymptotically stable and (X,Y)-uniformly ultimately bounded with $P(\varepsilon) = \alpha\varepsilon$ and $B(M_2) = M_2/\alpha$, respectively, where α is any given constant with $0 < \alpha < 1/K_2$.

REMARK 5.10. By Proposition 5.9, the stable D-operator introduced by Cruz-Hale [3] for NFDEs with finite delay is (C_r, R^n) (or (C_r, C_r))-uniformly asymptotically stable and uniformly ultimately bounded. Moreover, Melvin [17] showed that even for NFDEs with finite delay, the result obtained in Proposition 5.7 and 5.8 is very sharp in the sense that this is the best possible result such that the D-operator retains its stability under appropriate perturbation on r_i. For details we refer to [17].

6. **Uniform asymptotic stability and uniform ultimate boundedness: comparison theorems.** In this section, we provide a general result and several corollaries for uniform asymptotic stability and uniform ultimate boundedness of neutral equations with infinite delay.

To present an as general as possible comparison theorem, we introduce the following concept.

DEFINITION 6.1. Let $W: [0,\infty) \times [0,\infty) \longrightarrow R$ be continuous. The solutions of

$$\dot{z} = W(t,z,\delta) \tag{6.1}$$

are *strongly uniformly asymptotically convergent to zero*, if

(1) for any $\delta, \eta, M > 0$ there exists $S_1(\delta,\eta,M) > 0$ such that for any nonnegative solution $z(t)$ of (6.1) through $(t_0,z_0) \in [0,\infty) \times [0,M]$, we can find $\tau \in [t_0, t_0+S_1(\delta,\eta,M)]$ so that $z(\tau) < \eta$;

(2) for any $\delta, \sigma, M > 0$ there exists $S_2(\delta,\sigma,M) > 0$ such that for any nonnegative solution $z(t)$ of (6.1) through $(t_0,z_0) \in \big[S_2(\delta,\sigma,M),\infty\big) \times [0,M]$, we have $z(t) < z(t_0)+\sigma$ for $t \geq t_0$.

EXAMPLE 6.2. If $W(t,z,\delta) = -W(z)+g(t)$, where $W: [0,\infty) \longrightarrow [0,\infty), g: [0,\infty) \longrightarrow R$ are continuous, $W(x) > 0$ for all $x > 0$ and $\int_0^\infty g(s)\,ds < +\infty$, then the solutions of (6.1) are strongly uniformly asymptotically convergent to zero.

THEOREM 6.3. *Suppose that the operator D is (X,R^n)-pseudo uniformly asymptotically stable, and that there exist wedges $W_i (i = 1,2,3)$, continuous functions $V: [0,\infty) \times R^n \longrightarrow [0,\infty)$ and $W: [0,\infty) \times [0,\infty) \times [0,\infty) \longrightarrow R$ such that*

(i) for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0,\infty) \longrightarrow R^n$ being continuous, we have

$$|D(t,x_t)| \leq W_3\Big(\max\big\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\big\}\Big), \quad t \geq t_0;$$

(ii) $W_1(|x|) \leq V(t,x) \leq W_2(|x|)$;

(iii) for any $M > 0$ *and* $b \geq a > 0$ *there exist* $\delta > 0$ *and* $h > 0$ *such that for any* $x: R \longrightarrow R^n$ *with* $x_{t_0} \in X$ *and* $x: [t_0, \infty) \longrightarrow R^n$ *being continuous, at any* $t \geq t_0 + h$ *with* $\max\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\} \leq M, a \leq V\big(t, D(t,x_t)\big) \leq b$ *and* $V\big(s, x(s)\big) \leq W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t,x_t)\big)\Big) + \delta$ *for* $s \in [t-h, t]$, *we have*

$$\dot{V}_{(2.1)}\big(t, D(t,x_t)\big) \leq W\Big(t, V\big(t, D(t,x_t)\big), \delta\Big),$$

where P *is given in Definition 5.5;*

(iv) the solutions of (6.1) are strongly uniformly asymptotically convergent to zero.

Then the origin (X, R^n)*-attracts solutions of (2.1) uniformly.*

PROOF. Let $M, \eta > 0$ be given and $x(t)$ be a solution of (2.1) defined for $t \geq t_0$ with

$$\max\Big\{|x_{t_0}|_X, \sup_{s \geq t_0} |x(s)|\Big\} \leq M.$$

Then

$$V\big(t, D(t,x_t)\big) \leq W_2 \circ W_3(M)$$

for $t \geq t_0$. Choose $h = h(\eta, M)$ and $\delta = \delta(\eta, M) > 0$ such that if for some $t \geq t_0 + h$, we have

$$\frac{W_1\big(P(\eta)\big)}{2} \leq V\big(t, D(t,x_t)\big) \leq W_2 \circ W_3(M)$$

and

$$V\big(s, x(s)\big) \leq W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t,x_t)\big)\Big) + \delta$$

for $s \in [t-h, t]$, then

$$\dot{V}_{(2.1)}\big(t, D(t,x_t)\big) \leq W\Big(t, V\big(t, D(t,x_t)\big), \delta\Big).$$

For the $\delta > 0$ chosen above, find $\alpha = \alpha(\eta, M) > 0$ and $\beta = \beta(\eta, M) > 0$ so that $\beta < \frac{W_1(P(\eta))}{2}$ and $W_2 \circ P^{-1} \circ W_1^{-1}(s+u) - W_2 \circ P^{-1} \circ W_1^{-1}(s) \leq \delta$ for $\frac{W_1(P(\eta))}{2} \leq s \leq W_2 \circ W_3(M)$ and $0 \leq u \leq \alpha + 2\beta$. Let $N(\eta, M)$ be a positive integer such that

$$W_1\big(P(\eta)\big) + N(\alpha + \beta) \geq W_2 \circ W_3(M),$$

and let

$$\varepsilon_i = W_1\big(P(\eta)\big) + i(\alpha + \beta)$$

for $1 \leq i \leq N$. Then for $t \geq t_0$, we have

$$V\big(t, D(t,x_t)\big) \leq W_2 \circ W_3(M) \leq \varepsilon_N.$$

By the (X, R^n)-pseudo uniformly asymptotic stability, this implies that

$$|x(t)| \leq P^{-1} \circ W_1^{-1}(\varepsilon_N)$$

for $t \geq t_0 + T_1\big(P^{-1} \circ W_1^{-1}(\varepsilon_N), M\big)$.

If $N > 1$ and $V\big(t, D(t, x_t)\big) \geq \varepsilon_{N-1} - \beta$ for all $t \geq t_1^*$, where

$$t_1^* = t_0 + T_1\big(P^{-1} \circ W_1^{-1}(\varepsilon_N), M\big) + h + S_2$$

and

$$S_2 = S_2\big(\delta(\eta, M), \beta(\eta, M), W_2 \circ W_3(M)\big),$$

then we have

$$\frac{W_1\big(P(\eta)\big)}{2} \leq V\big(t, D(t, x_t)\big) \leq W_2 \circ W_3(M) \tag{6.2}$$

for $t \geq t_1^*$, and

$$\begin{aligned} V\big(s, x(s)\big) &\leq W_2\big(|x(s)|\big) \\ &\leq W_2 \circ P^{-1} \circ W_1^{-1}(\varepsilon_N) \\ &\leq W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t, x_t)\big)\Big) \\ &\qquad + W_2 \circ P^{-1} \circ W_1^{-1}(\varepsilon_N) - W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t, x_t)\big)\Big) \\ &\leq W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t, x_t)\big)\Big) + \delta \end{aligned}$$

for $s \in [t - h, t]$ and $t \geq t_1^*$. This implies

$$\dot{V}\big(t, D(t, x_t)\big) \leq W\Big(t, V\big(t, D(t, x_t)\big), \delta\Big)$$

for $t \geq t_1^*$, and thus by the well-known comparison principle (see, *cf.* [14]), we have

$$V\big(t, D(t, x_t)\big) \leq z\big(t; t_1^*, W_2 \circ W_3(M)\big)$$

for $t \geq t_1^*$,where $z\big(t; t_1^*, W_2 \circ W_3(M)\big)$ is the solution of (6.1) through $\big(t_1^*, W_2 \circ W_3(M)\big)$. By assumption (iv) there exists $t_1^{**} \in [t_1^*, t_1^* + S_1]$ such that

$$z\big(t_1^{**}; t_1^*, W_2 \circ W_3(M)\big) < \frac{W_1\big(P(\eta)\big)}{2},$$

where

$$S_1 = S_1\left(\delta(\eta, M), \frac{W_1\big(P(\eta)\big)}{2}, W_2 \circ W_3(M)\right).$$

This implies

$$V\big(t_1^{**}, D(t_1^{**}, x_{t_1^{**}})\big) < \frac{W_1\big(P(\eta)\big)}{2}$$

which is contrary to (6.2).

Therefore there must be a $\tau \in [t_1^*, t_1^* + S_1]$ such that

$$V\big(\tau, D(\tau, x_\tau)\big) < \varepsilon_{N-1} - \beta.$$

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If there exists a $\tau^* > \tau$ so that $V\big(\tau^*, D(\tau^*, x_{\tau^*})\big) \geq \varepsilon_{N-1}$, then there must be a $\tau^{**} \in [\tau, \tau^*]$ such that

$$V\big(\tau^{**}, D(\tau^{**}, x_{\tau^{**}})\big) = \varepsilon_{N-1} - \beta \leq V\big(t, D(t, x_t)\big)$$

for $t \in [\tau^{**}, \tau^*)$. Using the same argument as above, we can prove that

$$\dot{V}\big(t, D(t, x_t)\big) \leq W\Big(t, V\big(t, D(t, x_t)\big), \delta\Big)$$

for $t \in [\tau^{**}, \tau^*)$, thus by assumption (iv) we have

$$\begin{aligned} V\big(\tau^*, D(\tau^*, x_{\tau^*})\big) &\leq z\Big(\tau^*; \tau^{**}, V\big(\tau^{**}, D(\tau^{**}, x_{\tau^{**}})\big)\Big) \\ &< V\big(\tau^{**}, D(\tau^{**}, x_{\tau^{**}})\big) + \beta(\eta, M) = \varepsilon_{N-1}. \end{aligned}$$

This contradicts to $V\big(\tau^*, D(\tau^*, x_{\tau^*})\big) \geq \varepsilon_{N-1}$. Therefore

$$V\big(t, D(t, x_t)\big) < \varepsilon_{N-1}$$

holds for all $t \geq \tau$, hence for all $t \geq t_0 + T_1\big(P^{-1} \circ W_1^{-1}(\varepsilon_N), M\big) + h + S_2 + S_1$.

Following a similar argument, we can prove that

$$V\big(t, D(t, x_t)\big) < \varepsilon_{N-k}$$

for $t \geq t_0 + T_k^*(\eta, M)$, where

$$T_k^*(\eta, M) = \sum_{i=1}^{k} T_1\big(P^{-1} \circ W_1^{-1}(\epsilon_N - i + 1), M\big) + k[h + S_2 + S_1].$$

Thus

$$V\big(t, D(t, x_t)\big) \leq \varepsilon_0 = W_1\big(P(\eta)\big)$$

for $t \geq t_0 + T_N^*(\eta, M)$. It follows that $|x(t)| < \eta$ for $t \geq t_0 + T_N^*(\eta, M) + T_1(\eta, M)$. This completes the proof. ∎

Likewise, we can prove the following

THEOREM 6.4. *Suppose that the D-operator is (X, R^n)-pseudo uniformly ultimately bounded, and that there exists a constant $M > 0$, unbounded pseudo wedges $W_i (i = 1, 2, 3)$, continuous functions $V\colon [0, \infty) \times R^n \longrightarrow [0, \infty)$ and $W\colon [0, \infty) \times [0, \infty) \longrightarrow R$ such that*

(i) for any $x\colon R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x\colon [t_0, \infty) \longrightarrow R^n$ being continuous, we have

$$|D(t, x_t)| \leq W_3\Big(\max\big\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\big\}\Big), \quad t \geq t_0;$$

(ii) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$;

(iii) for any $M_1 > 0$ and $b \geq M$ there exist $\delta > 0$ and $h > 0$ such that for any $x\colon R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x\colon [t_0, \infty) \longrightarrow R^n$ being continuous, at any $t \geq t_0 + h$ with

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$\max\{|x_{t_0}|_X, \sup_{t_0\le s\le t}|x(s)|\} \le M_1$, $a \le V\big(t, D(t,x_t)\big) \le b$ *and* $V\big(s,x(s)\big) \le W_2 \circ B \circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big) + \delta$ *for* $s \in [t-h,t]$, *we have*

$$\dot V_{(2.1)}\big(t, D(t,x_t)\big) \le W\Big(t, V\big(t,D(t,x_t)\big), \delta\Big),$$

where B is given in Definition 5.6;

(iv) the solutions of (6.1) are strongly uniformly ultimately bounded.

Then solutions of (2.1) are (X,R^n)-weakly uniformly ultimately bounded.

Here, by *strongly uniformly ultimate boundedness* of solutions of (6.1), we mean that there exist constants $M^* \ge 0$ and $M^{**} > 0$ such that

(1) for any $\delta > 0$ and $M > M^*$ there exists $S_3(\delta, M) > 0$ such that for any nonnegative solution $z(t)$ of (6.1) through $(t_0, z_0) \in [0,\infty) \times [M^*, M]$, we can find a $\tau \in [t_0, t_0 + S_3(\delta, M)]$ so that $z(\tau) < M^{**}$;

(2) for any $\sigma, M > 0$ there exists $S_4(\sigma, M) > 0$ such that for any nonnegative solution $z(t)$ of (6.1) through $(t_0, z_0) \in [S_4(\sigma,M),\infty) \times [M^*, M]$, we have $z(t) < z(t_0) + \sigma$ for $t \ge t_0$.

We now present some utilizable corollaries of Theorem 6.3 and 6.4. First we notice that Theorem 6.3 contains the classical Liapunov-Razumikhin type theorem.

THEOREM 6.5. *Suppose that the operator D is (X,R^n)-pseudo uniformly asymptotically stable, and that there exist wedges $W_i (i = 1,2,3,4)$, continuous functions $V: [0,\infty) \times R^n \longrightarrow [0,\infty)$ and $q: [0,\infty) \longrightarrow [0,\infty]$ with $q(s) > s$ for $s > 0$, such that*

(i) for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0,\infty) \longrightarrow R^n$ continuous, we have

$$|D(t,x_t)| \le W_3\big(\max\{|x_{t_0}|_X, \sup_{t_0\le s\le t}|x(s)|\}\big), \quad t \ge t_0;$$

(ii) $W_1(|x|) \le V(t,x) \le W_2(|x|)$;

(iii) for any $M > 0$ and $b \ge a > 0$ there exists $h > 0$ such that for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0,\infty) \longrightarrow R^n$ being continuous, at any $t \ge t_0 + h$ with $\max\{|x_{t_0}|_X, \sup_{t_0\le s\le t}|x(s)|\} \le M$, $a \le V\big(t,D(t,x_t)\big) \le b$ and $V\big(s,x(s)\big) \le q \circ W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)$ for $s \in [t-h,t]$, we have

$$\dot V_{(2.1)}\big(t,D(t,x_t)\big) \le -W_4\Big(\big|D(t,x_t)\big|\Big).$$

Then the origin (X,R^n)-attracts solutions of (2.1) uniformly.

PROOF. For any $b \ge a > 0$, define

$$\delta = \inf\{q \circ W_2 \circ P^{-1} \circ W_1^{-1}(s) - W_2 \circ P^{-1} \circ W_1^{-1}(s); a \le s \le b\} > 0.$$

Obviously, if $a \le V\big(t,D(t,x_t)\big) \le b$ and if

$$V\big(s,x(s)\big) \le W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big) + \delta$$

for $s \in [t-h, t]$, then

$$V\big(s, x(s)\big) \le q \circ W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t, x_t)\big)\Big)$$

and thus

$$\dot{V}_{(2.1)}\big(t, D(t, x_t)\big) \le -W_4\big(|D(t, x_t)|\big) \le -W_4 \circ W_2^{-1}\Big(V\big(t, D(t, x_t)\big)\Big).$$

Therefore the origin (X, R^n)-attracts solutions of (2.1) uniformly by Theorem 6.3. ∎

The following theorem, also as a consequence of Theorem 6.3, contains a variant of the main result in [23].

THEOREM 6.6. *Suppose that the operator D is (X, R^n)-pseudo uniformly asymptotically stable and that there exist wedges $W_i (i = 1, 2, \ldots, 5)$, continuous functions $V: [0, \infty) \times R^n \longrightarrow [0, \infty)$, $F: [0, \infty) \times [0, \infty) \times [0, \infty) \longrightarrow R$, $k: [0, \infty) \longrightarrow [0, \infty)$ with $\int_0^\infty k(t)\, dt < \infty$ such that*

(i) for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0, \infty) \longrightarrow R^n$ being continuous, we have

$$|D(t, x_t)| \le W_3\Big(\max\big\{|x_{t_0}|, \sup_{t_0 \le s \le t} |x(s)|\big\}\Big), \quad t \ge t_0;$$

(ii) $W_1(|x|) \le V(t, x) \le W_2(|x|)$;

(iii) $F\big(t, V, W_2 \circ P^{-1} \circ W_1^{-1}(V)\big) \le -W_4(V)$;

(iv) $|F(t, V, N_1) - F(t, V, N_2)| \le W_5(|N_1 - N_2|) + k(t)|N_1 - N_2|$ for $t \ge 0, V \ge 0, N_1, N_2 \ge 0$;

(v) for any $\sigma > 0$ and $M > 0$, there exists $h > 0$ such that for any $N > 0$ and any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0, \infty) \longrightarrow R^n$ being continuous, at any $t \ge t_0 + h$ with $\max\{|x_{t_0}|X, \sup_{t_0 \le s \le t} |x(s)|\} \le M$ and $\sup_{-h \le s \le t} V\big(s, x(s)\big) \le N$, we have

$$\dot{V}_{(2.1)}\big(t, D(t, x_t)\big) \le F\Big(t, V\big(t, D(t, x_t)\big), N\Big) + \sigma.$$

Then the origin (X, R^n)-attracts the solutions of (2.1) uniformly.

PROOF. For any $b \ge a > 0$, choose positive constants $\delta = W_5^{-1}\big(\frac{1}{4}W_4(a)\big)$ and $\sigma = \frac{1}{4}W_4(a)$. Then for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0, \infty) \longrightarrow R^n$ being continuous, at any $t \ge t_0 + h$ with $\max\{|x_{t_0}|X, \sup_{t_0 \le s \le t} |x(s)|\} \le M$, $a \le V\big(t, D(t, x_t)\big) \le b$ and

$$\max_{t-h \le s \le t} V\big(s, x(s)\big) \le W_2 \circ P^{-1} \circ W_1^{-1}\Big(V\big(t, D(t, x_t)\big)\Big) + \delta,$$

we have

$$\begin{aligned}
\dot V_{(2.1)}\big(t,D(t,x_t)\big) &\le F\Big(t, V\big(t,D(t,x_t)\big), W_2\circ P^{-1}\circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)+\delta\Big)+\sigma\\
&\le F\Big(t, V\big(t,D(t,x_t)\big), W_2\circ P^{-1}\circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)\Big)\\
&\quad + F\Big(t, V\big(t,D(t,x_t)\big), W_2\circ P^{-1}\circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)+\delta\Big)\\
&\quad - F\Big(t, V\big(t,D(t,x_t)\big), W_2\circ P^{-1}\circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)\Big)+\sigma\\
&\le -W_4\Big(V\big(t,D(t,x_t)\big)\Big)+W_5(\delta)+\sigma+k(t)\delta\\
&\le -\frac{1}{2}W_4\Big(V\big(t,D(t,x_t)\big)\Big)+k(t)\delta.
\end{aligned}$$

Therefore the origin (X,R^n)-attracts the solutions of (2.1) uniformly by Example 6.2 and Theorem 6.3. ∎

Likewise, by Theorem 6.4, we can prove that

THEOREM 6.7. *Suppose that the D-operator is (X,R^n)-pseudo uniformly ultimately bounded, and that there exist unbounded wedges $W_i (i = 1,2,3,4)$, a constant $M > 0$, and a continuous function $V: [0,\infty)\times R^n \longrightarrow [0,\infty)$ such that (i) and (ii) of Theorem 6.6 hold. Moreover, suppose that either*

(i) for any $\beta > 0$ there exists $h > 0$ such that for any $x: R \longrightarrow R^n$ with $x_{t_0}\in X$ and $x: [t_0,\infty) \longrightarrow R^n$ being continuous, at any $t \ge t_0+h$ with $\max\{|x_{t_0}|_X, \sup_{t_0\le s\le t}|x(s)|\}\le\beta$ and $V\big(t,D(t,x_t)\big)\ge M$ and $V\big(s,x(s)\big)\le q\circ W_2\circ B\circ W_1^{-1}\Big(V\big(t,D(t,x_t)\big)\Big)$ for $s\in[t-h,t]$, we have

$$\dot V_{(2.1)}\big(t,D(t,x_t)\big)\le W_4\big(|D(t,x_t)|\big),$$

where $q: [0,\infty)\longrightarrow[0,\infty)$ is continuous and $q(s) > s$ for $s > 0$, or

(ii) for any $\sigma > 0$ and $\beta > 0$ there exists $h(\sigma) > 0$ such that for any $N > 0$ and any $x: R \longrightarrow R^n$ with $x_{t_0}\in X$ and $x: [t_0,\infty)\longrightarrow R^n$ being continuous, at any $t\ge t_0+h$ with $\max\{|x_{t_0}|_X, \sup_{t_0\le s\le t}|x(s)|\}\le\beta$ and $V\big(t,D(t,x_t)\big)\ge M$ and $V\big(s,x(s)\big)\le N$ for $s\in[t-h,t]$, we have

$$\dot V_{(2.1)}\big(t,D(t,x_t)\big)\le F\Big(t,V\big(t,D(t,x_t)\big),N\Big)+\sigma,$$

where $F: [0,\infty)\times[0,\infty)\times[0,\infty)\longrightarrow R$ satisfies the same conditions as those in Theorem 6.6.

Then the solutions of (2.1) are (X,R^n)-weakly uniformly ultimately bounded.

For illustrative purposes, we now consider the following neutral Volterra integrodifferential equations

$$\begin{aligned}
&\frac{d}{dt}\Big[x(t)-\sum_{i=1}^{\infty}B_i\big(t,x(t-r_i)\big)-\int_{-\infty}^{t}G\big(t,s,x(s)\big)\,ds-f(t)\Big]\\
&\qquad = Ax(t)+\sum_{i=1}^{\infty}A_i\big(t,x(t-r_i)\big)+\int_{-\infty}^{t}H\big(t,s,x(s)\big)\,ds+g(t),
\end{aligned}\tag{6.3}$$

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where

(1) A is an $n \times n$ stable constant matrix, and thus there exist a positive definite $n \times n$ matrix P and constants $\beta \geq \alpha > 0$ so that

$$A^T P + PA = -I, \quad \alpha^2 x^T x \leq x^T P x \leq \beta^2 x^T x;$$

(2) $f, g\colon [0,\infty) \longrightarrow R^n$ are continuous and there exist constants $M_1, M_2 \geq 0$ so that $|f(t)| \leq M_1$ and $|g(t)| \leq M_2$ for $t \geq 0$;

(3) $B_i, A_i\colon [0,\infty) \times R^n \longrightarrow R^n$ are continuous and $B_i(t,0) = A_i(t,0) = 0$;

(4) $G, H\colon [0,\infty) \times R \times R^n \longrightarrow R^n$ are continuous and $G(t,s,0) = H(t,s,0) = 0$.

Let

$$D(t,\phi) = \phi(0) - \sum_{i=1}^{\infty} B_i\big(t, \phi(-r_i)\big) - \int_{-\infty}^{0} G\big(t, t+s, \phi(s)\big)\, ds - f(t) \tag{6.4}$$

and $V(t,x) = x^T P x$. It follows that

$$\begin{aligned}
&\dot V_{(6.3)}\big(t, D(t,x_t)\big)\\
&\quad = -D^T(t,x_t)D(t,x_t) + 2D^T(t,x_t)P\Big\{\sum_{i=1}^{\infty}\big[A_i\big(t, x(t-r_i)\big) + AB_i\big(t, x(t-r_i)\big)\big]\\
&\qquad + \int_{-\infty}^{t}\big[H\big(t,s,x(s)\big) + AG\big(t,s,x(s)\big)\big]\, ds + g(t) + Af(t)\Big\}.
\end{aligned}$$

Therefore, if all conditions of Proposition 3.8 hold, then D is (X, R^n)-uniformly stable with $Q(\epsilon) = \frac{1-k_1}{1+K}\epsilon$ (if $f(t) = 0$) and (X, R^n)-uniformly bounded with $S(H) = \frac{(1+K)H+M_1}{1-k_1}$. Moreover, suppose that there exist $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ such that

(5) for any $t \geq t_0 \geq 0$ and $\phi \in X$, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sup_{-r_i \leq s \leq 0} \big|A_i\big(t, \phi(s)\big) + AB_i\big(t, \phi(s)\big)\big|\\
&\qquad + \int_{-\infty}^{0} \big|H\big(t, t_0+s, \phi(s)\big) + AG\big(t, t_0+s, \phi(s)\big)\big|\, ds\\
&\quad \leq \alpha_1 |\phi|_X.
\end{aligned}$$

(6) for any continuous $x\colon [t_0,\infty) \longrightarrow R^n$ we have

$$\sum_{i=1}^{N}\big|A_i\big(t, x(t-r_i)\big) + AB_i\big(t, x(t-r_i)\big)\big| + \int_{t_0}^{t}\big|H\big(t,s,x(s)\big) + AG\big(t,s,x(s)\big)\big|\, ds \leq \alpha_2 \sup_{t_0 \leq s \leq t} |x(s)|,$$

where N is an integer such that $r_N \leq t - t_0 < r_{N+1}$.

Then

$$\max\big\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\big\} \leq S\Big(\big[V(D(t,x_t))\big]^{1/2}/\alpha\Big)$$

implies

$$\max\big\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\big\} \leq \frac{(1+K)\beta|D(t,x_t)| + \alpha M_1}{\alpha(1-k_1)}.$$

Thus

$$
\begin{aligned}
\dot V_{(6.3)}\big(t, D(t,x_t)\big) &\le -D^T(t,x_t)D(t,x_t) + 2|D(t,x_t)||P|\Big[\alpha_2 \sup_{t_0\le s\le t}|x(s)| \\
&\quad + \alpha_1 |x_{t_0}|_X + M_2 + |A|M_1\Big] \\
&\le -D^T(t,x_t)D(t,x_t) \\
&\quad + 2|D(t,x_t)|\,|P|\Big[\frac{\beta(1+K)}{\alpha(1-k_1)}(\alpha_1+\alpha_2)|D(t,x_t)| \\
&\quad + \frac{M_1(\alpha_1+\alpha_2)}{1-k_1} + M_2 + |A|M_1\Big] \\
&= -\Big(1 - 2|P|\frac{\beta(\alpha_1+\alpha_2)(1+K)}{\alpha(1-k_1)}\Big)D^T(t,x_t)D(t,x_t) \\
&\quad + 2|P|\Big(M_2 + |A|M_1 + \frac{M_1(\alpha_1+\alpha_2)}{1-k_1}\Big)|D(t,x_t)|.
\end{aligned}
$$

Therefore by Theorem 4.1, we have

PROPOSITION 6.8. *Suppose that (1)–(6) and all conditions of Proposition 3.8 hold. Then*

(i) *if* $\alpha_1+\alpha_2 < \frac{(1-k_1)\alpha}{2(1+K)\beta|P|}$, *then the solutions of (6.3) are* (X, R^n)*-uniformly bounded;*

(ii) *if* $\alpha_1+\alpha_2 \le \frac{(1-k_1)\alpha}{2(1+K)\beta|P|}$ *and* $f(t) = g(t) = 0$, *then the solutions of (6.3) are* (X, R^n)*-uniformly bounded and the zero solution of (6.3) is* (X, R^n)*-uniformly stable.*

Now we consider the (X, R^n)-uniformly asymptotic properties of solutions of (6.3) under assumptions (1)–(4). Let all conditions of Proposition 5.7 hold and $f(t) = g(t) = 0$. Then the D-operator is (X, R^n)-pseudo uniformly asymptotically stable with $P(r) = qr$. Moreover, suppose that there exists a constant $\beta_1 > 0$, for any $\varepsilon > 0$ and $M > 0$ there exists an integer $K = K(\varepsilon, M) > 0$ such that

(7) for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X, x: [t_0, \infty) \longrightarrow R^n$ being continuous and

$$\max\Big\{|x_{t_0}|_X, \sup_{s\ge t_0}|x(s)|\Big\} \le M,$$

we have

$$
\begin{aligned}
&\int_{-\infty}^{-r_K}\Big|H\big(t,t+s,x(t+s)\big) + AG\big(t,t+s,x(t+s)\big)\Big|\,ds \\
&\qquad + \sum_{i=K+1}^{\infty}\Big|A_i\big(t,x(t-r_i)\big) + AB_i\big(t,x(t-r_i)\big)\Big| \le \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{-r_K}^{0}\big|H\big(t,t+s,x(t+s)\big) + AG\big(t,t+s,x(t+s)\big)\big|\,ds \\
&\qquad + \sum_{i=1}^{K}\Big|A_i\big(t,x(t-r_i)\big) + AB_i\big(t,x(t-r_i)\big)\Big| \le \beta_1 \sup_{t-r_K\le s\le t}|x(s)|
\end{aligned}
$$

for $t \ge t_0 + r_K$.

Now, for any $M > 0$ and $b \geq a > 0$, choose ε and $\delta > 0$ so that

$$\frac{2\sqrt{b}}{\alpha}|P|\Big(\varepsilon + \beta_1 \frac{\sqrt{\delta}}{\alpha}\Big) \leq \frac{a}{2\beta^2}$$

and let $h = r_K$, $K = K(\varepsilon, M)$. Then for any $x: R \longrightarrow R^n$ with $x_{t_0} \in X$ and $x: [t_0, \infty) \longrightarrow R^n$ being continuous, at any $t \geq t_0 + h$ with

$$\max\Big\{|x_{t_0}|_X, \sup_{t_0 \leq s \leq t} |x(s)|\Big\} \leq M, \quad a \leq V\big(t, D(t, x_t)\big) \leq b$$

and

$$V\big(s, x(s)\big) \leq \beta^2 \frac{V\big(t, D(t, x_t)\big)}{q^2\alpha^2} + \delta$$

for $s \in [t-h, t]$, we have

$$\begin{aligned}\max_{t-h \leq s \leq t} |x(s)| &\leq \Big(\frac{\beta^4 |D(t, x_t)|^2}{\alpha^2 q^2} + \delta\Big)^{1/2} \Big/ \alpha \\ &\leq \frac{\beta^2 |D(t, x_t)|}{\alpha^2 q} + \frac{\sqrt{\delta}}{\alpha},\end{aligned}$$

and thus

$$\begin{aligned}\dot V_{(6.3)}\big(t, D(t, x_t)\big) \leq{}& -D^T(t, x_t) D(t, x_t) \\ &+ 2|D(t, x_t)|\,|P| \Big\{ \sum_{i=1}^{K} \big|A_i\big(t, x(t - r_i)\big) + AB_i\big(t, x(t - r_i)\big)\big| \\ &+ \int_{-r_K}^{0} \big|H\big(t, t+s, x(t+s)\big) + AG\big(t, t+s, x(t+s)\big)\big|\, ds \\ &+ \sum_{i=K+1}^{\infty} \big|A_i\big(t, x(t - r_i)\big) + AB_i\big(t, x(t - r_i)\big)\big| \\ &+ \int_{-\infty}^{-r_K} \big|H\big(t, t+s, x(t+s)\big) + AG\big(t, t+s, x(t+s)\big)\big|\, ds \Big\} \\ \leq{}& -D^T(t, x_t) D(t, x_t) + 2|D(t, x_t)|\,|P| \big(\beta_1 \sup_{t - r_K \leq s \leq t} |x(s)| + \varepsilon\big) \\ \leq{}& -D^T(t, x_t) D(t, x_t) + 2|P|\beta_1 \frac{\beta^2}{\alpha^2 q} |D(t, x_t)|^2 \\ &+ 2|D(t, x_t)|\,|P| \Big(\epsilon + \beta_1 \frac{\sqrt{\delta}}{\alpha}\Big) \\ \leq{}& -\frac{1}{2} D(t, x_t) D(t, x_t) - \frac{a}{2\beta^2} \\ &+ 2|P|\beta_1 \frac{\beta^2}{\alpha^2 q} |D(t, x_t)|^2 + \frac{2\sqrt{b}}{\alpha}|P| \Big(\epsilon + \beta_1 \frac{\sqrt{\delta}}{\alpha}\Big) \\ \leq{}& -\Big(\frac{1}{2} - 2|P|\beta_1 \frac{\beta^2}{\alpha^2 q}\Big) |D(t, x_t)|^2.\end{aligned}$$

Therefore by Theorem 6.3 and 6.4 we have the following

PROPOSITION 6.9. *If all conditions of Proposition 3.8 hold and* $4|P|\beta_1\beta^2 < \alpha^2 q$, *then the origin* (X, R^n)*-attracts solutions of (6.3) uniformly. Moreover, if* $f(t)$ *and* $g(t)$ *are bounded, then the solutions of (6.3) are* (X, R^n)*-weakly uniformly ultimately bounded.*

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