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# FAMILIES OF SEQUENCES OF 0s AND 1s IN *FK*-SPACES

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ABSTRACT. A characterization is given of those families $\mathcal{F}$ of subsets of the positive integers for which $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete, where $\bar{\mathcal{F}}$ denotes the linear span of the characteristic functions of $\mathcal{F}$. Consequences include some known matrix results, such as Hahn's Theorem and conditions required for a matrix to "sum" the characteristic functions of the lacunary sets.

1. **Introduction.** Lacunary sets of positive integers have been of interest in various areas of mathematics. They have long been studied, for example, in Fourier analysis ([5] (Chapter 15), [6]). A recent investigation of their relationship to arithmetic progressions can be found in [4].

By the lacunary sets we shall mean here those infinite sets $A = (n_k)$ of positive integers for which $\lim_{k\to\infty}(n_{k+1} - n_k) = \infty$, together with the finite sets.

It was observed in [7] that the family $\mathcal{L}$ of lacunary sets, as defined above, satisfies the following two properties:

(i) if $(t_k)$ is a sequence of real numbers for which $\sum t_k$ converges on each lacunary set, then $\sum |t_k| < \infty$;

(ii) if $A \subseteq B \in \mathcal{L}$, then $A \in \mathcal{L}$.

The main result in [7] gave necessary and sufficient conditions for the convergence field of an infinite matrix to contain the characteristic functions of any family satisfying conditions (i) and (ii) above. Several known results, including Hahn's Theorem, followed as consequences.

The authors were unable to obtain the results in [7] without imposing condition (ii), there called hereditary.

Here we are able to identify the essential property underlying the work in [7] – it is sequential completeness of a particular weak topology on the space $\ell_1$ of absolutely convergent series. Our work has been largely influenced by that of Bennett and Kalton ([2], [3]), especially with regard to the study of weak topologies on $\ell_1$.

Theorems 1 and 2 characterize those families, both without the hereditary restriction and with, for which the sequential completeness holds. Special cases are obtained by using the fact that the convergence field of any matrix is a separable $FK$-space. One of the special cases is the main result in [7].

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2. **Full families.** We assume throughout that any family $\mathcal{F} \subseteq 2^{\mathbf{N}}$ being considered contains the family $\mathcal{F}_0$ of finite subsets of the positive integers. Corresponding to any $\mathcal{F}$, we are interested in the set of characteristic functions of its members, that is, $\{\chi_S : S \in \mathcal{F}\}$. For notational convenience we usually deal with the linear span of this set of sequences of 0s and 1s, which we will denote by $\bar{\mathcal{F}}$. A family $\mathcal{F}$ will be called *hereditary* in case $A \subseteq B \in \mathcal{F}$ implies $A \in \mathcal{F}$. Recall that the $\beta$-dual of any set of sequences $G$ is defined by

$$G^{\beta} = \left\{ (x_i) : \sum_{i=1}^{\infty} x_i y_i \text{ converges for each } (y_i) \in G \right\}.$$

With the restriction of containing the finite sets, the definition of a full family given in [7] is equivalent to

Definition 1. *A family $\mathcal{F} \subseteq 2^{\mathbf{N}}$ is full in case*
(i) $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$
*and*
(ii) $\mathcal{F}$ *is hereditary.*

As observed in [7], the family of subsets of the positive integers that are lacunary is a full family. Another example is provided by the family of sets having density zero.

Lemma 1. *If $\mathcal{F} \subseteq 2^{\mathbf{N}}$ is full, then $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete.*

Proof.The assumption that $\mathcal{F}$ is hereditary forces $\bar{\mathcal{F}}$ to be a monotone sequence space, i.e., a sequence space algebraically closed under coordinatewise multiplication by all sequences of 0s and 1s. Thus, by [1] (Proposition 3), $\sigma(\bar{\mathcal{F}}^{\beta}, \bar{\mathcal{F}})$ is sequentially complete. The assumption that $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$ forces $\bar{\mathcal{F}}^{\beta} = \ell_1$ (since always $\ell_1 \subseteq \bar{\mathcal{F}}^{\beta}$), and the result follows. □

The proof of Proposition 3 of [1] is rather complicated (It uses, for example, the Grothendieck completion theorem). It seems instructive, therefore, to observe that Lemma 1 can be proved in the following elementary way.

Alternate Proof (of Lemma 1). Assume that $\mathcal{F}$ is full and let $(t^n)$ be a $\sigma(\ell_1, \bar{\mathcal{F}})$-Cauchy sequence in $\ell_1$. Define a matrix $A = (a_{nk})$ by

$$a_{nk} = t_k^n \qquad n, k = 1, 2, \ldots.$$

Then

$$\lim_{n \to \infty} \sum_{k \in S} a_{nk}$$

exists for each $S \in \mathcal{F}$ and it follows from Proposition 6 of [7] that $\lim_{n \to \infty} a_{nk} = a_k$ exists, $k = 1, 2, \ldots, (a_k) \in \ell_1$, and

$$\lim_{n \to \infty} \sum_{k \in S} |a_{nk} - a_k| = 0 \text{ for each } S \in \mathcal{F}.$$

Clearly then $(t^n) \longrightarrow (a_k)$ in $\sigma(\ell_1, \bar{\mathcal{F}})$. The proof of Proposition 6 of [7] uses only elementary gliding hump arguments. □

The converse to Lemma 1 does not hold. The following partial converse is, however, useful in the sequel.

LEMMA 2. *If $\mathcal{F} \subseteq 2^{\mathbf{N}}$ is any family for which $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete, then $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$.*

PROOF. Assume that $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete, and let $t = (t_i) \in \bar{\mathcal{F}}^{\beta}$. Define $(t^n) \subseteq \ell_1$ by $t^n = (t_1, t_2, \ldots, t_n, 0, 0, 0, \ldots)$, $n = 1, 2, \ldots$. Then $(t^n)$ is $\sigma(\ell_1, \bar{\mathcal{F}})$-Cauchy, hence there exists $s = (s_i) \in \ell_1$ for which $(t^n) \longrightarrow s$ in $\sigma(\ell_1, \bar{\mathcal{F}})$. Since $\mathcal{F}$ contains the singleton sets, we have $t_i = \lim_{n\to\infty} t_i^n = s_i$, $i = 1, 2, \ldots$. Thus $t = s$ and $t \in \ell_1$. □

3. **Sequential completeness of $\sigma(\ell_1, \bar{\mathcal{F}})$.** We can now characterize those families $\mathcal{F}$, without the hereditary restriction, for which $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete.

Recall that an $FK$-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate linear functionals are continuous. $FK$-spaces possess many of the properties of Banach spaces, and include most of the matrix domains studied in Summability. For any $FK$-space $E$,

$$W_E = \left\{ x \in E : \sum_{k=1}^{n} x_k \delta^k \longrightarrow x \text{ (weakly)} \right\},$$

$$S_E = \left\{ x \in E : \sum_{k=1}^{n} x_k \delta^k \longrightarrow x \right\}, \quad \text{where } \delta_i^k = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

THEOREM 1. *Let $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq 2^{\mathbf{N}}$. The following are equivalent:*
(i) *$\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete;*
(ii) *if $E$ is any separable $FK$-space containing $\bar{\mathcal{F}}$, then $c_0 \cup \bar{\mathcal{F}} \subseteq W_E$.*

PROOF. ((i) ⇒ (ii)). If $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete then, by Lemma 2, $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$. Thus $\bar{\mathcal{F}}^{\beta} = \ell_1$ and $\sigma(\bar{\mathcal{F}}^{\beta}, \bar{\mathcal{F}})$ is sequentially complete. By Theorem 5 ((i) ⇒ (iv)) of [2], it follows that $\bar{\mathcal{F}} \subseteq W_E$. We claim that, in addition, $c_0 \subseteq W_E$. To see this, let $f \in E'$ and $S \in \mathcal{F}$. Since $\chi_S \in W_E$, we can write

$$f(\chi_S) = \sum_{k=1}^{\infty} (\chi_S)_k f(\delta^k) = \sum_{k \in S} f(\delta^k).$$

In particular, $\sum_{k\in S} f(\delta^k)$ converges for each $S \in \mathcal{F}$ and for each $f \in E'$, that is, $(f(\delta^k)) \in \bar{\mathcal{F}}^{\beta}$ for each $f \in E'$. Again observing that $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$ (Lemma 2), we have $(f(\delta^k)) \in \ell_1$ for each $f \in E'$. It follows that $c_0 \subseteq E$ ([8], p. 138). Since, for any $FK$ space $E$ containing $c_0, c_0 \subseteq W_E$ ([8], p. 164), the proof is complete.

((ii) ⇒ (i)). We first show that the condition (ii) implies that $\bar{\mathcal{F}}^{\beta} \subseteq \ell_1$. To this end let $t = (t_i) \in \bar{\mathcal{F}}^{\beta}$, and define a matrix $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} t_k & 1 \leqq k \leqq n \\ 0 & k > n. \end{cases}$$

Then, for $S \in \mathcal{F}$,

$$\lim_{n\to\infty} (A(\chi_S))_n = \lim_{n\to\infty} \sum_{\substack{i=1 \\ i\in S}}^{n} t_i$$

exists, since $t \in \bar{\mathcal{F}}^{\beta}$. We thus have $\bar{\mathcal{F}} \subseteq c_A$. Since a convergence field is a separable $FK$-space, condition (ii) implies that $c_0 \subseteq c_A$. It follows that

$$\sum_{i=1}^{\infty} |t_i| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < +\infty,$$

that is, $t \in \ell_1$. Thus $\bar{\mathcal{F}}^{\beta} = \ell_1$ and we can apply Theorem 5 ((iv) ⇒ (i)) of [2] to conclude that $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete. □

COROLLARY 1.1 *Let $\mathcal{F} \subseteq 2^{\mathbf{N}}$ be a family for which $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete, and let $A = (a_{nk})$ be an infinite matrix. Then $\lim_{n\to\infty} \sum_{k\in S} a_{nk}$ exists for each $S \in \mathcal{F}$ if and only if*

(*i*) $$\lim_{n\to\infty} a_{nk} (= a_k) \text{ exists}, \quad k = 1, 2, \ldots$$

(*ii*) $$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < +\infty$$

*and*

(*iii*) $$\lim_{n\to\infty} \sum_{k\in S} a_{nk} = \sum_{k\in S} a_k \text{ for each } S \in \mathcal{F}.$$

PROOF. For any matrix $A$, the convergence field $c_A$ is a separable $FK$-space, and conditions (i) and (ii) are equivalent to $c_0 \subseteq c_A$. Condition (iii) follows from $\chi_S \in W_{c_A}$ for each $S \in \mathcal{F}$, by using the continuous linear functional $\lim_A$. □

If we impose the hereditary restriction on the family $\mathcal{F}$, we can state

THEOREM 2. *Let $\mathcal{F}_0 \subseteq \mathcal{F} \subseteq 2^{\mathbf{N}}$ and assume that $\mathcal{F}$ is hereditary. The following are equivalent:*

(i) *$\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete;*

(ii) *if $E$ is any separable $FK$-space containing $\bar{\mathcal{F}}$, then $c_0 \cup \bar{\mathcal{F}} \subseteq S_E$.*

PROOF. If $\mathcal{F}$ is hereditary, then $\bar{\mathcal{F}}$ is a monotone sequence space and Theorem 6 of [2] can be applied (see the remarks on p. 519) to give the condition $\bar{\mathcal{F}} \subseteq S_E$. The

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rest of the proof is essentially the same as the proof of Theorem 1, noting that if $E$ is any $FK$-space containing $c_0$, then $c_0 \subseteq S_E$ ([8], p. 164).

COROLLARY 2.1. ([7], *Proposition* 6). *Let* $\mathcal{F} \subseteq 2^{\mathbf{N}}$ *be a full class of subsets of the positive integers, and let* $A = (a_{nk})$ *be an infinite matrix. Then* $\lim_{n\to\infty} \sum_{k\in S} a_{nk}$ *exists for each* $S \in \mathcal{F}$ *if and only if*

$$(i) \qquad \lim_{n\to\infty} a_{nk} (= a_k) \text{ exists, } k = 1, 2, \ldots;$$

$$(ii) \qquad \sup_n \sum_{k=1}^{\infty} |a_{nk}| < +\infty;$$

*and*

$$(iii) \qquad \lim_{n\to\infty} \sum_{k\in S} |a_{nk} - a_k| = 0 \text{ for each } S \in \mathcal{F}.$$

PROOF. The sufficiency of conditions (i), (ii), and (iii) is clear. To prove the necessity, suppose that $\lim_{n\to\infty} \sum_{k\in S} a_{nk}$ exists for each set $S$ in the full class $\mathcal{F}$. Then $c_A$ is a separable $FK$-space containing $\bar{\mathcal{F}}$ and, since $\mathcal{F}$ is full, $\sigma(\ell_1, \bar{\mathcal{F}})$ is sequentially complete by Lemma 1. Thus, by Theorem 2, $c_0 \cup \bar{\mathcal{F}} \subseteq S_{c_A}$. The containment $c_0 \subseteq c_A$ implies conditions (i) and (ii), and the containment $\bar{\mathcal{F}} \subseteq S_{c_A}$ implies that

$$\lim_{r\to\infty} \sum_{\substack{k=r \\ k\in S}}^{\infty} a_{nk} = 0$$

uniformly in $n = 1, 2, \ldots$ ([8], p. 190, Theorem 7, with $Y = c$ and $z = \chi_S$). However, since $\mathcal{F}$ is hereditary, we have

$$\lim_{r\to\infty} \sum_{\substack{k=r \\ k\in T}}^{\infty} a_{nk} = 0$$

uniformly in $n = 1, 2, \ldots$ for each subset $T \subset S$, and this stronger statement is easily seen to be the same as

$$(*) \qquad \lim_{r\to\infty} \sum_{\substack{k=r \\ k\in S}}^{\infty} |a_{nk}| = 0 \text{ uniformly in } n = 1, 2, \ldots \text{ for each } S \in \mathcal{F}.$$

Condition $(*)$, together with conditions (i) and (ii), implies condition (iii). □

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