ON THE GENERAL THEORY OF DIFFERENTIABLE MANIFOLDS WITH ALMOST TANGENT STRUCTURE

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Some of the most important G-Structures of the first kind [1] are those defined by linear operators satisfying algebraic relations. If the linear operator J acting on the complexified space of a differentiable manifold V satisfies a relation of the form

$$J^2 = -I ,$$

where I is the identity operator, the manifold has an almost complex structure ([2] [3]). The structures defined by

$$J^2 = I$$

are the almost product structures ([3] [4]). In the present paper we investigate the structures defined by nilpotent operators of degree 2, that is by relations of the form

$$J^2 = 0 .$$

Some of the results of this investigation are stated in [5]. Recently, an attempt has been made to study the more general case

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$$J^r = 0$$

where $r \geq 2$. So far, only the integrability of such structures has been studied [6].

1. General definitions. We consider a differentiable manifold V_{2n} of class C^∞. Let T_x be the tangent space at any point $x \in V_{2n}$, and T_x^c the complexified space of T_x. We assume that a field of class C^∞ of linear operators J_x is defined on V_{2n}, such that, at each point $x \in V_{2n}$, J_x maps T_x^c into itself; moreover J_x is of rank n everywhere in V_{2n}, and it satisfies the relation

$$J_x^2 = 0$$

for any $x \in V_{2n}$, where 0 is the null operator. In this case we say that J defines an almost tangent structure on the manifold V_{2n}.

PROPOSITION 1. The image $J(T_x^c)$ and the $\mathrm{Ker} J$ coincide with the space of the eigenvectors of J.

Proof. Let $a \in T_x^c$. Ja is an eigenvector of J, since

$$J(Ja) = J^2 a = 0 .$$

Hence, the image $J(T_x^c)$ is composed of the eigenvectors of J. On the other hand, every vector of $J(T_x^c)$ is mapped unto the zero vector; therefore

$$J(T_x^c) = \mathrm{Ker} J .$$

If S_x is the complementary space of $\mathrm{Ker} J$ with respect

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to T^c_x, we have

$$T^c_x = \mathrm{Ker}J \oplus S_x ,$$

and J induces an isomorphism between S_x and KerJ.
Let $(e_1, e_2, \ldots, e_n; e_{n+1}, \ldots, e_{2n})$ be a basis of T^c_x, where $(e_1, \ldots, e_n)$ is a basis of KerJ and $(e_{n+1}, \ldots, e_{2n})$ is a basis of S_x. We shall write briefly $(e_\alpha, e_{\alpha*})$ where $\alpha = 1, 2, \ldots, n$, $\alpha* = \alpha+n$ (Greek indices take the values $1\ldots n$ and Latin indices the values $1\ldots 2n$). We can always arrange that

$$e_\alpha = Je_{\alpha*} .$$

We call the basis $(Je_{\alpha*}, e_{\alpha*})$ a basis adapted to the almost tangent structure or briefly an adapted basis. Let $(e_{\alpha'}, e_{\alpha'*})$ be another adapted basis; we have

$$e_{\alpha'} = A^\beta_{\alpha'} e_\beta$$

$$e_{\alpha'*} = B^\beta_{\alpha'*} e_\beta + A^{\beta*}_{\alpha'*} e_{\beta*} .$$

From the latter we have

$$Je_{\alpha'*} = A^{\beta*}_{\alpha'*} Je_{\beta*} ,$$

and hence

$$A^{\beta*}_{\alpha'*} = A^\beta_{\alpha'} .$$

Therefore

(1.1) $$e_{\alpha'} = A^\beta_{\alpha'} e_\beta , \quad e_{\alpha'*} = B^\beta_{\alpha'*} e_\beta + A^{\beta*}_{\alpha'*} e_{\beta*} ,$$

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with $A^{\beta *}_{\alpha' *} = A^{\beta}_{\alpha'}$. The transformation matrix for the adapted bases is of the form

$$\alpha = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix},$$

where $A \in GL(n, C)$, B is an (n, n) matrix and 0 the null (n, n) matrix.

LEMMA 1. The set of all the matrices α is a group under multiplication, which will be denoted by $G\binom{n}{nn}$.

Proof. For any two matrices α, $\alpha_1 \in G\binom{n}{nn}$, we have, is we use multiplication by blocks,

$$\alpha \alpha_1 = \begin{pmatrix} AA_1 & 0 \\ BA_1+AB_1 & AA_1 \end{pmatrix} \in G\binom{n}{nn}$$

and also

$$\alpha^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -A^{-1}BA^{-1} & A^{-1} \end{pmatrix} \in G\binom{n}{nn}.$$

Hence $G\binom{n}{nn}$ is a subgroup of the group $GL(2n, C)$. It is moreover a Lie group.

Consider the operator J_x or J; to this operator there corresponds a tensor F^i_j defined by

$$(1.2) \qquad (Jv)^i = F^i_j v^j,$$

and if we use the relation $J^2 = 0$, we obtain

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(1.3) $$F_i^j F_j^k = 0 .$$

If the vector space T_x^c is referred to an adapted basis, the components of the tensor F_i^j are given by

(1.4) $$F_\alpha^\beta = F_\alpha^{\beta*} = F_{\alpha*}^{\beta*} = 0 , \qquad F_{\alpha*}^\beta = \delta_\alpha^\beta = \delta_{\alpha*}^{\beta*} .$$

Hence F_i^j is represented by a matrix of the form

(1.5) $$\begin{pmatrix} 0 & 0 \\ E_n & 0 \end{pmatrix},$$

where E_n denotes the unit matrix of order n. Since the matrix (1.5) commutes with all the elements of $G\binom{n}{nn}$, J will have the form (1.5) with respect to any adapted basis.

Note. The group $G\binom{n}{nn}$ is composed of all the elements of $GL(2n, C)$ which commute with the matrix (1.5).

For any vector $v \in T_x^c$ referred to an adapted basis we have

$$v = v^\alpha e_\alpha + v^{\alpha*} e_{\alpha*} ,$$

and hence

$$Jv = v^\alpha Je_\alpha + v^{\alpha*} Je_{\alpha*} = v^{\alpha*} e_\alpha$$

or

$$(Jv)^\alpha = v^{\alpha*} , \quad (Jv)^{\alpha*} = 0 .$$

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2. <u>The dual space.</u> Let us consider the dual space $(T_x^c)^*$ of the complexified space T_x^c at a point x of the differentiable manifold V_{2n}. If (e_i), $(e_{j'})$ are two adapted bases at x, to these there correspond two dual bases (θ^i), $(\theta^{j'})$. From the relation

$$e_{j'} = A_{j'}^i e_i$$

for the adapted bases, we have the relation

$$\theta^\alpha = A_{\beta'}^\alpha \theta^{\beta'}$$

for the dual bases. Hence

$$\theta^\alpha = A_{\beta'}^\alpha \theta^{\beta'} + A_{\beta'*}^\alpha \theta^{\beta'*}$$

and

$$\theta^{\alpha*} = A_{\beta'*}^{\alpha*} \theta^{\beta'*} .$$

We thus see that if the transformation matrix for the adapted bases is given by (1.1), the transformation matrix for the dual bases is of the form

$$(t_\alpha)^{-1} = \begin{pmatrix} tA & tB \\ 0 & tA \end{pmatrix}^{-1} = \begin{pmatrix} A_1 & B_1 \\ 0 & A_1 \end{pmatrix} .$$

On the other hand, if $(\mathrm{Ker}\, J)^*$ is the dual space of $\mathrm{Ker} J$, to the basis (e_α) of $\mathrm{Ker} J$ there corresponds the dual basis $(\theta^{\alpha*})$ of $(\mathrm{Ker} J)^*$.

PROPOSITION 2. An almost tangent structure is defined in the dual space $(T_x^c)^*$ by the space $(\mathrm{Ker} J)^*$, that is, there exists a linear operator J_1 of rank n satisfying $J_1^2 = 0$, such that

$$J_1[(T_x^c)^*] = (\operatorname{Ker} J)^* .$$

<u>Proof.</u> Let us define in $(T_x^c)^*$ an operator J_1, which with respect to an adapted basis is represented by the matrix

$$J_1 = \begin{pmatrix} 0 & E_n \\ 0 & 0 \end{pmatrix} .$$

This operator has the same representation with respect to any other adapted dual basis. Indeed we have

$$\begin{pmatrix} B & B_1 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & E_n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B & B_1 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} B & B_1 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & E_n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B^{-1} & -B^{-1}B_1B^{-1} \\ 0 & B^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} B & B_1 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & B^{-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_n \\ 0 & 0 \end{pmatrix} .$$

J_1 has therefore an intrinsic meaning. Since $\det(E_n) = 1$ and all the other submatrices of J_1 of order greater than n are singular, rank $J_1 = n$. In the other hand $J_1^2 = 0$. It is easy to see, by using the components of the vectors θ^α and $\theta^{\alpha*}$, that

$$J_1\theta^\alpha = \theta^{\alpha*} , \qquad J_1\theta^{\alpha*} = 0 .$$

For any element $v \in (T_x^c)^*$ with components v_i with respect to the basis (θ^i) we have

$$v = v_\alpha\theta^\alpha + v_{\alpha*}\theta^{\alpha*} ,$$

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and hence

$$J_1 \bar{v} = \bar{v}_\alpha J_1 \theta^\alpha + v_{\alpha*} J_1 \theta^{\alpha*} = v_\alpha \theta^{\alpha*} \in (\mathrm{Ker} J)^* .$$

We thus see that the elements of $(\mathrm{Ker}J)^*$ are the images under J_1 of the elements of $(T_x^c)^*$.

3. Connections in V_{2n}. Let $E_T(V_{2n})$ be the set of all the adapted bases at the different points of V_{2n}, and p the canonical mapping

$$p : E_T(V_{2n}) \to V_{2n} ,$$

which associates with an adapted basis at x the point x itself. $E_T(V_{2n})$ has, with respect to p, a natural structure of a principal fibre bundle of base V_{2n} and structural group the sub-group $G(^n_{nn})$ of $GL(2n, C)$.

Definition. We will call an almost tangent connection (briefly A.T. connection) on V_{2n}, every infinitesimal connection defined on the fibre bundle of the adapted bases.

For the definition of an infinitesimal connection one may consult [7].

Given a covering of V_{2n} by neighbourhoods endowed with local cross sections of $E_T(V_{2n})$, an A.T. connection may be defined in each neighbourhood U by a form w_U with values in the Lie Algebra of the group $G(^n_{nn})$; such a form may be represented at x by means of a matrix of order $2n$ whose elements are complex-valued linear forms at x; it will be denoted by

$$\pi_U = (\pi_i^j) .$$

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Hence an A. T. connection is represented by the matrix

$$\begin{pmatrix} \pi^{\alpha}_{\beta} & 0 \\ \pi^{\alpha}_{\beta*} & \pi^{\alpha*}_{\beta*} = \pi^{\alpha}_{\beta} \end{pmatrix}.$$

PROPOSITION 3. With respect to an A. T. connection we have

$$\nabla J = 0 .$$

Proof. We refer the tensor $J = (F^j_i)$ to an adapted basis. We have

$$\nabla F^{\alpha}_{\beta} = dF^{\alpha}_{\beta} + \pi^{\alpha}_{\rho} F^{\rho}_{\beta} + \pi^{\alpha}_{\rho*} F^{\rho*}_{\beta} - \pi^{\rho}_{\beta} F^{\alpha}_{\rho} - \pi^{\rho*}_{\beta} F^{\alpha}_{\rho*} = -\pi^{\rho*}_{\beta} \delta^{\alpha*}_{\rho*}$$

$$= - \pi^{\alpha*}_{\beta} = 0 ,$$

$$\nabla F^{\alpha*}_{\beta*} = dF^{\alpha*}_{\beta*} + \pi^{\alpha*}_{\rho} F^{\rho}_{\beta*} + \pi^{\alpha*}_{\rho*} F^{\rho*}_{\beta*} - \pi^{\rho}_{\beta*} F^{\alpha*}_{\rho} - \pi^{\rho*}_{\beta*} F^{\alpha*}_{\rho*}$$

$$= \pi^{\alpha*}_{\rho} \delta^{\rho}_{\beta} = \pi^{\alpha*}_{\beta} = 0 ,$$

(3.1)

$$\nabla F^{\alpha}_{\beta*} = dF^{\alpha}_{\beta*} + \pi^{\alpha}_{\rho} F^{\rho}_{\beta*} + \pi^{\alpha}_{\rho*} F^{\rho*}_{\beta*} - \pi^{\rho}_{\beta*} F^{\alpha}_{\rho} - \pi^{\rho*}_{\beta*} F^{\alpha}_{\rho*}$$

$$= \pi^{\alpha}_{\rho} \delta^{\rho}_{\beta} - \pi^{\rho*}_{\beta*} \delta^{\alpha*}_{\rho*} = \pi^{\alpha}_{\beta} - \pi^{\alpha*}_{\beta*} = 0 ,$$

$$\nabla F^{\alpha*}_{\beta} = dF^{\alpha*}_{\beta} + \pi^{\alpha*}_{\rho} F^{\rho}_{\beta} + \pi^{\alpha*}_{\rho*} F^{\rho*}_{\beta} - \pi^{\rho}_{\beta} F^{\alpha*}_{\rho} - \pi^{\rho*}_{\beta} F^{\alpha*}_{\rho*} = 0 .$$

$E_T(V_{2n})$ may be considered as a sub-bundle of the fibre bundle $E_c(V_{2n})$ of the complex bases. An A. T. connection defines canonically a complex linear connection with which it

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may be identified. Conversely, let us consider a complex linear connection and a covering of V_{2n} by neighbourhoods equipped with local cross sections of $E_T(V_{2n})$. This connection may be defined on each neighbourhood by a local form,, with values in the Lie Algebra of GL(2n, C), represented by a matrix (w_i^j) whose elements are complex-valued local Pfaffian forms. In order that the given connection may be identified with an A.T. connection it is necessary and sufficient that (w_i^j) belongs in the Lie Algebra of the structural group $G(^{n}_{nn})$ of $E_T(V_{2n})$. That is,

$$w_\alpha^\beta = w_{\alpha*}^{\beta*}, \quad w_\alpha^{\beta*} = 0 .$$

Comparing with (3.1), we obtain the following

PROPOSITION 4. In order that a complex linear connection may be identified with an A.T. connection it is necessary and sufficient that the tensor $J = (F_i^j)$ have a zero absolute differential with respect to this connection.

We shall now consider any complex linear connection referred to an adapted basis. Let

$$w = (w_i^j)$$

be the matrix representing this connection. Under transformations of bases the forms w_i^j transform according to

$$w_{m'}^{\ell'} = A_a^{\ell'} w_b^a A_{m'}^b + A_s^{\ell'} dA_{m'}^s$$

or

$$A_{\ell'}^j w_{m'}^{\ell'} = A_{m'}^b w_b^j + dA_{m'}^j . \tag{3.2}$$

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If we apply the above relation for $j = \zeta^*$ and $m = \mu$ we obtain

$$(3.3) \qquad w^{\lambda'*}_{\mu'} = A^{\lambda'*}_{\iota*} A^{\zeta}_{\mu'} w^{\iota*}_{\zeta} .$$

On the other hand, substituting in (3.2), first $j = \zeta$, $m = \mu$, then $j = \zeta^*$, $m = \mu^*$, and substracting the second equation from the first we obtain

$$(3.4) \quad A^{\zeta}_{\lambda'} w^{\lambda'}_{\mu'} - A^{\zeta}_{\lambda'*} w^{\lambda'*}_{\mu'} - A^{\zeta*}_{\lambda'*} w^{\lambda'*}_{\mu'*} = w^{\lambda}_{\zeta} A_{\mu'} - w^{\lambda*}_{\zeta} A^{\zeta}_{\mu'*} - w^{\lambda*}_{\zeta*} A^{\zeta*}_{\mu'*} .$$

If we consider the transformation relations given in the Appendix, we see that the quantities

$$(3.5) \qquad t^{\lambda}_{\mu} = 0 , \quad t^{\lambda*}_{\mu} = t^{\lambda}_{\mu*} = w^{\lambda*}_{\mu} , \quad t^{\lambda*}_{\mu*} = w^{\lambda*}_{\mu*} - w^{\lambda}_{\mu} ,$$

are the components of a tensor form of type (1, 1). We call it the tensor form associated to the linear connection. From the relation (3.5) we have

PROPOSITION 5. In order that a complex linear connection on V_{2n} be an A.T. connection it is necessary and sufficient that the associated tensor form be equal to zero.

4. The operators C and M in an A.T. manifold.

As in the theory of almost complex manifolds and the almost product manifolds, we may introduce, in the theory of manifolds with A.T. structure, operators C and M.

Let V_{2n} be such manifold and let us denote by $\Lambda^c_r(V_{2n})$ the vector space of all the complex-valued exterior r-forms defined on V_{2n}. We associate with the A.T. structure two operators C and M defined on (V_{2n}) in the following way:

If $(v_1, v_2, \ldots, v_r)$ are r vectors of T^c_x and f an r-form, we denote by $f(v_1, \ldots, v_x)$ the value of f for

$v_1 \wedge v_2 \wedge \dots \wedge v_r$. C is then defined by the relation

$$(4.1) \qquad Cf(v_1, \dots, v_r) = f(Jv_1, Jv_2, \dots, Jv_r) .$$

If the components of f are $f_{j_1 j_2 \dots j_r}$ the components of Cf will be

$$(4.2) \qquad (Cf)_{i_1 i_2 \dots i_r} = F_{i_1}^{j_1} F_{i_2}^{j_2} \dots F_{i_r}^{j_r} f_{j_1 j_2 \dots j_r} .$$

It is obvious from (1.1) that C satisfies the relation

$$(4.3) \qquad C^2 = 0 .$$

Definition. A pure form f of type r, can be written

$$f = \frac{1}{r!} f_{\alpha_1 \alpha_2 \dots \alpha_r} \theta^{\alpha_1^*} \wedge \theta^{\alpha_2^*} \wedge \dots \wedge \theta^{\alpha_r^*} .$$

It is obvious that this definition is independent of the adapted basis to which this form is referred.

PROPOSITION 6. C maps every r-form of $\Lambda_r^c(V_{2n})$ into a pure r-form.

Proof. The relation (4.2) written in an adapted basis provides

$$(Cf)_{\alpha_1 \alpha_2 \dots \alpha_r} = F_{\alpha_1}^{j_1} F_{\alpha_2}^{j_2} \dots F_{\alpha_r}^{j_r} f_{j_i \dots j_r} = 0 ,$$

$$(Cf)_{\alpha_1 \dots \alpha_k^* \dots \alpha_r} = F_{\alpha_1}^{j_1} \dots F_{\alpha_k^*}^{j_k} \dots F_{\alpha_r}^{j_r} f_{j_i \dots j_2} = 0 ,$$

$$(Cf)_{\alpha_1^* \alpha_2^* \dots \alpha_r^*} = F_{\alpha_1^*}^{j_1} F_{\alpha_2^*}^{j_2} \dots F_{\alpha_r^*}^{j_r} f_{j_1 \dots j_r} =$$

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$$= \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} \cdots \delta_{\alpha_r}^{\beta_r} f_{\beta_1 \cdots \beta_r} = f_{\alpha_1 \cdots \alpha_r} ,$$

and hence

$$Cf = \frac{1}{r!} f_{\alpha_1 \cdots \alpha_r} \theta^{\alpha_1^*} \wedge \theta^{\alpha_2^*} \wedge \cdots \wedge \theta^{\alpha_r^*} .$$

The operator M will be defined in the following way: For any $v_1, v_2, \ldots, v_r \in T_x^c$, M is defined by the relation

$$(4.4) \quad Mf(v_1, v_2, \ldots, v_r) = \sum_{k=1}^{r} f(v_1, v_2, \ldots, v_{k-1}, Jv_k, v_{k+1}, \ldots, v_r) ,$$

where the right-hand side obviously defines an r-form.

PROPOSITION 7.

$$(4.5) \qquad C = \frac{1}{r!} M^r \text{ and } M^{r+1} = 0 .$$

Proof. By repeated application of the operator M on the form f we obtain

$$(4.6) \quad M^r f(v_1, v_2, \ldots, v_r) = r! f(Jv_1, Jv_2, \ldots, Jv_r) ;$$

using (4.1), we find

$$M^r f(v_1, v_2, \ldots, v_r) = r! Cf(v_1, v_2, \ldots, v_r).$$

The above relation holds for any r-form f, hence

$$C = \frac{1}{r} M^r .$$

The same relation (4.6) provides

$$M^{r+1} f(v_1, v_2, \ldots, v_r) = 0 ,$$

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hence

$$M^{r+1} = 0 .$$

From the relations (4.5) we obtain

(4.7) $$MC = CM = 0 .$$

If f admits the components $f_{i_1 i_2 \ldots i_r}$, the form Mf has the components

(4.8) $$(Mf)_{i_1 i_2 \ldots i_r} = \sum_k f_{i_1 i_2 \ldots i_{k-1} s i_{k+1} \ldots i_r} F^s_{ik}$$

$$= \frac{1}{(r-1)!} \varepsilon^{j_1 j_2 \ldots j_r}_{i_1 i_2 \ldots i_r} F^s_{j_1} f_{s j_2 \ldots j_r} ,$$

where $\varepsilon^{j_1 \ldots j_r}_{i_1 \ldots i_r}$ is the Kronecker tensor.

PROPOSITION 8. For any 1-form f we have

(4.9) $$Cdf - MdCf = f \circ T$$

where T is the tensor of the structure of the manifold V_{2n}.

Proof. We consider a 1-form f defined in a neighbourhood U of V_{2n} by

$$f = f_i \theta^i .$$

Its exterior differential is

$$df = df_i \wedge \theta^i + f_i d\theta^i ,$$

or

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$$df = \frac{1}{2}(\partial_i f_j - \partial_j f_i)(\theta^i \wedge \theta^j) + \frac{1}{2} f_i c^i_{jk} \theta^j \wedge \theta^k ,$$

where c^i_{jk} are the coefficients of $d\theta^i$ in the decomposition

$$d\theta^i = \frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k . \qquad (c^i_{jk} + c^i_{kj} = 0) .$$

Operating by C, we find

$$Cdf = \frac{1}{2}(\partial_\alpha f_\beta - \partial_\beta f_\alpha)(\theta^{\alpha *} \wedge \theta^{\beta *}) + \frac{1}{2} f_i c^i_{\alpha\beta} \theta^{\alpha *} \wedge \theta^{\beta *} .$$

Similarly,

$$Cf = f_\alpha \theta^{\alpha *}$$

and

$$dCf = df_\alpha \wedge \theta^{\alpha *} + f_\alpha d\theta^{\alpha *}$$

$$= \frac{1}{2}(\partial_\lambda f_\alpha - \partial_\alpha f_\lambda)\theta^\lambda \wedge \theta^{\alpha *} + \frac{1}{2}(\partial_{\lambda *} f_\alpha - \partial_{\alpha *} f_\lambda)\theta^{\lambda *} \wedge \theta^{\alpha *}$$

$$+ \frac{1}{2} f_\alpha c^{\alpha *}_{ij} \theta^i \wedge \theta^j .$$

Let us put $dCf = \alpha$. Then $(M\alpha)_{ij} = \varepsilon^{k\ell}_{ij} F^s_k \alpha_{s\ell} = \varepsilon^{u*\ell}_{ij} F^\sigma_{u*} \alpha_{\sigma\ell}$

$$= \varepsilon^{\sigma * \ell}_{ij} \alpha_{\sigma \ell} ,$$

$$(M\alpha)_{\iota\zeta} = 0$$

$$(M\alpha)_{\iota *\zeta} = \varepsilon^{\sigma * \ell}_{\iota * \zeta} \alpha_{\sigma\ell} = \varepsilon^{\sigma * \lambda}_{\iota * \zeta} \alpha_{\sigma\lambda} = \alpha_{\iota\zeta} = 0 ,$$

$$(M\alpha)_{\iota *\zeta *} = \varepsilon^{\sigma * \ell}_{\iota * \zeta *} \alpha_{\sigma\ell} = \varepsilon^{\sigma * \lambda *}_{\iota * \zeta *} \alpha_{\sigma \lambda *} = \alpha_{\iota\zeta *} - \alpha_{\zeta\iota *}$$

$$= \frac{1}{2}(\partial_\iota f_\zeta - \partial_\zeta f_\iota) .$$

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In a similar manner, if we put $f_\alpha C^{\alpha*}_{ij} = \beta_{ij}$ and $\beta = \frac{1}{2}(f_\alpha C^{\alpha*}_{ij}\, \theta^i \wedge \theta^i)$ we obtain

$$(M\beta)_{\iota\zeta} = 0 ,$$

$$(M\beta)_{\iota *\zeta} = \beta_{\iota\zeta} = f_\alpha c^{\alpha*}_{\iota\zeta} ,$$

$$(M\beta)_{\iota *\zeta *} = \beta_{\iota\zeta *} - \beta_{\zeta\iota *} = \beta_{\iota\zeta *} + \beta_{\iota *\zeta} ;$$

hence
$$M\beta = (f_\alpha c^{\alpha*}_{\iota\zeta})\theta^{\iota *} \wedge \theta^{\zeta} + \beta_{\iota\zeta *}\, \theta^{\iota *} \wedge \theta^{\zeta *}$$
$$= (f_\alpha c^{\alpha*}_{\iota\zeta})\theta^{\iota *} \wedge \theta^{\zeta} + f_\alpha c^{\alpha*}_{\iota *\zeta}\, \theta^{\iota *} \wedge \theta^{\zeta *} ,$$

and

$$MdCf = \frac{1}{2}(\partial_\iota f_\zeta - \partial_\zeta f_\iota)\theta^{\iota *} \wedge \theta^{\zeta *} + f_\alpha [c^{\alpha*}_{\iota\zeta}\theta^{\iota *} \wedge \theta^{\zeta} + c^{\alpha*}_{\iota *\zeta}\theta^{\iota *} \wedge \theta^{\zeta *}]$$

Therefore

$$Cdf - MdCf = f_\alpha\{c^{\alpha*}_{\iota\zeta}\theta^{\zeta} \wedge \theta^{\iota *} + \frac{1}{2}(c^{\alpha}_{\iota\zeta} - [c^{\alpha*}_{\iota *\zeta} + c^{\alpha*}_{\iota\zeta *}])\theta^{\iota *} \wedge \theta^{\zeta *}\}$$
$$+ \frac{1}{2} f_{\alpha *} c^{\alpha*}_{\iota\zeta}\, \theta^{\iota *} \wedge \theta^{\zeta *} .$$

In the other hand, if we apply the relations of p. 5328 of [6] for $p = q = \frac{1}{2}n$, $r = 1$, we obtain for the components of the tensor of structure

$$t^{\alpha}_{\beta *\lambda *} = c^{\alpha}_{\beta\lambda} - c^{\alpha*}_{\beta *\lambda} - c^{\alpha*}_{\beta\lambda *} , \quad t^{\alpha}_{\beta *\lambda} = c^{\alpha*}_{\beta\lambda} , \quad t^{\alpha}_{\beta\lambda} = 0 ,$$

$$t^{\alpha*}_{\beta *\lambda *} = c^{\alpha*}_{\beta\lambda} , \quad t^{\alpha*}_{\beta *\lambda} = 0 , \quad t^{\alpha*}_{\beta\lambda} = 0 ;$$

the structure form is then given by

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$$T^{\alpha} = c^{\alpha *}_{\iota\zeta} \theta^{\zeta} \wedge \theta^{\iota *} + \frac{1}{2}(c^{\alpha}_{\iota\zeta} - c^{\alpha *}_{\iota *\zeta} - c^{\alpha *}_{\iota\zeta *})\theta^{\iota *} \wedge \theta^{\zeta *},$$

$$T^{\alpha *} = \frac{1}{2} c^{\alpha *}_{\beta} \theta^{\beta *} \wedge \theta^{\lambda *}.$$

and finally

$$Cdf - MdCf = f_{\alpha} T^{\alpha} + f_{\alpha *} T^{\alpha *} = f \circ T .$$

The relation (4.9) can be used in order to obtain, in local coordinates, an expression of the tensor of the structure in terms of the tensor F^{j}_{i} of the almost tangent structure.

Indeed, (4.9) provides, in local coordinates,

$$(4.10) \qquad (Cdf - MdCf)_{jk} = \frac{1}{2} t^{i}_{jk} f_{i} .$$

The form df has, as components in local coordinates,

$$(df)_{jk} = \frac{1}{2}(\partial_{j} f_{k} - \partial_{k} f_{j}) ,$$

also

$$(Cdf)_{jk} = \frac{1}{2} F^{a}_{j} F^{b}_{k} (\partial_{a} f_{b} - \partial_{b} f_{a}) .$$

Hence for MdCf we have, according to the relation (4.8),

$$(MdCf)_{jk} = \frac{1}{2} F^{a}_{j} [\partial_{a} (F^{b}_{k} f_{b}) - \partial_{k} (F^{b}_{a} f_{b})] - \frac{1}{2} F^{a}_{k} [\partial_{a} (F^{b}_{j} f_{b}) - \partial_{j} (F^{b}_{a} f_{b})] .$$

Using the relation $F^{a}_{i} F^{j}_{a} = 0$, we obtain

$$(MdCf)_{jk} = \frac{1}{2} F^{a}_{j} \partial_{a} F^{b}_{k} f_{b} + \frac{1}{2} F^{a}_{j} F^{b}_{k} \partial_{a} f_{b} - \frac{1}{2} F^{a}_{j} \partial_{k} F^{b}_{a} f_{b}$$

$$- \frac{1}{2} F^{a}_{k} \partial_{a} F^{b}_{j} f_{b} - \frac{1}{2} F^{a}_{k} F^{b}_{j} \partial_{a} F^{b}_{j} f_{b} - \frac{1}{2} F^{a}_{k} F^{b}_{j} \partial_{a} f_{b} + \frac{1}{2} F^{a}_{k} \partial_{j} F^{b}_{a} f_{b} ,$$

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and

$$(Cdf-MdCf)_{jk} = \frac{1}{2}\{ F^a_j(\partial_k F^b_a - \partial_a F^b_k) + F^a_k(\partial_a F^b_j - \partial_j F^b_a)\} f_b .$$

From (4.10) we see that

$$t^i_{jk} = F^a_j(\partial_k F^i_a - \partial_a F^i_k) + F^a_k(\partial_a F^i_j - \partial_j F^i_a) ,$$

or

$$t^i_{jk} = F^i_a(\partial_j F^a_k - \partial_k F^a_j) + F^a_k \partial_a F^i_j - F^a_j \partial_a F^i_k ,$$

since

$$F^a_j \partial_k F^i_a = - F^i_a \partial_k F^a_j .$$

PROPOSITION 9. For the almost tangent structures the Nijenhuis tensor, is the negative of the tensor of the structure.

Proof. The Nijenhuis tensor is defined [8] by

$$N(u,v) = [Ju, Jv] + J^2[u,v] - J[Ju,v] - J[u,Jv] ,$$

for any vector fields u, v. For A.T. structures we have $J^2 = 0$, hence

$$(4.11) \qquad N(u,v) = [Ju, Jv] - J[Ju,v] - J[u,Jv] .$$

The relation (4.11) may be written explicitly

$$[N(u,v)]^k = [Ju, Jv]^k - F^k_\ell [Ju,v]^\ell - F^k_\ell [u, Jv]^\ell ,$$

where

$$[Ju,v]^\ell = F^m_a u^a \partial_m v^\ell - v^m \partial_m (F^\ell_a u^a) ,$$

$$[u,Jv]^\ell = u^m \partial_m (F^\ell_b v^b) - F^m_a v^a \partial_m u^\ell ,$$

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$$[Ju, Jv]^k = F^m_a u^a \partial_m (F^k_b v^b) - F^m_a v^a \partial_m (F^k_b u^b)$$

$$= F^m_a F^k_b u^a \partial_m v^b - F^m_a F^k_b v^a \partial_m u^b + F^m_a \partial_m F^k_b u^a v^b$$

$$- F^m_a \partial_m F^k_b v^a u^b .$$

Hence, after cancellations of opposite terms and rearrangement of indices, we obtain

$$N(u, v)^k = [F^k_r \partial_\ell F^r_m - F^k_r \partial_m F^r_\ell + F^r_m \partial_r F^k_\ell - F^r_\ell \partial_r F^k_m] v^\ell u^m .$$

Therefore

$$N^k_{m\ell} = F^k_r (\partial_\ell F^r_m - \partial_m F^r_\ell) + F^r_m \partial_r F^k_\ell - F^r_\ell \partial_r F^k_m ,$$

and

$$N^k_{m\ell} = - t^k_{m\ell} .$$

COROLLARY. In order that an A. T. structure be completely integrable it is necessary and sufficient that the Nijenhuis tensor be equal to zero.

5. Curvature tensor of the almost tangent connection. Given an A. T. connection, the curvature of this connection is defined by the relation

(5.1) $$\Omega^j_i = d\pi^j_i + \pi^j_\ell \wedge \pi^\ell_i ,$$

where the tensor 2-form (5.1) is the curvature form of the connection and it satisfies Bianchi's identity

(5.2) $$d\Omega^j_i = \Omega^j_\ell \wedge \pi^\ell_i - \pi^j_\ell \wedge \Omega^\ell_i .$$

From the relation (5.1) we have

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$$\Omega^{\beta}_{\alpha} = d\pi^{\beta}_{\alpha} + \pi^{\beta}_{\lambda} \wedge \pi^{\lambda}_{\alpha} + \pi^{\beta}_{\lambda*} \wedge \pi^{\lambda*}_{\alpha} = d\pi^{\beta}_{\alpha} + \pi^{\beta}_{\lambda} \wedge \pi^{\lambda}_{\alpha}$$

$$\Omega^{\beta*}_{\alpha*} = d\pi^{\beta*}_{\alpha*} + \pi^{\beta*}_{\lambda} \wedge \pi^{\lambda}_{\alpha*} + \pi^{\beta*}_{\lambda*} \wedge \pi^{\lambda*}_{\alpha*} = d\pi^{\beta*}_{\alpha*} + \pi^{\beta*}_{\lambda*} \wedge \pi^{\lambda*}_{\alpha*} .$$

(Hence $\Omega^{\beta*}_{\alpha*} = \Omega^{\beta}_{\alpha}$.)

$$\Omega^{\beta*}_{\alpha} = d\pi^{\beta*}_{\alpha} + \pi^{\beta*}_{\lambda} \wedge \pi^{\lambda}_{\alpha} + \pi^{\beta*}_{\lambda*} \wedge \pi^{\lambda*}_{\alpha} = 0 ,$$

$$\Omega^{\beta}_{\alpha*} = d\pi^{\beta}_{\alpha*} + \pi^{\beta}_{\lambda} \wedge \pi^{\lambda}_{\alpha*} + \pi^{\beta}_{\lambda*} \wedge \pi^{\lambda*}_{\alpha*} .$$

By contraction on α and β we obtain

$$(5.3) \qquad \Omega^{\alpha}_{\alpha} = d\pi^{\alpha}_{\alpha} \qquad \text{or} \qquad \Omega^{\alpha*}_{\alpha*} = d\pi^{\alpha*}_{\alpha*} ,$$

$$\left(\pi^{\alpha}_{\alpha} = \pi^{\alpha*}_{\alpha*} = \frac{1}{2}\pi^{i}_{i} \qquad \text{and} \qquad \Omega^{\alpha}_{\alpha} = \Omega^{\alpha*}_{\alpha*} = \frac{1}{2}\Omega^{i}_{i}\right) .$$

If we consider a covering of V_{2n} by neighbourhoods $U, V, \ldots$ equipped with local cross sections of $E_T(V_{2n})$, we see that

$$\psi = \Omega^{\alpha}_{\alpha} = \Omega^{\alpha*}_{\alpha*}$$

is a complex 2-form. We call ψ the characteristic form of the A.T. connection. We deduce from (5.3) that ψ is a closed form. π^{α}_{α} defines on the fibre bundle $E_T(V_{2n})$ a complex-valued 1-form and if $p^*\psi$ is the inverse image of ψ in $E_T(V_{2n})$ by the projection, we may write

$$p^*\psi = d\phi .$$

Thus, $p^*\psi$ is homologous to 0 on $E_T(V_{2n})$. The cohomology class on V_{2n} of the form ψ does not depend on the connection

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under consideration. If $(\hat{\pi}^{\alpha}_{\beta})$ is another A. T. connection,

$$\hat{\psi} - \psi = d(\hat{\pi}^{\alpha}_{\alpha} - \pi^{\alpha}_{\alpha}) ,$$

where $\hat{\pi}^{\alpha}_{\alpha} - \pi^{\alpha}_{\alpha}$ defines a complex-valued 1-form on V_{2n}. Then $\hat{\psi} - \psi$ is homologous to 0 in V_{2n}. The form ψ defines an integral cohomology class of degree 2.

6. The Holonomy group of the A. T. connections. Let V_{2n} be a manifold endowed with an A. T. connection. The holonomy group of this connection is a sub-group of the structural group $G(^{n}_{nn})$ of the fibre bundle $E_T(V_{2n})$ [(2), p. 62]. Conversely, let V_{2n} be a differentiable manifold endowed with a linear complex connection. Let us consider a point $x \in V_{2n}$ and let us assume that there exists at x, a complex basis b such that the holonomy group of the connection ψ_b at b, is a subgroup of $G(^{n}_{nn})$; the elements of ψ_b are matrices of the form

$$\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} .$$

Let us now consider, at the point x, the tensor whose components with respect to the basis b are

$$F^{\beta}_{\alpha} = F^{\beta *}_{\alpha} = F^{\beta *}_{\alpha *} = 0 , \qquad F^{\beta}_{\alpha *} = \delta^{\beta}_{\alpha} = \delta^{\beta *}_{\alpha *} ,$$

that is, the tensor represented by the matrix

$$\begin{pmatrix} 0 & 0 \\ E_n & 0 \end{pmatrix} .$$

It will be invariant under transformations by the elements of ψ_b, since

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$$\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ E_n & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ B & A \end{pmatrix},$$

and its components satisfy, at x, the relation

$$F^j_k F^k_i = 0 \ . \tag{6.1}$$

From this tensor, we obtain by parallel transport in V_{2n}, a tensor F^j_i defined on the whole manifold V_{2n} with absolute differential equal to zero [(2), p. 113]. Moreover the relation (6.1) remains true at every point of V_{2n}. An A.T. structure is thus defined on V_{2n}. Since $\nabla F^j_i = 0$, by proposition 4, the given connection may be identified with an A.T. connection. We may thus state the following proposition:

PROPOSITION 10. A necessary and sufficient condition in order that a complex linear connection in a manifold V_{2n} be an A.T. connection of an A.T. structure is that the holonomy group of the linear connection be a sub-group of $G(^n_{nn})$.

7. The restricted holonomy group. Before studying the restricted holonomy group we shall prove the following lemma.

LEMMA. The set $SG(^n_{nn})$ of all the matrices of the form $\alpha = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$ with $\det A = 1$ is an invariant subgroup of $G(^n_{nn})$.

Proof. If

$$\alpha = \begin{pmatrix} A & 0 \\ B & A \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} A_1 & 0 \\ B_1 & A_1 \end{pmatrix},$$

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with $\det A = \det A_1 = 1$, we have

$$\alpha\alpha_1^{-1} = \begin{pmatrix} AA_1^{-1} & 0 \\ * & AA_1^{-1} \end{pmatrix}$$

and $\det(AA^{-1}) = \det A \det(A_1^{-1}) = 1$. Hence $\alpha\alpha_1^{-1} \in SG\binom{n}{nn}$ and and $SG\binom{n}{nn}$ is a subgroup of $G\binom{n}{nn}$. It is an invariant subgroup, because for any $A \in SG\binom{n}{nn}$ and any $A_1 \in G\binom{n}{nn}$ we have $\det(A_1^{-1}AA_1) = \det(A_1^{-1})(\det A)(\det A_1) = 1$. Hence $\alpha_1^{-1}\alpha\alpha_1 \in SG\binom{n}{nn}$. The $SG\binom{n}{nn}$ is obviously a Lie subgroup of the Lie group $G\binom{n}{nn}$.

Without changing notations we shall now pass to the universal covering of V_{2n}.

Let b be an adapted basis at the point $x_o \in V_{2n}$, and let us assume that the restricted holonomy group σ_b is a subgroup of $SG\binom{n}{nn}$. Then this assumption will be true at every point of $E_T(V_{2n})$. We introduce at the point x_o the covariant tensor t_o of order n, whose components with respect to the base b are

$$t_{i_1 i_2 \ldots i_n} = \varepsilon^{1*2*\ldots n*}_{i_1 i_2 \ldots i_n} .$$

The tensor t_o is invariant under σ_b. Indeed,

$$t_{j'_1 j'_2 \ldots j'_n} = A^{i_1}_{j'_1} A^{i_2}_{j'_2} \ldots A^{i_n}_{j'_n} \varepsilon^{1*2*\ldots n*}_{i_1 i_2 \ldots i_n} .$$

Hence

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$$t_{\zeta'_1\zeta'_2\ldots\zeta'_n} = t_{\zeta'_1\zeta'_2{}^*\ldots\zeta'_n} = 0 ,$$

$$t_{\zeta'_1{}^*\zeta'_2{}^*\ldots\zeta'_n{}^*} = A^{\alpha^*_1}_{\zeta'_1{}^*} A^{\alpha^*_2}_{\zeta'_2{}^*} \ldots A^{\alpha^*_n}_{\zeta_n{}^*} \varepsilon^{1*2*\ldots n*}_{\alpha^*_1\alpha^*_2\ldots\alpha^*_n}$$

$$= \varepsilon^{1*2*\ldots n*}_{\zeta'_1{}^*\zeta'_2{}^*\ldots\zeta'_n{}^*} .$$

Therefore

$$t_{j'_1j'_2\ldots j'_n} = \varepsilon^{1*2*\ldots n*}_{j'_1j'_2\ldots j'_n} .$$

By parallel displacement, t_o generates a tensor t defined on the whole V_{2n} and $\nabla t = 0$. If U is an open neighbourhood of V_{2n} endowed with a local cross section of $E_T(V_{2n})$, there exists a differentiable function e^f with complex values $\neq 0$, defined on U, such that we have in U

$$(7.1) \qquad t_{i_1i_2\ldots i_n} = \varepsilon^{1*2*\ldots n*}_{i_1i_2\ldots i_n} e^f .$$

From (7.1) we obtain

$$\nabla t_{i_1i_2\ldots i_n} = (de^f)\,\varepsilon^{1*\ldots n*}_{i_1\ldots i_n} + e^f \nabla\varepsilon^{1*\ldots n*}_{i_1\ldots i_n} ;$$

but

$$\nabla\varepsilon^{1*\ldots n*}_{i_1i_2\ldots i_n} = -\pi^\rho_{i_1}\varepsilon^{1*2*\ldots n*}_{\rho\, i_2\ldots i_n} - \pi^\rho_{i_2}\varepsilon^{1*\ldots n*}_{i_1\rho i_2\ldots i_n} - \ldots$$

$$- \pi^\rho_{i_n}\varepsilon^{1*\ldots n*}_{i_1\ldots i_{n-1}\rho}$$

or

$$\nabla \varepsilon^{1^*\dots n^*}_{\alpha^*_1\dots\alpha^*_n} = -\pi^{\rho^*}_{\alpha^*_1}\varepsilon^{1^*\dots n^*}_{\rho^*\alpha^*_2\dots\alpha^*_n} - \pi^{\rho^*}_{\alpha^*_2}\varepsilon^{1^*\dots n^*}_{\alpha^*_1\rho^*\dots\alpha^*_n} - \dots$$

$$- \pi^{\rho^*}_{\alpha^*_n}\varepsilon^{1^*\dots n^*}_{\alpha^*_1\dots\alpha^*_{n-1}\rho}$$

$$= -(\pi^{\alpha^*_1}_{\alpha^*_1} + \pi^{\alpha^*_2}_{\alpha^*_2} + \dots + \pi^{\alpha^*_n}_{\alpha^*_n})\,\varepsilon^{1^*2^*\dots n^*}_{\alpha^*_1\alpha^*_2\dots\alpha^*_n} = -\pi^{\alpha^*}_{\alpha^*}\varepsilon^{1^*\dots n^*}_{\alpha^*_1\dots\alpha^*_n} .$$

Finally

$$\nabla\varepsilon^{i^*\dots n^*}_{i_1\dots i_n} = -\pi^{\alpha^*}_{\alpha^*}\varepsilon^{i^*2^*\dots n^*}_{i_1 i_2\dots i_n} .$$

We may thus write (7.2)

$$\nabla t_{i_1 i_2\dots i_n} = e^f(df - \pi^{\alpha^*}_{\alpha^*})\,\varepsilon^{1^*\dots n^*}_{i_1\dots i_n}$$

and obtain

$$\pi^{\alpha^*}_{\alpha^*} = \pi^{\alpha}_{\alpha} = df ,$$

or

$$\psi = d\pi^{\alpha}_{\alpha} = d^2 f = 0 .$$

The characteristic form is everywhere equal to zero.

Conversely, let us consider a differentiable manifold V_{2n}, equipped with an A. T. connection and let us assume that the characteristic form ψ is zero at every point of V_{2n}. With respect to any local cross section of $E_T(V_{2n})$, we have

$$d\pi^{\alpha}_{\alpha} = d\pi^{\alpha^*}_{\alpha^*} = 0 .$$

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To every point x of V_{2n} we may associate an open neighbourhood $U(x)$ and a complex-valued function f defined in U such that, with respect to the cross section,

$$\pi^{\alpha}_{\alpha} = df .$$

We now consider the covariant tensor of order n defined in U, whose components with respect to the local cross section are

$$t_{i_1 i_2 \dots i_n} = \varepsilon^{1^* \dots n^*}_{i_1 \dots i_n} e^{f} ;$$

its absolute differential is given by

$$\nabla t_{i_1 i_2 \dots i_n} = \varepsilon^{1^* \dots n^*}_{i_1 \dots i_n} e^{f}(df - \pi^{\alpha}_{\alpha}) = 0 .$$

If b_x is an adapted basis at x, the holonomy group σ_{b_x} of the connection at b_x is, as we have seen previously, a subgroup of $G(^{n}_{nn})$. Since $\nabla t = 0$ in U, the elements of σ_{b_x} which we obtain by developing the loops at x situated in U, leave t invariant. Therefore they belong in $SG(^{n}_{nn})$. Since we may associate to every point x, such a neighbourhood U, it follows from the factorization lemma [(2), p. 52], that for every $b \in E_T(V_{2n})$, σ_b is a subgroup of $SG(^{n}_{nn})$. We may thus state:

PROPOSITION 11. In order that a manifold with an A. T. connection has an holonomy group σ as sub-group of $SG(^{n}_{nn})$, it is necessary and sufficient that the characteristic form of the connection be equal to zero.

Some interesting topics of the theory of manifolds with A. T. structures are: 1. The compatibility of Euclidean and

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A. T. structures. 2. Compatibility of Hermitian and A. T. structures. 3. The automorphisms of such manifolds. The first topic is already studied in [9], the other two will be investigated in another paper.

APPENDIX

If t_i^j and $t_{\ell'}^{m'}$ are the components of a tensor of type (1, 1) with respect to two different adapted bases, we have the following relations:

$$t_{\lambda'}^{\mu'} = A_{\lambda'}^{\alpha} A_{\zeta}^{\mu'} t_{\alpha}^{\zeta} + A_{\lambda'}^{\alpha} B_{\zeta *}^{\mu'} t_{\alpha}^{\zeta *} ,$$

$$t_{\lambda' *}^{\mu' *} = B_{\lambda' *}^{\alpha} A_{\zeta *}^{\mu' *} t_{\alpha}^{\zeta *} + A_{\lambda' *}^{\alpha *} A_{\zeta *}^{\mu' *} t_{\alpha *}^{\zeta *} ,$$

$$t_{\lambda'}^{\mu' *} = A_{\lambda'}^{\alpha} A_{\zeta *}^{\mu'} t_{\alpha}^{\zeta *} ,$$

$$t_{\lambda' *}^{\mu'} = B_{\lambda' *}^{\alpha} A_{\zeta}^{\mu'} t_{\alpha}^{\zeta} + B_{\lambda' *}^{\alpha} A_{\zeta *}^{\mu'} t_{\alpha}^{\zeta *} + A_{\lambda' *}^{\alpha *} A_{\zeta}^{\mu'} t_{\alpha *}^{\zeta} + A_{\lambda' *}^{\alpha *} B_{\zeta *}^{\mu'} t_{\alpha *}^{\zeta *} .$$

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