Integer triangles with integer circumradii

EMRYS READ

Introduction

When studying properties of the circumcircle of an integer triangle, it quickly becomes evident that the radius of such a circle (*circumradius*) need not itself be an integer. When it is not an integer, the circumradius can still be rational but it can also be irrational, as exemplified in the following examples. It is left to the reader to verify that the triangle with sides 10, 24, 26 has circumradius 13, and that the corresponding values for the triangles with sides 13, 14, 15 and 1, 1, 1 are 65/8 and $\sqrt{3}/3$ respectively. It is shown in Theorem 1 below that a necessary condition for the circumradius to be an integer is that the area of the triangle is itself an integer (Heronian triangle) but this condition is not in itself sufficient. A simple counterexample is given by the 13, 14, 15 triangle above which has area 84. However, as a consequence of Theorem 1, we can restrict ourselves to considering Heronian triangles, and relevant properties of such triangles, proved in [1], are given in Theorems 2 and 3 below. We also need to quote some well-known results involving the sums of two squares (see e.g. [2]) and these are listed in Lemma 3. In all that follows, we will use the convention that if T is a triangle with sides a, b, c and $z > 0$, then zT will denote the triangle, similar to T, with sides za, zb, zc.

Some results, similar to ones proved here but enunciated in terms of the circumdiameter appear in [3]. Our main results are given in Theorems 4, 5, 6 and 7. In Theorem 4, we show that within each equivalence class of similar Heronian triangles, there exists a *minimal* triangle M which is such that every triangle in this class with integer circumradius is of the form qM, for some positive integer q. Furthermore, the circumradius of M is an odd number which is the product of primes of the form $4n + 1$. Finally, in Theorems 5, 6 and 7, we show how to construct such a mimimal triangle having a given circumradius.

Preliminary results on Heronian triangles

Lemma 1

Let the triangle T with sides a, b, c have area Δ and circumradius R. Then $\Delta = \dfrac{abc}{4R}$.

Proof:

This is a well-known result whose proof is straightforward and left to the reader.

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Lemma 2

Any integer triangle whose area is known to be rational is in fact a Heronian triangle.

Proof:

This possibly unexpected result, a proof of which may be found in Lemma 3 of [3], is a well-known property of Heronian triangles.

Theorem 1

Let T be an integer triangle whose circumradius is also an integer. Then T must be a Heronian triangle.

Proof:

We see from Lemma 1 that the area of T must be rational. The result now follows from Lemma 2.

Theorem 2

Let β, δ be rationals such that $0 < \beta < 1$ and $0 < \delta < 1$. Write $\beta = \beta_1 / \beta_2$, $\delta = \delta_1 / \delta_2$, where β_1, β_2, δ_1, δ_2 are positive integers such that $\gcd(\beta_1, \beta_2) = \gcd(\delta_1, \delta_2) = 1$. Let $h = \gcd(\beta_1\beta_2, \delta_1\delta_2)$, $k = \gcd(\beta_2\delta_2 - \beta_1\delta_1, \beta_1\delta_2 + \beta_2\delta_1)$ and $d = hk$.

(a) The triangle $T_{\{\beta,\delta\}}$, with sides $a = \beta_1\beta_2(\delta_1^2 + \delta_2^2)$, $b = \delta_1\delta_2(\beta_1^2 + \beta_2^2)$, $c = (\beta_2\delta_2 - \beta_1\delta_1)(\beta_1\delta_2 + \beta_2\delta_1)$ is a Heronian triangle.

(b) $T_{\{\beta,\delta\}}$ has area $\beta_1\beta_2\delta_1\delta_2(\beta_2\delta_2 - \beta_1\delta_1)(\beta_1\delta_2 + \beta_2\delta_1)$.

(c) $P_{\{\beta,\delta\}} = \frac{1}{d}T_{\{\beta,\delta\}}$ is a *primitive* Heronian triangle (in the sense that the gcd of its sides is 1).

(d) All primitive Heronian triangles are of the form $P_{\{\beta,\delta\}}$ for some choice of rational β, δ such that $0 < \beta < 1$ and $0 < \delta < 1$.

(e) $P_{\{\beta,\delta\}}$ (and hence $T_{\{\beta,\delta\}}$) is right-angled if, and only if, $\beta_2\delta_2 - \beta_1\delta_1 = \beta_1\delta_2 + \beta_2\delta_1$.

Proof:

All these results are either proved in [1] or are immediate consequences of results proved therein.

Theorem 3

Using the same notation as in Theorem 2,

(a) $\gcd(h, \beta_1^2 + \beta_2^2) = \gcd(h, \delta_1^2 + \delta_2^2) = 1$,

(b) $R_P = \dfrac{(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)}{4hk}$, where R_P denotes the circumradius of $P_{\{\beta,\delta\}}$.

Proof:

(a) Let p be a prime factor of h. Then p divides either β_1 or β_2 but not both and thus p does not divide $\beta_1^2 + \beta_2^2$. It follows that $\gcd(h, \beta_1^2 + \beta_2^2) = 1$ and in the same way $\gcd(h, \delta_1^2 + \delta_2^2) = 1$.

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(b) Let R_T denote the circumradius of $T_{\{\beta,\delta\}}$. If we now substitute in Lemma 1 the expressions for a, b, c and Δ given in Theorem 2(a), we see that $R_T = \frac{1}{4}\left(\beta_1^2 + \beta_2^2\right)\left(\delta_1^2 + \delta_2^2\right)$.

Since $P_{\{\beta,\delta\}} = \frac{1}{d}T_{\{\beta,\delta\}}$, it follows that $R_P = \dfrac{\left(\beta_1^2 + \beta_2^2\right)\left(\delta_1^2 + \delta_2^2\right)}{4hk}$.

Minimal Heronian triangles

In order to be able to prove our main results, we need two further lemmas, each of which involves properties of positive integers.

Lemma 3

(a) Let u, v be positive integers such that $\gcd(u, v) = 1$. Then $u^2 + v^2$ is not divisible by any prime of the form $4n + 3$.

(b) Let N be the product of odd primes each of the form $4n + 1$. Then

 (i) there exist positive integers u, v, with $u < v$ and $\gcd(u, v) = 1$ such that $u^2 + v^2 = N$,

 (ii) the number of different expressions for N of the above form is 2^{t-1}, where t is the number of distinct primes in the decomposition of N.

Proof:

Both are immediate consequences of the result given in Theorem 3.22 of [2].

Lemma 4

Let u, v, x, y be positive integers such that $u < v$, $x < y$ and $\gcd(u, v) = \gcd(x, y) = 1$. Let $w = \gcd(vy - ux, uy + vx)$. Then

(a) $\gcd(w, u) = \gcd(w, v) = \gcd(w, x) = \gcd(w, y) = 1$,

(b) $w \mid u^2 + v^2$ and $w \mid x^2 + y^2$.

Proof:

(a) Let $z = \gcd(w, y)$. Then $z \mid ux$ and $z \mid vx$ and, since $\gcd(x, y) = 1$, we must have $z \mid u$ and $z \mid v$. The fact that $\gcd(u, v) = 1$ now implies that $\gcd(w, y) = z = 1$. In the same way, $\gcd(w, u) = \gcd(w, v) = \gcd(w, x) = 1$.

(b) $w \mid v(vy - ux) + u(uy + vx)$ and thus $w \mid y\left(u^2 + v^2\right)$. But by (a), $\gcd(w, y) = 1$ and thus $w \mid u^2 + v^2$. In the same way $w \mid x^2 + y^2$.

Theorem 4

Let the notation be as above with β, δ as in Theorem 2. Then there exists a Heronian triangle $M_{\{\beta,\delta\}}$ having the following properties.

(a) The circumradius R_M of $M_{\{\beta,\delta\}}$ is an odd integer.

(b) If H is a Heronian triangle with integer circumradius R and H is similar to $P_{\{\beta,\delta\}}$ then $H = qM_{\{\beta,\delta\}}$ for some positive integer q.

(c) $R_M \geqslant 5$.

(d) R_M is the product of odd primes of the form $4n + 1$.

We shall call such a triangle $M_{\{\beta,\delta\}}$ a *minimal* Heronian triangle.

Proof:

In order to prove our results, we need to consider three different possibilities.

Type 1: one of β_1, β_2, δ_1, δ_2 is even, the other three odd,

Type 2: all four of β_1, β_2, δ_1, δ_2 are odd,

Type 3: one of β_1, β_2 is even, one odd, one of δ_1, δ_2 is even, one odd.

First of all, assume that β_1, β_2, δ_1, δ_2 are of Type 1 and, without loss of generality, that one of β_1, β_2 is even and one is odd.

(a) We note that β_1, β_2, δ_1, δ_2 satisfy the conditions of Lemma 4. It follows that $k \mid (\beta_1^2 + \beta_2^2)$ and thus k must be odd. Also $(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2) \equiv 2 \pmod 4$. Let $A = \frac{1}{2k}(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)$ Then A must be an odd integer.

Furthermore, Theorem 3 tells us that $R_P = \frac{1}{4hk}(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)$ and also that $\gcd\left(h, (\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)\right) = 1$. It follows that $R_P = \frac{A}{2h}$ is a fraction in its lowest terms. Now define $M_{\{\beta,\delta\}}$ to be $2hP_{\{\beta,\delta\}}$. Then $R_M = A$, an odd integer.

(b) Since H is similar to $P_{\{\beta,\delta\}}$, there exists an integer z such that $H = zP_{\{\beta,\delta\}}$ and thus $R = zR_P = z\frac{A}{2h}$. Since R is an integer and $\gcd(A, 2h) = 1$, there exists an integer q such that $z = 2hq$. Consequently $H = 2hqP_{\{\beta,\delta\}}$ and thus $H = qM_{\{\beta,\delta\}}$.

(c) $R_M = \frac{1}{2k}(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)$. As in part (a), we note that $k \mid (\delta_1^2 + \delta_2^2)$, and, as k is odd and $(\delta_1^2 + \delta_2^2)$ even, we see that $2k \mid (\delta_1^2 + \delta_2^2)$. Thus $R_M \geqslant (\beta_1^2 + \beta_2^2)$ and as one of β_1, β_2 is even and the other odd, we must have $R_M \geqslant 5$.

(d) From Lemma 3(a), we know that $(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)$ cannot be divisible by any prime of the form $4n + 3$ and thus neither can R_M. The result now follows, since R_M is odd and cannot equal 1 since it must be $\geqslant 5$ by part (c).

The proof is now complete when β, δ are of Type 1. The proofs for β, δ of Types 2 and 3 although not identical follow similar lines and are left as an exercise for the reader. The important details are given in the table below.

	Type 1	Type 2	Type 3
β_1, β_2, δ_1, δ_2	3 odd, 1 even	4 odd	2 odd, 2 even
$(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)$	$\equiv 2 \pmod 4$	$\equiv 4 \pmod 8$	odd
k	odd	$\equiv 2 \pmod 4$	odd
$A = \dfrac{(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)}{2k}$	A odd integer	A odd integer	$2A$ odd integer
R_P in its lowest terms	$\dfrac{A}{2h}$	$\dfrac{A}{2h}$	$\dfrac{(2A)}{4h}$
$M_{\{\beta,\delta\}}$	$2hP_{\{\beta,\delta\}}\left(= \frac{2}{k}T_{\{\beta,\delta\}}\right)$	$2hP_{\{\beta,\delta\}}\left(= \frac{2}{k}T_{\{\beta,\delta\}}\right)$	$4hP_{\{\beta,\delta\}}\left(= \frac{4}{k}T_{\{\beta,\delta\}}\right)$
$\boldsymbol{R_M}$	$\boldsymbol{A}$	$\boldsymbol{A}$	$\boldsymbol{2A}$
R_M	$\geqslant 5$	$\geqslant 5$	$\geqslant 5$

TABLE 1

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Corollary 1

(a) No primitive integer triangle can have an integer circumradius.

(b) If the integer R is the circumradius of the integer triangle H, then R must be divisible by some odd prime of the form $4n + 1$. In particular, no prime of the form $4n + 3$ can be the value of the circumradius of any integer triangle.

Proof:

(a) Note that each of the expressions for R_P given in Table 1 above has an odd numerator and an even denominator and thus R_P can never be an integer.

(b) $H = qM_{\{\beta,\delta\}}$ for some β, δ, where q is an integer, and thus $R = qR_M$. But, by part (d), R_M is the product of odd primes of the form $4n + 1$ and thus some such prime must divide R.

Theorem 5

Let N be an odd integer which is the product of primes of the form $4n + 1$.

(a) There exists a minimal right-angled triangle $M_{\{\beta,\delta\}}$ such that $R_M = N$.

(b) The number of distinct such right-angled triangles $M_{\{\beta,\delta\}} = 2^{t-1}$, where t is the number of distinct primes in the decomposition of N.

Proof:

(a) By Lemma 3(b)(i), there exist positive integers u, v with $u < v$ and $\gcd(u, v) = 1$ such that $u^2 + v^2 = N$. Since N is odd, one of u, v must be even and the other odd. Put $\beta_1 = u$, $\beta_2 = v$, $\delta_1 = v - u$, $\delta_2 = v + u$. We note that $\gcd(\delta_1, \delta_2) = 1$ and it may easily be verified that $\beta_1\delta_2 + \beta_2\delta_1 = \beta_2\delta_2 - \beta_1\delta_1 = u^2 + v^2$. Thus by Theorem 2(e) $T_{\{\beta,\delta\}}$ is right-angled. Furthermore $h = 1$, $k = v^2 + u^2$ and we see that $P_{\{\beta,\delta\}}$ has sides $v^2 - u^2$, $2uv$, $v^2 + u^2$. It is well known that a triangle with these sides is a primitive right-angled triangle and that every primitive right-angled triangle is of this form for appropriate values of u, v. Since only one of β_1, β_2, δ_1, δ_2 is even (Type 1), it follows that the right-angled triangle $M_{\{\beta,\delta\}} = 2hP_{\{\beta,\delta\}}$ has sides $2(v^2 - u^2)$, $4uv$, $2(v^2 + u^2)$ and that

$$R_M = \frac{(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)}{2k} = \frac{(u^2 + v^2) \times 2(u^2 + v^2)}{2 \times (u^2 + v^2)} = u^2 + v^2 = N.$$

(b) This now follows immediately from Lemma 3(b)(ii).

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Theorem 6

Let the integer triangle ABC have circumradius p, where p is a prime. Then ABC is a right-angled triangle.

Proof:

By Theorem 1, ABC is Heronian and thus $\Delta = \dfrac{abc}{4p}$ is an integer. Consequently p must divide one of a, b, c and without loss of generality we may assume that p divides a. Now $a \leqslant 2p$, the diameter of the circumcircle, and thus either $a = p$ or $a = 2p$. Let O denote the circumcentre of ABC. If $a = p$, BOC would be an equilateral triangle and angle BAC would thus be 30°. However, the cosine rule tells us that this is impossible in an integer triangle. Thus $a = 2p$ and BC is a diameter of the circumcircle, which implies that ABC must be a right-angled triangle.

Theorem 7

Let N be an odd integer which is the product of at least two primes (not necessarily distinct) of the form $4n + 1$. There exists a minimal Heronian triangle $M_{\{\beta,\delta\}}$ which is *not* right-angled such that $R_M = N$.

Proof:

Assume $N = Qp$, where $Q > 1$ is a product of primes of the form $4n + 1$ and p is a prime of the form $4n = 1$.

By Lemma 3(b)(i), there exist positive integers u, v, x, y with $u < v$, $x < y$ and $\gcd(u, v) = \gcd(x, y) = 1$ such that $u^2 + v^2 = Q$ and $x^2 + y^2 = p$. Since Q and p are odd, one of u, v must be even and the other odd and one of x, y must be even and the other odd.

Let $w = \gcd(vy - ux, uy + vx)$. By Lemma 4(b), $w \mid (x^2 + y^2)$ and thus $w = 1$ or p.

Case 1: $w = 1$.

Put $\beta_1 = u, \beta_2 = v, \delta_1 = x, \delta_2 = y$.

Then one of β_1, β_2 is even, one is odd, one of δ_1, δ_2 is even, one is odd and $k = w = 1$.

Thus, from Table 1 (Type 3), we see that

$$R_M = \frac{(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)}{k} = \frac{Qp}{1} = N.$$

Case 2: $w = p$.

Let $f = u + v$, $g = v - u$. Then both f and g are odd. Also $g < f$ and $\gcd(f, g) = 1$ and thus g, f, x, y satisfy the conditions of Lemma 4. Let $z = \gcd(fy - gx, gy + fx)$.

By Lemma 4(b), $z \mid (x^2 + y^2)$ and thus $z = 1$ or p.

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If $z = p$, it follows that $p \mid fy - gx + gy + fx$ and, after substituting for f, g, we see that $p \mid yv + xu$,

But since $w = p$, we also know that $p \mid yv - xu$ and thus we have $p \mid xu$ and $p \mid yv$.

The fact that $x < p$ and $y < p$, implies that $p \mid u$ and $p \mid v$. This contradicts the fact that $\gcd(u, v) = 1$ and thus we must have $z = 1$.

Now put $\beta_1 = g = v - u, \beta_2 = f = v + u, \delta_1 = x, \delta_2 = y$.

Then $\left(\beta_1^2 + \beta_2^2\right) = \left(g^2 + f^2\right) = 2\left(u^2 + v^2\right)$.

Also, both of β_1, β_2 are odd, one of δ_1, δ_2 is even, one is odd and $k = z = 1$.

Therefore, in this case, we see from Table 1 (Type 1) that

$$R_M = \frac{\left(\beta_1^2 + \beta_2^2\right)\left(\delta_1^2 + \delta_2^2\right)}{2k} = \frac{2 \times Q \times p}{2 \times 1} = Qp = N.$$

Let us now consider the two following examples.

Example 1

Find a minimal non-right-angled Heronian triangle with circumradius 3125. Noting that $3125 = 625 \times 5$, we have $Q = 625$, $p = 5$ and thus $u = 7, v = 24, x = 1, y = 2$. Since $w = \gcd(41, 38) = 1$, we put $\beta_1 = 7, \beta_2 = 24, \delta_1 = 1, \delta_2 = 2$, so that β, δ are of Type 3. We also have $k = w = 1$ and $h = \gcd(168, 2) = 2$ and thus, by Theorem 2(c), $P_{\{\beta,\delta\}}$ has sides 420, 625, 779. Consequently, $M_{\{\beta,\delta\}} = 4hP_{\{\beta,\delta\}}$ has sides 3360, 5000, 6232. Moreover, $R_M = \dfrac{\left(7^2 + 24^2\right)\left(1^2 + 2^2\right)}{1} = \dfrac{625 \times 5}{1} = 3125$.

Example 2

Find a minimal non-right-angled Heronian triangle with circumradius 2197. Noting that $2197 = 169 \times 13$, we have $Q = 169$, $p = 13$ and thus $u = 5$, $v = 12$, $x = 2$, $y = 3$. Since $w = \gcd(26, 39) = 13 \neq 1$, we put $\beta_1 = v - u = 7$, $\beta_2 = u + v = 17$, $\delta_1 = 2$, $\delta_2 = 3$, so that β, δ are of Type 1. We also have $k = z = \gcd(37, 55) = 1$ and $h = \gcd(119, 6) = 1$ and thus, from Theorem 2(c), $P_{\{\beta,\delta\}}$ has sides 1547, 2028, 2035. Consequently $M_{\{\beta,\delta\}} = 2hP_{\{\beta,\delta\}}$ has sides 3094, 4056, 4070. Moreover,

$$R_M = \frac{\left(7^2 + 17^2\right)\left(2^2 + 3^2\right)}{2 \times 1} = \frac{338 \times 13}{2 \times 1} = 2197.$$

Particular case

In each of the above examples N is a prime power. The following alternative method for finding $M_{\{\beta,\delta\}}$ is valid when the prime decomposition of N contains at least two distinct primes. In this case we may write $N = QL$, where $Q > 1, L > 1$ and $\gcd(Q, L) = 1$.

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By Lemma 3(b)(i), there exist integers u, v, x, y with $u < v, x < y$ and $\gcd(u, v) = \gcd(x, y) = 1$ such that $u^2 + v^2 = Q$ and $x^2 + y^2 = L$. Let $w = \gcd(vy - ux, uy + vx)$.

Then by Lemma 4(b), $w \mid (u^2 + v^2)$ and $w \mid (x^2 + y^2)$ and thus $w \mid \gcd(Q, L) = 1$.

Put $\beta_1 = u, \beta_2 = v, \delta_1 = x, \delta_2 = y$. Then one of β_1, β_2 is even, one is odd, one of δ_1, δ_2 is even, one is odd and $k = w = 1$. Thus, from Table 1 (Type 3), we see that $R_M = \dfrac{(\beta_1^2 + \beta_2^2)(\delta_1^2 + \delta_2^2)}{k} = \dfrac{QL}{1} = N$.

We illustrate this alternative approach in the following example.

Example 3

Find a minimal non-right-angled Heronian triangle with circumradius 65. Noting that $N = 65 = 5 \times 13$, we have $Q = 5$, $L = 13$ and thus $u = 1$. $v = 2$, $x = 2$, $y = 3$, so that β, δ are of Type 3. We also have $k = w = \gcd(4, 7) = 1$, $h = \gcd(2, 6) = 2$ and thus, from Theorem 2(c), $P_{\{\beta,\delta\}}$ has sides 13, 15, 14. Consequently $M_{\{\beta,\delta\}} = 4hP_{\{\beta,\delta\}}$ has sides 104, 120, 112. Moreover, $R_M = \dfrac{(1^2 + 2^2)(2^2 + 3^2)}{1} = \dfrac{5 \times 13}{1} = 65$.

Finally, we note that each of the three above procedures does in fact yield a triangle which is not right-angled. By Theorem 2(e), $P_{\{\beta,\delta\}}$, and consequently $M_{\{\beta,\delta\}}$, will be right-angled if, and only if, $\beta_2\delta_2 - \beta_1\delta_1 = \beta_1\delta_2 + \beta_2\delta_1$.

Thus $k = \gcd(\beta_2\delta_2 - \beta_1\delta_1, \beta_1\delta_2 + \beta_2\delta_1)$ will equal $\beta_1\delta_2 + \beta_2\delta_1$, which is greater than 1. But in each of the above constructions $k = 1$, and thus $M_{\{\beta,\delta\}}$ cannot be right-angled.

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Gwynedd,
LL55 1UW
e-mail: *mairacemrys@btinternet.com*